Multiple Binomial Sums

Joint work with Alin Bostan and Pierre Lairez

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Examples of Binomial Sums

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n^3} \]  
\text{Dixon}

\[ \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2 \]  
\text{Andrews-Paule}

\[ \sum_{j,k} (-1)^{j+k} \binom{j+k}{k+1} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l} \]  
\text{Graham-Knuth-Patashnik}

\[ \sum_{r_1 \geq 0, r_2 \geq 0, r_3 \geq 0} \frac{(r_2+1)(r_3+1)(r_2+r_3+2)}{(n+1)(n+2)} \binom{n+2}{r_1} \binom{n+2}{r_1+r_2+1} \binom{n+2}{r_1+r_2+r_3+2} = 2^{n-1} \binom{2n+1}{n} \]  
\text{Essam-Guttmann}

**Aims:**

1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable
Result

An efficient algorithm computing a linear recurrence for binomial sums.

\[
\sum_{n \geq 0} \sum_{u_n} \left( \ldots \right) z^n \quad \rightarrow \quad \left( \begin{array}{c} \ldots \\ \ldots \end{array} \right)
\]

\[u_{n+p} + \cdots + (\ldots)u_n = 0 \quad \rightarrow \quad y^{(m)}(z) + \cdots + ()y(z) = 0\]
Example: Apéry’s sequence

from his proof that $\zeta(3) \notin \mathbb{Q}$

$$U_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n + k}{k}^2$$

$$U(z) := \sum_{n \geq 0} U_n z^n$$

$$U(z) = \frac{1}{(2\pi i)^3} \oint \frac{dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1 - t_1 t_2 - t_1 t_2 t_3) - (1 + t_1)(1 + t_2)(1 + t_3)z}$$

$$z(z^2 - 34z + 1)U'''(z) + \cdots + (z - 5)U(z) = 0$$

$$(n + 2)^3 U_{n+2} - (2n + 3)(17n^2 + 51n + 39)U_{n+1} + (n + 1)^3 U_n = 0$$
Why is this a solution?

With a linear recurrence, it is easy to:

• find hypergeometric closed-forms (Petkovšek’s algo.);
• compute efficiently the first \( n \) terms, the \( n \)th term;
• (less easy) study the asymptotic behavior;
• build other sequences by closure properties;
• prove identities.

This is what summing means in symbolic computation
I. Creative Telescoping
Creative telescoping is:

- Differentiation under the integral sign plus integration by parts, made algorithmic;
- and the analogue for sums;
- and for multiple sums and multiple integrals.
Example: Apéry’s sequence

\[ U_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_{n,k} \]

1. Find a telescoping relation

\[
(n + 2)^3 f_{n+2,k} + \cdots + (n + 1)^3 f_{n,k} = g_{n,k+1} - g_{n,k}
\]

with

\[
g_{n,k} = \frac{\text{Pol}(n,k)}{(n + 1 - k)^2(n + 2 - k)^2} f_{n,k}
\]

2. Sum over \( k \) and telescope (with care)

\[
(n + 2)^3 U_{n+2} + \cdots + (n + 1)^3 U_n = 0
\]
An identity of Andrews & Paule
(difficult for creative telescoping)

\[ \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i + j}{i} \binom{4n - 2i - 2j}{2n - 2i} = (2n + 1) \binom{2n}{n} \]

Needed: \[ \sum a_k(n) f_{i,j,n+k} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n}) \]

. Andrews & Paule (93): two proofs with human insight;
. Wegschaider (97): via a generalization of Sister Celine’s technique;
. Koutschan’s code (2010):

\[ f_{i,j,n} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n}) \]

\[ \frac{g, h \text{ are}}{pol(i,j,n)} \frac{f_{i,j,n}}{(i + 1)(j + 1)(i + j - 2n)} \]

more work → result
An identity of Andrews & Paule
(with the new algorithm)

\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i} \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2
\]

\[
F(z) := \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f_{i,j,n} z^n
= \frac{1}{(2\pi i)^2} \int \frac{u_2(1+u_2)^2 du_1du_2}{(u_2^2 - z(1+u_2)^4)(u_1(u_2^2 + 2u_2 - u_1) - z(1+u_1)(1+u_2)^4)}
\]

\[
F^{(6)}(z) + \cdots + (\ldots)F(z) = 0
\]

\[
S_{n+4} + \cdots + (\ldots)S_n = 0 \quad \text{plus initial conditions}
\]

\[
\rightarrow (\text{Petkovšek’s algorithm}) \quad S_n = (2n+1) \binom{2n}{n}^2
\]

No human help. Total time < 2sec.
Telescoping Ideal

\[ T_t(f) := \left( \text{Ann} f + \partial_t \mathbb{Q}(x, t) \langle \partial_x, \partial_t \rangle \right) \cap \mathbb{Q}(x) \langle \partial_x \rangle \].

Approximated by:

1. Reducing the search space
   - restrict \text{int. by parts} to \( \mathbb{Q}(x) \langle \partial_x, \partial_t \rangle \) and use Gröbner bases. (The « holonomic » approach) [Wilf-Zeilberger, also Sister Celine].

2. Proceeding by increasing slices (and indeterminate coeffs)
   - hypergeometric summation: \( \text{dim}=1 + \)
   - finite dim, Ore algebras & GB (with F. Chyzak & M. Kauers)
   - infinite dim & GB
   - param. Gosper. [Zeilberger]
Examples of applications

- **Hypergeometric**: binomial sums, hypergeometric series;
  \[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3} \]

- **Higher dimension**: classical orthogonal polynomials, special functions like Bessel, Airy, Struve, Weber, Anger, hypergeometric and generalized hypergeometric,…

  \[ J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} \, dt \]

- **Infinite dimension**: Bernoulli, Stirling or Eulerian numbers, incomplete Gamma function,…

  \[ \int_0^\infty \exp(-xy) \Gamma(n, x) \, dx = \frac{\Gamma(n)}{y} \left( 1 - \frac{1}{(y + 1)^n} \right) \]
II. Faster Creative Telescoping
Certificates are too big

\[ C_n := \sum_{r,s} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{r} \binom{n+r}{s} \binom{2n-r-s}{n} f_{n,r,s} \]

\[(n + 2)^3 C_{n+2} - 2(2n + 3)(3n^2 + 9n + 7)C_{n+1} - (4n + 3)(4n + 4)(4n + 5)C_n = 180 \text{ kB} \sim 2 \text{ pages} \]

\[
I(z) = \int \frac{(1 + t_3)^2 dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1 + t_3 (1 + t_1))(1 + t_3 (1 + t_2)) + z(1 + t_1)(1 + t_2)(1 + t_3)^4}
\]

\[
z^2(4z + 1)(16z - 1)l'''(z) + 3z(128z^2 + 18z - 1)l''(z) + (444z^2 + 40z - 1)l'(z) + 2(30z + 1)l(z) = 1080 \text{ kB} \sim 12 \text{ pages} \]

Next, in \( T_t(f) := \left( \text{Ann } f + \partial_t Q(x, t) \langle \partial_x, \partial_t \rangle \right) \cap \left( Q(x) \langle \partial_x \rangle \right) \)

we restrict to rational \( f \) and \( \partial_t Q(x)[t, 1/\text{den } f] \langle \partial_x, \partial_t \rangle \)
Algorithm: \( R_0 := \left[ \frac{P}{Q^m} \right] \)  
for \( i=1,2,... \) do \( R_i := \left[ \partial_t R_{i-1} \right] \)  
when there is a relation \( c_0(t)R_0+...+c_i(t)R_i=0 \)  
return \( c_0+...+c_i\partial_t^i \)
More variables: Griffiths-Dwork reduction

\[ I(t) = \int \frac{P(t, x)}{Q^m(t, x)} \, dx \]

1. Control degrees by homogenizing \((x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_n)\)

2. If \(m=1\), \([P/Q] := P/Q\)

3. If \(m > 1\), reduce modulo Jacobian ideal \(J := \langle \partial_0 Q, \ldots, \partial_n Q \rangle\)

\[
P = r + v_0 \partial_0 Q + \cdots + v_n \partial_n Q
\]

\[
\frac{P}{Q^m} = \frac{r}{Q^m} - \frac{1}{m-1} \left( \partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}} \right) + \frac{1}{m-1} \underbrace{\partial_0 v_0 + \cdots + \partial_n v_n}_{A_{m-1}}
\]

\[
\left[ \frac{P}{Q^m} \right] := \frac{r}{Q^m} + [A_{m-1}]
\]

**Thm.** [Griffiths] In the regular case \((\mathbb{Q}(t)[x]/J)\) (finite dim), if \(R = P/Q^m\) hom of degree \(-n-1\), \([R] = 0 \iff \int Rdx = 0\).

→ SAME ALGORITHM.
Size and complexity

\[ I(t) = \int_{\mathbb{Q}(t,x)} \frac{P(t,x)}{Q^m(t,x)} \, dx \]

\[ \mathbf{Q} := \deg_x Q, \quad d_t := \max(\deg_t Q, \deg_t P) \]

**Thm.** [Bostan-Lairez-S. 2013] A linear differential equation for \( I(t) \) can be computed in \( \mathcal{O}(e^{3nN^8d_t}) \) operations in \( \mathbb{Q} \). It has order \( \leq N^n \) and degree \( \mathcal{O}(e^nN^3d_t) \).

Note: generically, the certificate has at least \( N^{n^2/2} \) monomials.

Non-regular case by deformation, better way in Pierre Lairez’s poster.
III. Multiple Binomial Sums
Definition

Multiple binomial sums are obtained from
indicators \( \mathbb{I}_n \)
geometric sequences \( n \mapsto C^n, \quad C \in \mathbb{Q} \)
binomial coefficients \( (n, k) \mapsto \binom{n}{k} \)

defined as the coeff of \( x^k \)
in \((1+x)^n\).

by

\(+, \times, \) multiplication by scalars,

indefinite summation,

affine transformations of the indices.
Examples

• All the sums in this talk;
• the Catalan numbers: \[ \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \]
• more generally: coefficients of algebraic series \( P(x,y)=0 \);
• more generally: coefficients of diagonals of rational series.

\[ \Delta \sum_{i} f_{i}z^{i} := \sum_{i} f_{i},...,iz^{i} \]

**Thm.** The sequence \( u_{n} \) is a multiple binomial sum **iff** its generating function is the diagonal of a rational series.

**Christol’s conjecture:** if \( f \in \mathbb{Z}[t] \) is convergent and solution of a LDE, then it is the diagonal of a rational series.
### Generating Function Dictionary

\[ u_i \mapsto U(z) = \sum_{i} u_i z^i \]

<table>
<thead>
<tr>
<th>Function</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>indicator ( \mathbb{I}_n )</td>
<td>( \frac{1}{1-tz} )</td>
</tr>
<tr>
<td>geometric sequence ( C^n )</td>
<td>( \frac{1}{1-Cz} )</td>
</tr>
<tr>
<td>binomial coefficient</td>
<td>( \frac{1}{1-t(1+z)} )</td>
</tr>
<tr>
<td>( \Sigma : (u_n)<em>n \mapsto \left( \sum</em>{k=0}^{n} u_k \right)_n )</td>
<td>( \frac{1}{1-z} U(z) )</td>
</tr>
<tr>
<td>( \times : ((u_n)_n, (v_n)_n) \mapsto (u_n v_n)_n )</td>
<td>( U(z) \odot V(z) = \Delta(U(z_1)V(z_2)) )</td>
</tr>
<tr>
<td>( \text{CT} : (u_n)_n \mapsto u_0 )</td>
<td>( U(z) \odot 1 )</td>
</tr>
<tr>
<td>( \text{Shift} : (u_n)<em>n \mapsto (u</em>{n+b})_n )</td>
<td>( z^{-b}U(z) )</td>
</tr>
<tr>
<td>( \text{Dilat} : (u_n)<em>n \mapsto (u</em>{kn})_n )</td>
<td>( \frac{1}{k}(U(z^{1/k}) + U(\omega z^{1/k}) + \cdots + U(\omega^{k-1}z^{1/k})), \quad \omega^k = 1 )</td>
</tr>
<tr>
<td>( \Delta : (u_{n,k})<em>{n,k} \mapsto (u</em>{n,n})_n )</td>
<td>( \Delta U(z_1, z_2) = \frac{1}{2\pi i} \oint U\left(\frac{z_1}{z_1}, \frac{z_2}{z_1}\right) \frac{dz_1}{z_1} )</td>
</tr>
<tr>
<td>( A : (u_n)<em>n \mapsto (u</em>{A \cdot n})_n, \ A \text{ invertible} )</td>
<td>( U(z^{A^{-1}}) )</td>
</tr>
</tbody>
</table>

Multiple binomial sums reduce to multiple integrals of rational functions.
Example: Dixon’s identity

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

decomposes as

$$\binom{n}{k} \rightarrow (n)^3 \rightarrow (-1)^k \binom{n}{k}^3 \mapsto \sum_{k=0}^m (-1)^k \binom{n}{k}^3 \xrightarrow{\Delta} \sum_{k=0}^n (-1)^k \binom{n}{k}^3 \xrightarrow{\text{Dilat}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3$$

which translates into

$$\text{Dilat}\Delta \frac{1}{1-t} \left( \frac{1}{1-t(1+z)} \otimes_{t,z} \frac{1}{1-t(1+z)} \otimes_{t,z} \frac{1}{1-t(1+z)} \otimes_{t,z} \frac{1}{(1+t)(1-z)} \right)$$

which is

$$\frac{1}{(2\pi i)^7} \int \text{Rat}(z, t_1, \ldots, t_7) dt_1 \cdots dt_7$$

Many tricks and optimizations possible
Demo?
Summary

Symbolic summation and integration deserve more attention in terms of complexity;

We propose algorithms that are much faster than usual creative telescoping, in particular by avoiding certificates;

Restricted to basic (but common!) classes, for now.