Multiple Binomial Sums

Joint work with Alin Bostan and Pierre Lairez

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Examples of Binomial Sums

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

Dixon

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$

Andrews-Paule

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}$$

Graham-Knuth-Patashnik

$$\sum_{r_1\geq 0, r_2 \geq 0, r_3 \geq 0} \frac{(r_2+1)(r_3+1)(r_2+r_3+2)}{(n+1)(n+2)} \binom{n+2}{r_1} \binom{n+2}{r_1+r_2+1} \binom{n+2}{r_1+r_2+r_3+2} = 2^{n-1} \binom{2n+1}{n}$$

Essam-Guttmann

Aims:

1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable

Result

$$\boxed{\text{An efficient algorithm computing a linear recurrence for binomial sums.}}$$

$$\sum_{n\geq 0} \underbrace{\sum \binom{}{}\binom{}{}\cdots\binom{}{}}_{u_n} z^n \longrightarrow \oiint \frac{()\cdots()}{()\cdots()}$$

$$u_{n+p} + \cdots + (\dots)u_n = 0 \longleftarrow y^{(m)}(z) + \cdots + ()y(z) = 0$$

Example: Apéry's sequence

from his proof that $\zeta(3) \notin \mathbb{Q}$

$$U_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad U(z) := \sum_{n\geq 0} U_n z^n$$

$$\Downarrow$$

$$U(z) = \frac{1}{(2\pi i)^3} \oint \frac{dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1 - t_1 t_2 - t_1 t_2 t_3) - (1+t_1)(1+t_2)(1+t_3) z}$$

$$\Downarrow$$

$$z(z^2 - 34z + 1)U'''(z) + \cdots + (z-5)U(z) = 0$$

$$\Downarrow$$

$$(n+2)^3 U_{n+2} - (2n+3)(17n^2 + 51n + 39)U_{n+1} + (n+1)^3 U_n = 0$$

Why is this a solution?

With a linear recurrence, it is easy to:

- find hypergeometric closed-forms (Petkovšek's algo.);
- compute efficiently the first n terms, the nth term;
- (less easy) study the asymptotic behavior;
- build other sequences by closure properties;
- prove identities.

This is what summing means in symbolic computation

I. Creative Telescoping

Creative telescoping is:

- Differentiation under the integral sign plus integration by parts, made algorithmic;

- and the analogue for sums;

- and for multiple sums and multiple integrals.

Example: Apéry's sequence

$$U_n := \sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{f_{n,k}}$$

1. Find a telescoping relation

Certificate

$$(n+2)^3 f_{n+2,k} + \cdots + (n+1)^3 f_{n,k} = g_{n,k+1} - g_{n,k}$$

shift under the sum sign

summation by parts

with $\quad g_{n,k} = \dfrac{\mathrm{Pol}(n,k)}{(n+1-k)^2(n+2-k)^2} f_{n,k}$

2. Sum over k and telescope (with care)

$$(n+2)^3 U_{n+2} + \cdots + (n+1)^3 U_n = 0$$

An identity of Andrews & Paule

(difficult for creative telescoping)

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\underbrace{\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i}}_{f_{i,j,n}} = (2n+1)\binom{2n}{n}^2$$

$$\boxed{\text{Needed: } \sum a_k(n) f_{i,j,n+k} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n})}$$

. Andrews & Paule (93): two proofs with human insight;

. Wegschaider (97): via a generalization of Sister Celine's technique;

. Koutschan's code (2010):

$$f_{i,j,n} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n})$$

g, h are $\dfrac{\mathrm{pol}(i,j,n)}{(i+1)(j+1)(i+j-2n)} f_{i,j,n}$

more work $\rightarrow$ result

An identity of Andrews & Paule

(with the new algorithm)

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\underbrace{\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i}}_{f_{i,j,n}} = (2n+1)\binom{2n}{n}^2$$

$$F(z) := \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} f_{i,j,n} z^n$$

$$= \frac{1}{(2\pi i)^2}\oint \frac{u_2(1+u_2)^2\,du_1 du_2}{(u_2^2 - z(1+u_2)^4)(u_1(u_2^2+2u_2-u_1) - z(1+u_1)(1+u_2)^4}$$

$$F^{(6)}(z) + \cdots + (\ldots)F(z) = 0$$

$S_{n+4} + \cdots + (\ldots)S_n = 0$ plus initial conditions

$\to$ (Petkovšek's algorithm) $S_n = (2n+1)\binom{2n}{n}^2$

No human help. Total time < 2sec.

Telescoping Ideal

$$\mathsf{T}_t(\mathsf{f}) := \left(\operatorname{Ann}\mathsf{f} + \underbrace{\partial_t\mathbb{Q}(\boldsymbol{x},t)\langle\boldsymbol{\partial}_{\boldsymbol{x}},\partial_t\rangle}_{\text{int. by parts}}\right) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial}_{\boldsymbol{x}}\rangle}_{\text{diff. under }\int}.$$

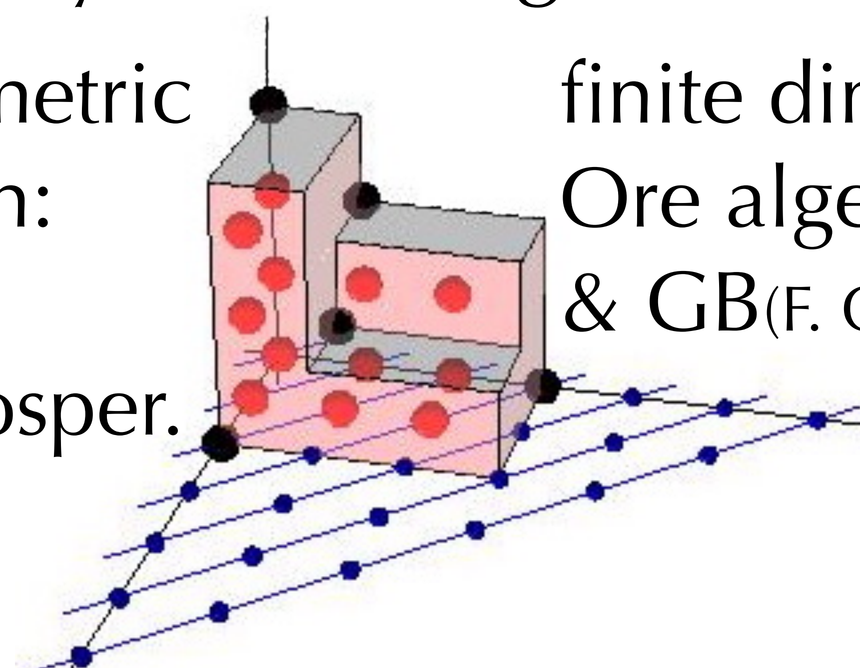
Approximated by:

1. Reducing the search space

   restrict int. by parts to $\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial}_{\boldsymbol{x}},\partial_t\rangle$ and use Gröbner bases. (The « holonomic » approach) [Wilf-Zeilberger, also Sister Celine].

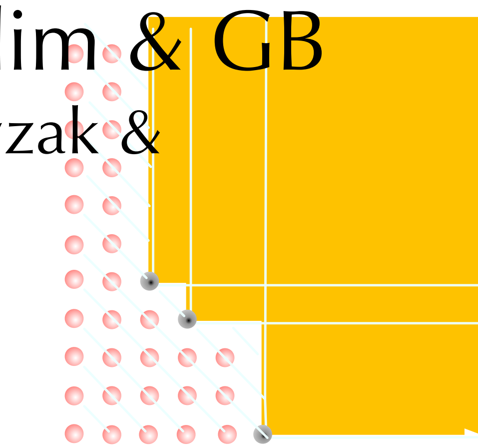
2. Proceeding by increasing slices (and indeterminate coeffs)

hypergeometric summation: dim=1 + param. Gosper. [Zeilberger]



finite dim, Ore algebras & GB(F. Chyzak)

infinite dim & GB (with F. Chyzak & M. Kauers)



Examples of applications

- **Hypergeometric**: binomial sums, hypergeometric series;

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

- **Higher dimension**: classical orthogonal polynomials, special functions like Bessel, Airy, Struve, Weber, Anger, hypergeometric and generalized hypergeometric,...

$$J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} \, dt$$

- **Infinite dimension**: Bernoulli, Stirling or Eulerian numbers, incomplete Gamma function,...

$$\int_0^\infty \exp(-xy)\Gamma(n,x)\, dx = \frac{\Gamma(n)}{y}\left(1 - \frac{1}{(y+1)^n}\right)$$

II. Faster Creative Telescoping

Certificates are too big

$$\mathsf{C_n} := \sum_{r,s} \underbrace{(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$$(n+2)^3 C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n = \color{blue}{180\text{ kB} \simeq 2\text{ pages}}$$

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1t_2t_3(1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z) = \color{blue}{1\,080\text{ kB}}$$

$\color{blue}{\simeq 12\text{ pages}}$

Next, in $\mathsf{T_t(f)} := \left(\operatorname{Ann} f + \underbrace{\color{blue}{\partial_t\mathbb{Q}(\boldsymbol{x},t)\langle\boldsymbol{\partial_x},\partial_t\rangle}}_{\text{int. by parts}}\right) \cap \underbrace{\color{red}{\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial_x}\rangle}}_{\text{diff. under }\int}.$

we restrict to rational f and $\partial_t\mathbb{Q}(\boldsymbol{x})[t, 1/\operatorname{den} f]\langle\boldsymbol{\partial_x},\partial_t\rangle$

Bivariate integrals by Hermite reduction

[BostanChenChyzakLi10]

$$I(t) = \oint \frac{P(t,x)}{Q^m(t,x)}\,dx$$

Q square-free

Int. over a cycle where Q≠0.

If m=1, Euclidean division: P=aQ+r, $\deg_x r < \deg_x Q$

$$\frac{P}{Q} = \frac{r}{Q} + \partial_x \int a$$

Def. Reduced form: $\left[\frac{P}{Q}\right] := \frac{r}{Q}$

If m>1, Bézout identity and integration by parts

$$P = uQ + v\partial_x Q \quad \rightarrow \quad \frac{P}{Q^m} = \underbrace{\frac{u + \frac{\partial_x v}{m-1}}{Q^{m-1}}}_{A_{m-1}} + \partial_x \frac{v/(1-m)}{Q^{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := [A_{m-1}]$$

Algorithm: $R_0 := [P/Q^m]$

for i=1,2,… do $R_i := [\partial_t R_{i-1}]$

when there is a relation $c_0(t)R_0 + \ldots + c_i(t)R_i = 0$

return $c_0 + \ldots + c_i \partial_t^i$

More variables: Griffiths-Dwork reduction

$$I(t) = \oint \frac{P(t,\underline{x})}{Q^m(t,\underline{x})}\, d\underline{x}$$

Q square-free

Int. over a cycle where Q≠0.

1. Control degrees by homogenizing $(x_1,\dots,x_n)\mapsto(x_0,\dots,x_n)$
2. If $m=1$, $[P/Q]:=P/Q$
3. If $m>1$, reduce modulo Jacobian ideal $J := \langle \partial_0 Q, \dots, \partial_n Q\rangle$

$$P = r + v_0\partial_0 Q + \cdots + v_n \partial_n Q$$

$$\frac{P}{Q^m} = \frac{r}{Q^m} - \frac{1}{m-1}\left(\partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}}\right) + \underbrace{\frac{1}{m-1}\,\frac{\partial_0 v_0 + \cdots + \partial_n v_n}{Q^{m-1}}}_{A_{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := \frac{r}{Q^m} + [A_{m-1}]$$

$\boxed{\text{\textbf{Thm.} [Griffiths] In the regular case } (\mathbb{Q}(t)[\underline{x}]/J \text{ finite dim}), \text{ if } R=P/Q^m \text{ hom of degree -n-1, } [R]=0 \Leftrightarrow \oint R\,d\underline{x} = 0.}$

→ SAME ALGORITHM.

Size and complexity

$$I(t) = \oint \underbrace{\frac{P(t,\underline{x})}{Q^m(t,\underline{x})}}_{\in \mathbb{Q}(t,\underline{x})} \, d\underline{x}$$

no regularity assumed

$$N := \deg_{\underline{x}} Q, \quad d_t := \max(\deg_t Q, \deg_t P)$$

$\deg_x P$ not too big

Thm. [Bostan-Lairez-S. 2013] A linear differential equation for $I(t)$ can be computed in $O(e^{3n}N^{8n}d_t)$ operations in $\mathbb{Q}$.
It has order $\leq N^n$ and degree $O(e^n N^{3n} d_t)$.

tight

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

Non-regular case by deformation, better way in Pierre Lairez's poster.

III. Multiple Binomial Sums

Definition

Multiple binomial sums are obtained from

indicators $\mathbb{1}_n$

geometric sequences $n \mapsto C^n, \quad C \in \mathbb{Q}$

binomial coefficients $(n, k) \mapsto \binom{n}{k}$

defined as the coeff of x^k in $(1+x)^n$.

by

+, ×, multiplication by scalars,
indefinite summation,
affine transformations of the indices.

Examples

- All the sums in this talk;
- the Catalan numbers: $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$
- more generally: coefficients of algebraic series ($P(\underline{x},y)=0$);
- more generally: coefficients of diagonals of rational series.

Diagonals:

$$\Delta \sum_{\underline{i}} f_{\underline{i}} \underline{z}^{\underline{i}} := \sum_{i} f_{i,\dots,i} z^{i}$$

Thm. The sequence u_n is a multiple binomial sum **iff** its generating function is the diagonal of a rational series.

Christol's conjecture: if $f \in \mathbb{Z}[t]$ is convergent and solution of a LDE, then it is the diagonal of a rational series.

Generating Function Dictionary

$$\underline{u}_{\underline{i}} \mapsto U(\underline{z}) = \sum_{\underline{i}} u_{\underline{i}} \underline{z}^{\underline{i}}$$

indicator $\mathbb{1}_n$	$\frac{1}{1 - tz}$
geometric sequence C^n	$\frac{1}{1 - Cz}$
binomial coefficient	$\frac{1}{1 - t(1+z)}$
$\Sigma : (u_n)_n \mapsto \left(\sum_{k=0}^{n} u_k\right)_n$	$\frac{1}{1-z} U(z)$
$\times : ((u_n)_n, (v_n)_n) \mapsto (u_n v_n)_n$	$U(z) \odot V(z) = \Delta(U(z_1)V(z_2))$
$\mathrm{CT} : (u_n)_n \mapsto u_0$	$U(z) \odot 1$
$\mathrm{Shift} : (u_n)_n \mapsto (u_{n+b})_n$	$z^{-b} U(z)$
$\mathrm{Dilat} : (u_n)_n \mapsto (u_{kn})_n$	$\frac{1}{k}(U(z^{1/k}) + U(\omega z^{1/k}) + \cdots + U(\omega^{k-1} z^{1/k})), \quad \omega^k = 1$
$\Delta : (u_{n,k})_{n,k} \mapsto (u_{n,n})_n$	$\Delta U(z_1, z_2) = \frac{1}{2\pi i} \oint U\left(z_1, \frac{z_2}{z_1}\right) \frac{dz_1}{z_1}$
$A : (u_{\underline{n}})_{\underline{n}} \mapsto (u_{A \cdot \underline{n}})_{\underline{n}}, \quad A$ invertible	$U(\underline{z}^{A^{-1}})$

Multiple binomial sums reduce to multiple integrals of rational functions

Example: Dixon's identity

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

decomposes as

$$\binom{n}{k} \overset{\times}{\mapsto} \binom{n}{k}^3 \overset{\times}{\mapsto} (-1)^k \binom{n}{k}^3 \overset{\Sigma}{\mapsto} \sum_{k=0}^{m} (-1)^k \binom{n}{k}^3 \overset{\Delta}{\mapsto} \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \overset{\text{Dilat}}{\mapsto} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3$$

which translates into

$$\text{Dilat}\Delta \frac{1}{1-t} \left(\frac{1}{1-t(1+z)} \odot_{t,z} \frac{1}{1-t(1+z)} \odot_{t,z} \frac{1}{1-t(1+z)} \odot_{t,z} \frac{1}{(1+t)(1-z)} \right)$$

which is

$$\frac{1}{(2\pi i)^7} \oint \text{Rat}(z, t_1, \dots, t_7) dt_1 \cdots dt_7$$

Many tricks and optimizations possible

Demo?

Summary

Symbolic summation and integration deserve more attention in terms of complexity;

We propose algorithms that are much faster than usual creative telescoping, in particular by avoiding certificates;

Restricted to basic (but common!) classes, for now.

THE END