Symbolic-Numeric Tools for Multivariate Asymptotics

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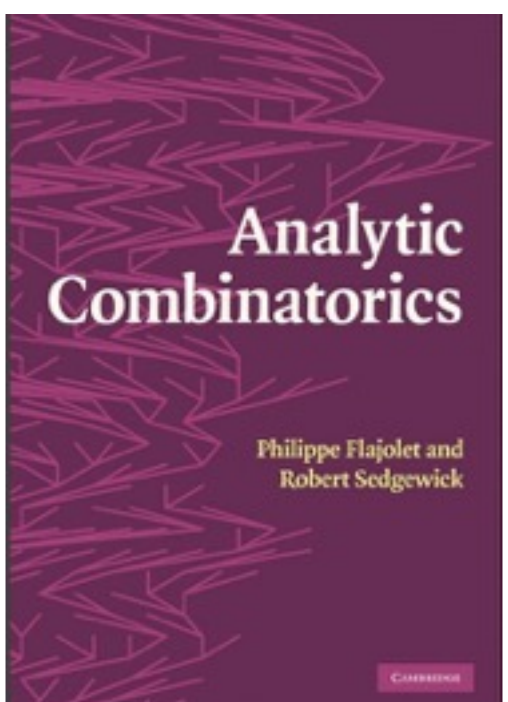
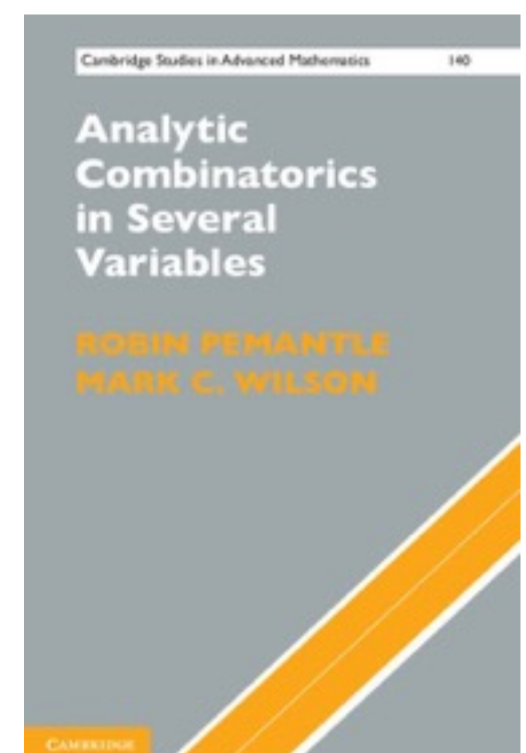
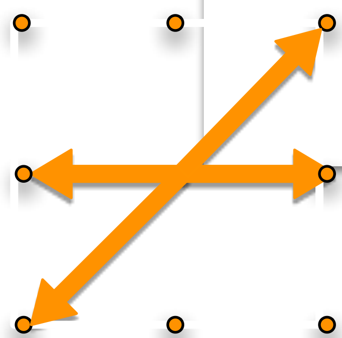
Joint work with Stephen Melczer
to appear in Proc. ISSAC 2016

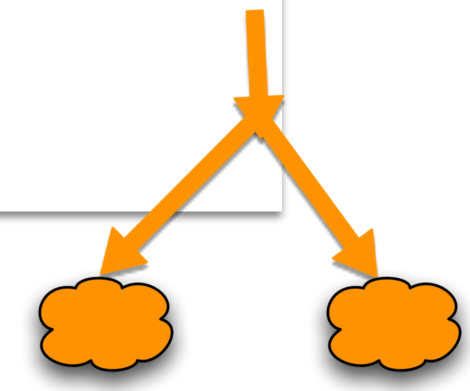
FastRelax meeting
May 25, 2016





Combinatorics, Randomness and Analysis

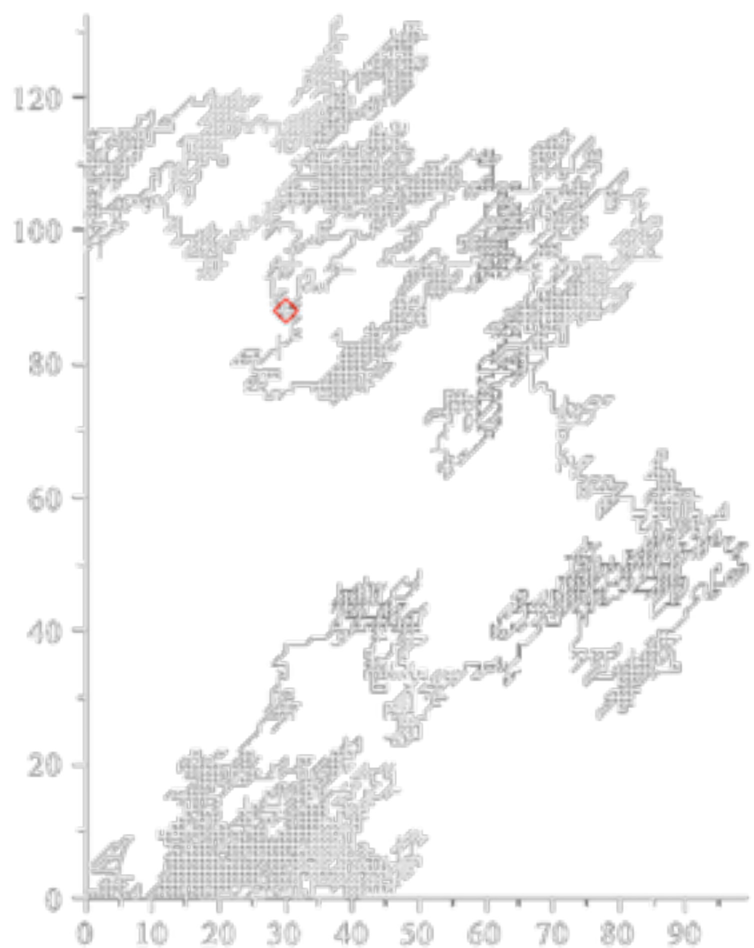


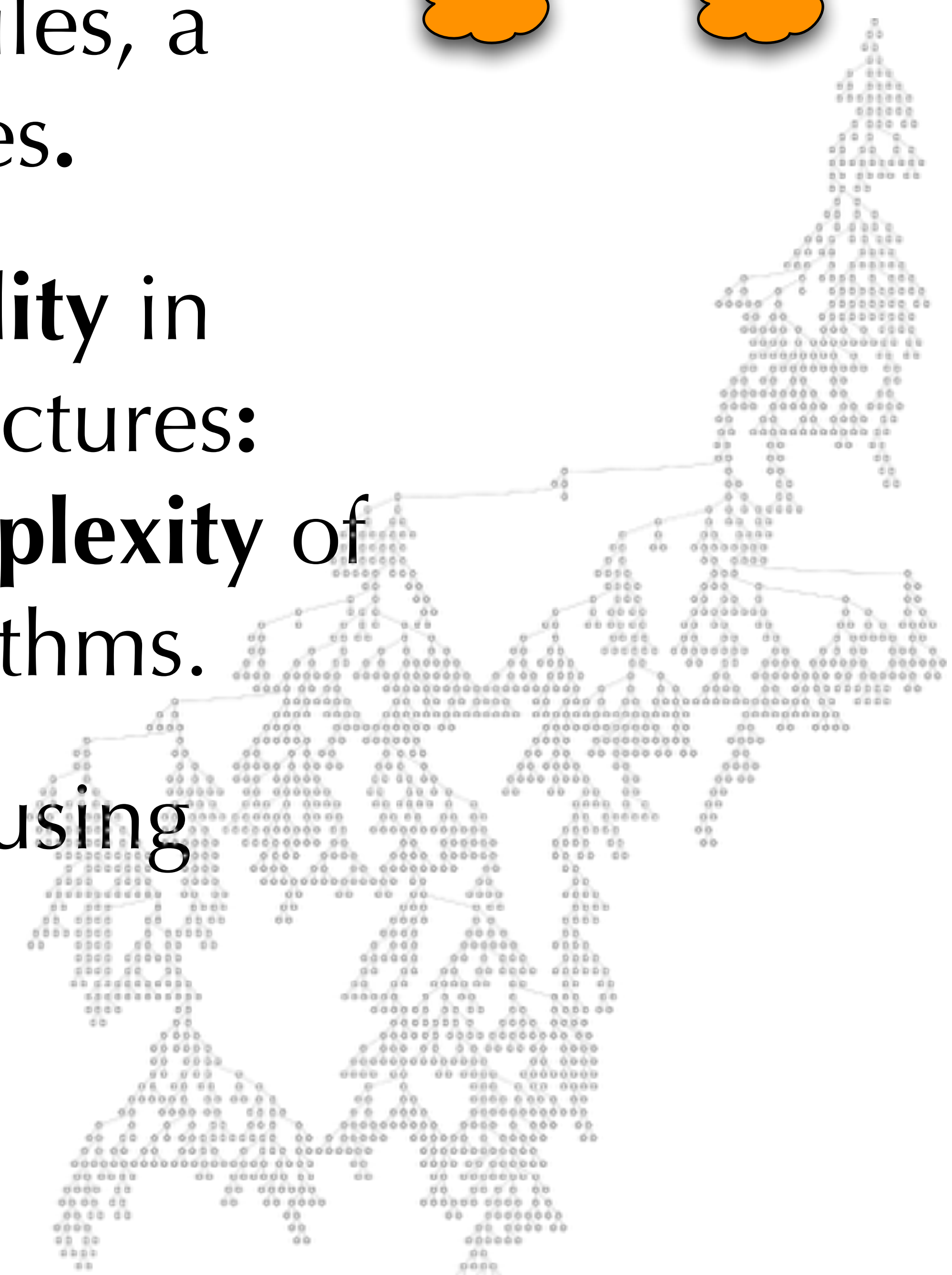


From **simple local** rules, a **global** structure arises**.**

A quest for **universality** in random discrete structures**:**
➡ **probabilistic complexity** of structures and algorithms.

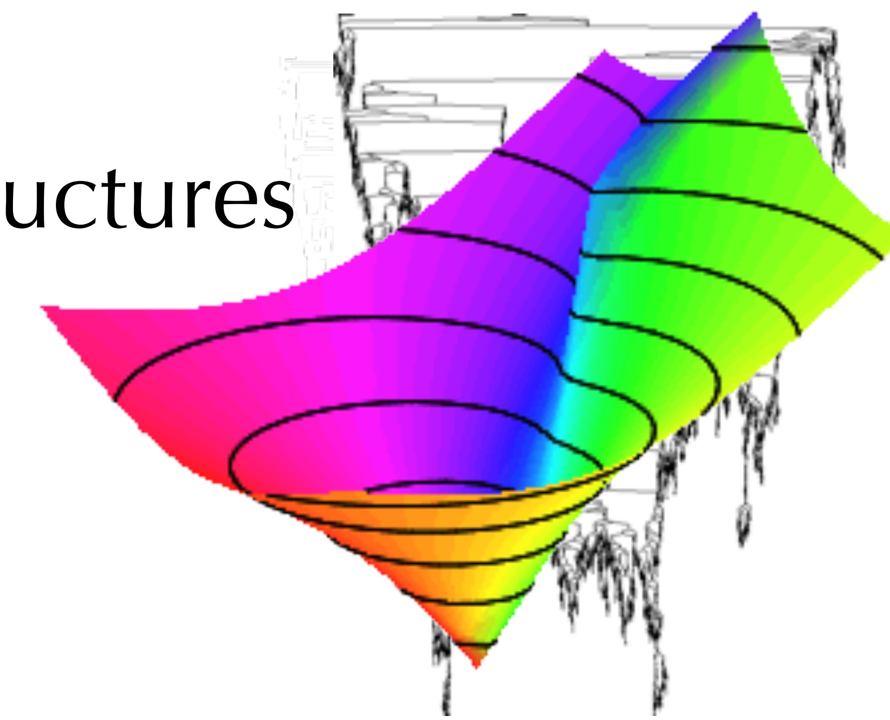
Quantitative results using **complex analysis.**





Overview

Ex.: binary trees



- Equations over combinatorial structures
- Generating functions

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

- Complex analysis

$$f_n \sim \cdots, \quad n \to \infty$$

$$\mathcal{B} = \mathcal{Z} \cup \mathcal{B} \times \mathcal{B}$$

$$B(z) = z + B(z)^2$$

$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Aim: several vars, automatically.

I. Families of Generating Functions

Rational Generating Functions

(regular languages, linear recurrences with const. coeffs.)

Example

Generating function of 1,2,4,8,... is $\frac{1}{1-2z}$.

Example

Fibonacci numbers 1,1,2,3,5,8,... have gf $\frac{1}{1-z-z^2}$.

Example (Simple lattice walks)

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j} c_{i,j} x^i y^j = \sum_{i,j} \binom{i+j}{i} x^i y^j$$

$$= 1 + x + 2xy + x^2 + y^2 + x^3 + y^3 + 3x^2y + 3xy^2 + 6x^2y^2 + \cdots$$

Here $c_{i,j}$ counts the number of ways to walk from (0,0) to (i,j) using i steps (0,1) and j steps (1,0).

Rational Generating Functions

(regular languages, linear recurrences with const. coeffs.)

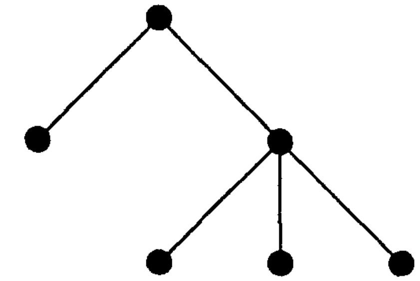
Example (Restricted factors in words)

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

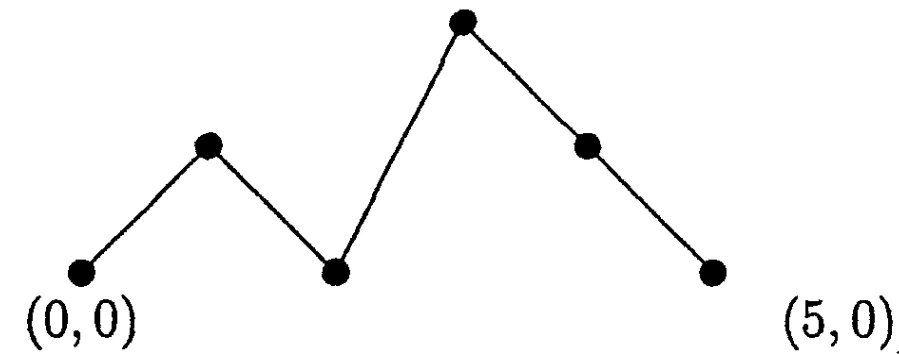
Here $c_{i,j}$ counts the number of binary words with *i* zeroes and *j* ones that do not contain 10101101 or 1110101.

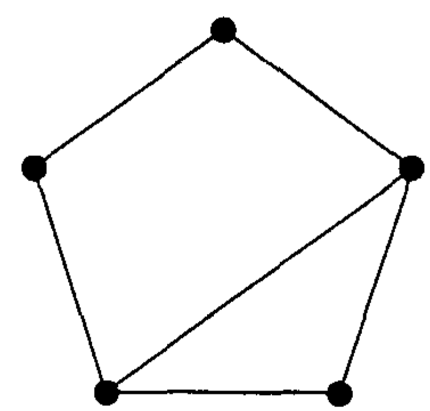
Algebraic Generating Functions

- Plane trees with n nodes and given arity set;
- Bracketings of a word of length n;
- Paths from (0,0) to (n,0) using steps (1,k) and (1,-1);
- Dissections of a convex n-gon…



(x(xxx))





Diagonals of Rational Functions

Input: Rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ with power series

$$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} c_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

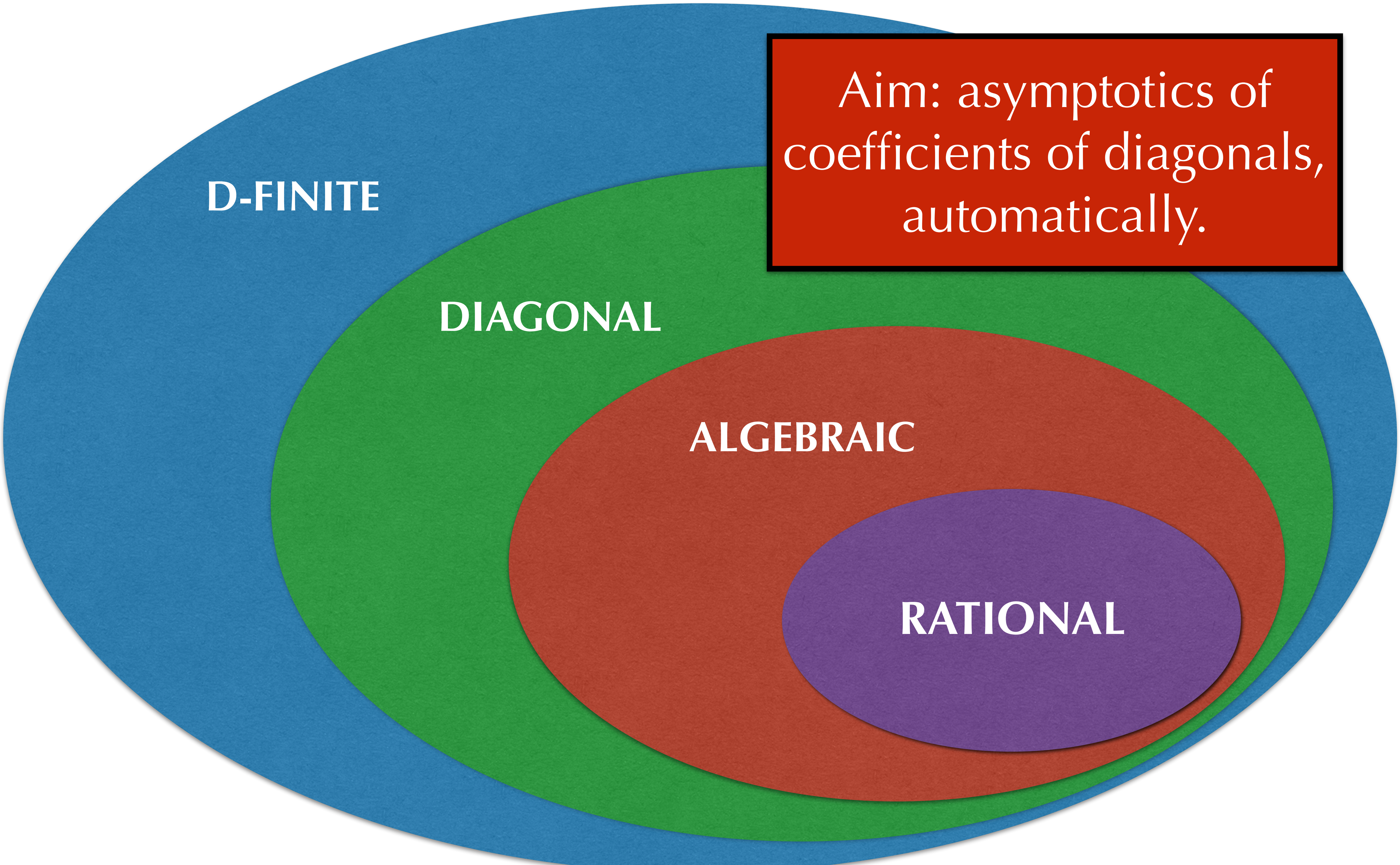
Goal: Asymptotics of the *diagonal sequence* $c_{k,k,\ldots,k}$ as $k \to \infty$

Example (Apéry)

$$F(a,b,c,z) = \frac{1}{1 - z(1+a)(1+b)(a+c)(1+b+c+bc+abc)}$$

Here $(c_{k,k,k,k,k})_{k \geq 0}$ determines Apéry's sequence, related to his celebrated proof of the irrationality of $\zeta(3)$.

Univariate Generating Functions



II. Analytic Combinatorics

Cauchys' formula

Thm. If $f = \sum_{i_1,\ldots,i_n \geq 0} c_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n}$ is convergent in the neighborhood of 0, then

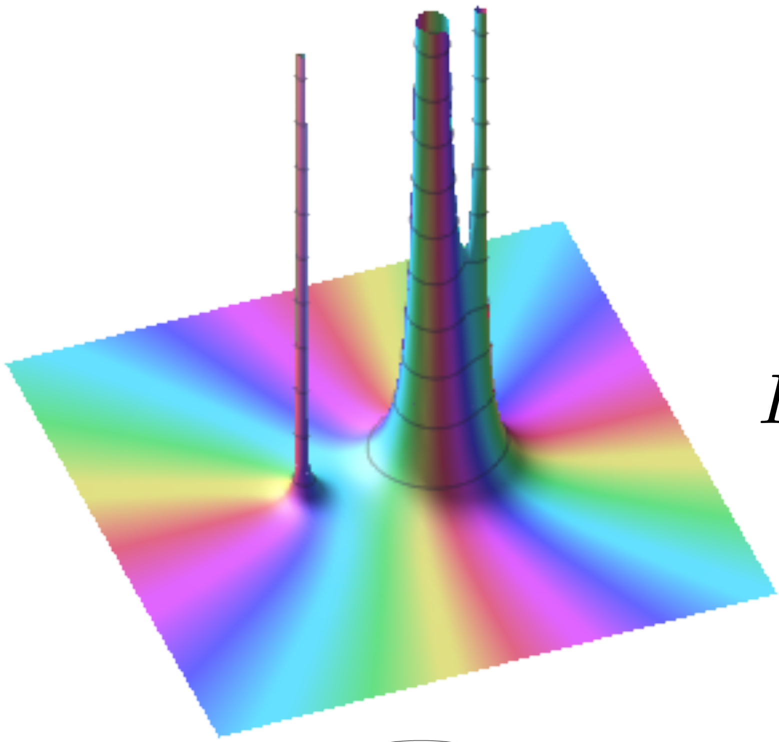
$$c_{i_1,\ldots,i_n} = \left(\frac{1}{2\pi i}\right)^n \int_T f(z_1,\ldots,z_n) \frac{dz_1 \cdots dz_n}{z_1^{i_1+1} \cdots z_n^{i_n+1}}$$

for any small torus T ($|z_j| = re^{i\theta_j}$) around 0.

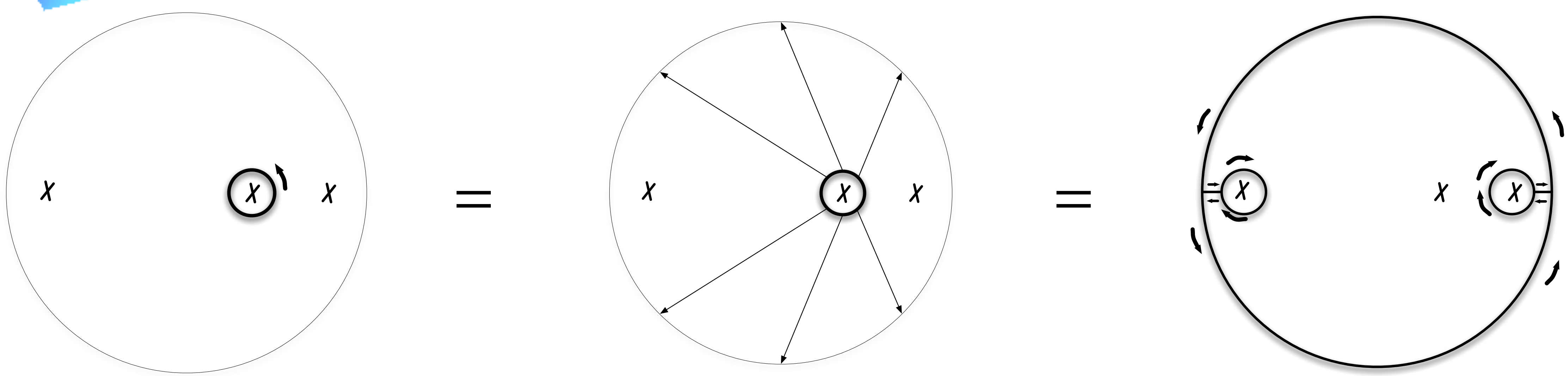
Asymptotics: deform the torus to pass where the integral concentrates asymptotically.

Coefficients of Univariate Rational Functions

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$



$$F_1 = 1 = \frac{1}{2\pi i} \oint \frac{1}{1 - z - z^2} \frac{dz}{z^2}$$



As n increases, the smallest singularities dominate.

$$F_n = \frac{\phi^{-n-1}}{1+2\phi} + \frac{\overline{\phi}^{-n-1}}{1+2\overline{\phi}}$$

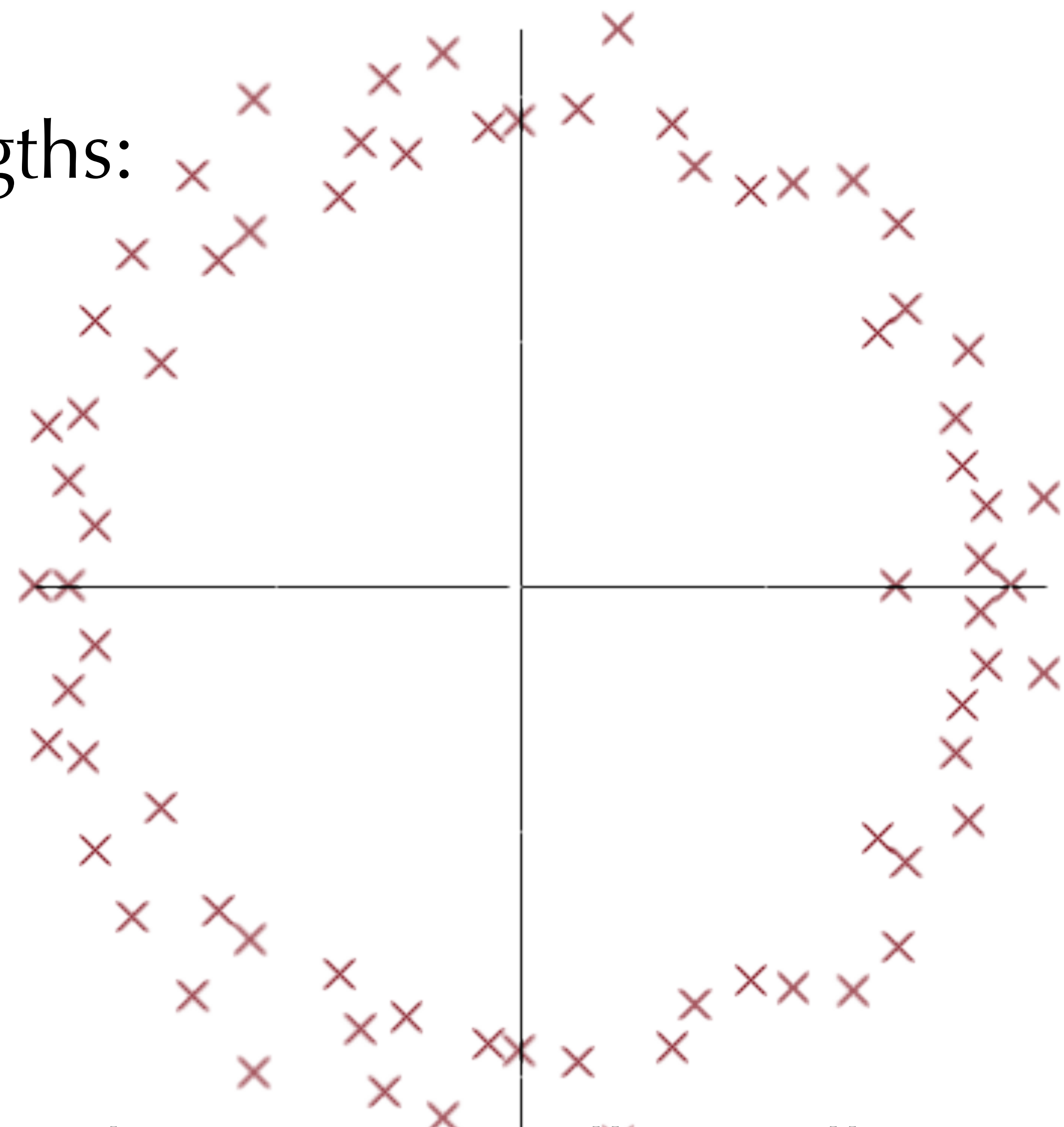
Conway's sequence

1,11,21,1211,111221,...

Generating function for lengths:
$f(z)=P(z)/Q(z)$
with deg $Q=72$.

Smallest singularity:
$\delta(f) \simeq 0.7671198507$

$\rho = 1/\delta(f) \simeq 1.30357727$



$$\ell_n \simeq 2.04216\, \rho^n$$

ρ Res$(f, \delta(f))$

remainder exponentially small

Coefficients of Multivariate Rational Functions

Def. $F(z_1,\ldots,z_n)$ is ***combinatorial*** if every coefficient is ≥ 0.

Assuming F=G/H is combinatorial + generic conditions, asymptotics are determined by the points in $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\}$ such that

partial derivatives

$$\begin{cases} (1) \quad z_1 H_{z_1}(\mathbf{z}) = \cdots = z_n H_{z_n}(\mathbf{z}) \\ (2) \quad \mathbf{z} \text{ on boundary of domain of convergence} \end{cases}$$

Critical Points
$\mathbf{z} \mapsto (z_1 \cdots z_n)$

$\mathbf{z}$ is coordinate-wise minimal in $\mathcal{V}$

Aim: find these points, in good complexity.

(1) is an **algebraic** condition.
(2) is a **semi-algebraic** condition (can be expensive).

III. Symbolic-Numeric Computation

Kronecker Representation

Algebraic part: ``compute'' the solutions of the system

Suppose
$$H(\mathbf{z}) = 0, \qquad z_1 H_{z_1}(\mathbf{z}) = \cdots = z_n H_{z_n}(\mathbf{z})$$

$$\deg(H) = d, \qquad \max\ \mathrm{coeff}(H) \leq 2^h \qquad D := d^n$$

Under genericity assumptions, in $\tilde{O}(hD^3)$ bit ops there is a prob. algorithm to find:

$$\left.\begin{aligned} P(u) &= 0 \\ P'(u)z_1 - Q_1(u) &= 0 \\ &\vdots \\ P'(u)z_n - Q_n(u) &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{Degree} \leq D \\ &\text{Height} \leq \tilde{O}(hD) \end{aligned}$$

History and Background:
see Castro, Pardo, Hägele,
and Morais (2001)

Giusti, Lecerf, Salvy (2001)
Schost (2001)

System reduced to
a univariate polynomial.

Example (Lattice Path Model)

The number of walks from the origin taking steps *{NW,NE,SE,SW}* and staying in the first quadrant has

$$F(x,y,t) = \frac{(1+x)(1+y)}{1-t(x^2+y^2+x^2y^2+1)}$$

One can calculate the Kronecker representation

$$P(u) = 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920$$

Next, find the *minimal* critical points, that lie on the boundary of the domain of convergence.

$$Q_t(u) = 4u^3 + 39u^2 - (4339/2)u + 4669/2$$

so that the critical points are given by:

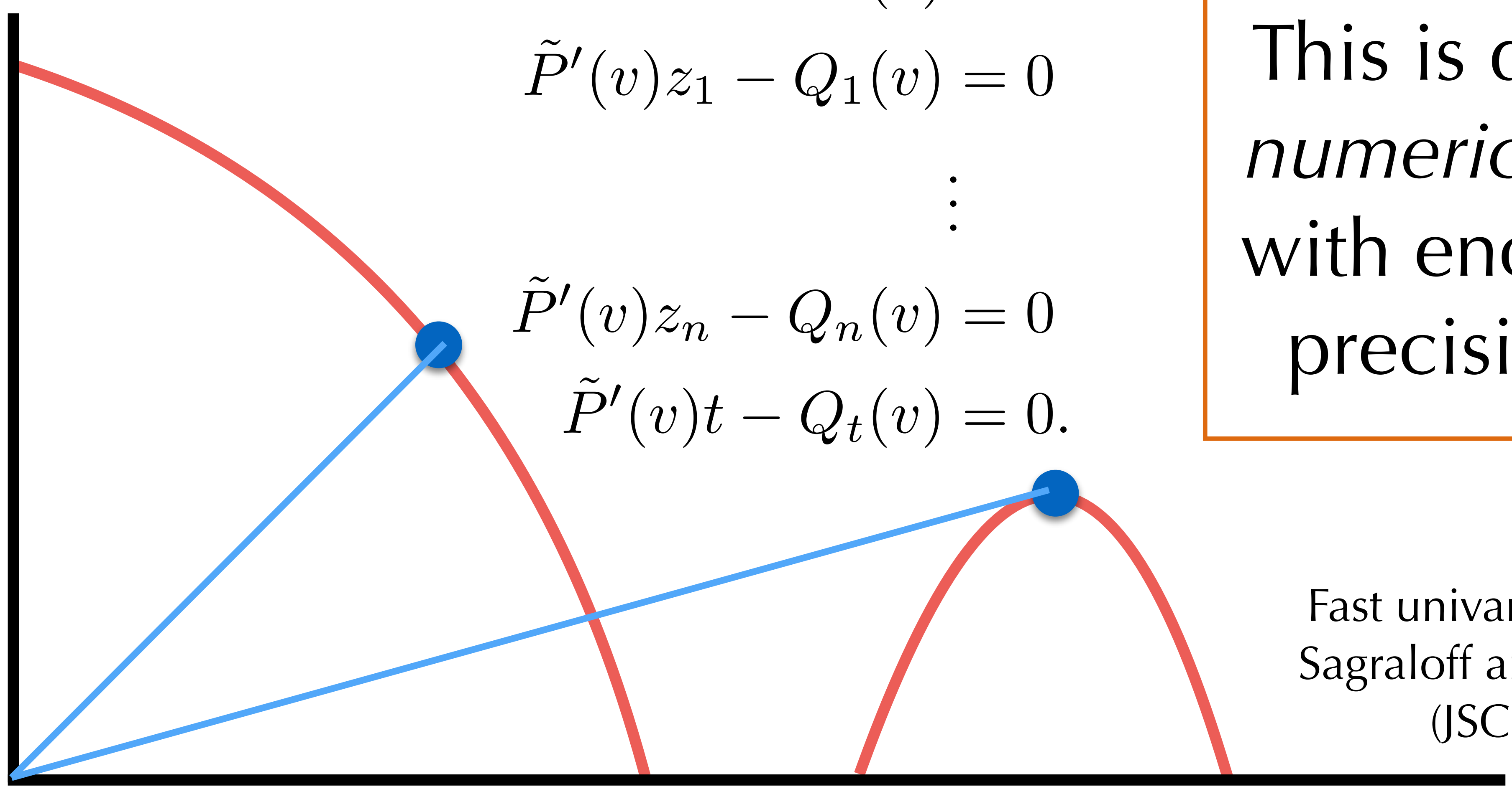
$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Testing Minimality

Semi-algebraic part: In the combinatorial case, it is enough to examine only the positive real points on the variety to determine ***minimality***.

Thus, we add the equation $H(tz_1, \ldots, tz_n) = 0$ for a new variable t and select the positive real point(s) $\mathbf{z}$ with no $t \in (0, 1)$ from a new Kronecker representation:

$$
\begin{aligned}
\tilde{P}(v) &= 0 \\
\tilde{P}'(v) z_1 - Q_1(v) &= 0 \\
&\vdots \\
\tilde{P}'(v) z_n - Q_n(v) &= 0 \\
\tilde{P}'(v) t - Q_t(v) &= 0.
\end{aligned}
$$



This is done *numerically*, with enough precision.

Fast univariate solving: Sagraloff and Mehlhorn (JSC 2016)

How many digits do we need?

Suppose $\alpha \neq \beta$ are roots of square-free poly $A(z)$ of degree δ, height σ

For the Kronecker Representation, we have polynomials of

degree $D = d^n$

height $\tilde{O}(hD)$

so we need a precision of $2^{-\tilde{O}(hD^2)}$ to rigorously decide signs, find exact zeroes, and identify equal coordinates.

First Complexity Results for ACSV

Theorem (M. and Salvy, 2016)

Under generic conditions, and assuming $F(\mathbf{z})$ is combinatorial, the points contributing to dominant diagonal asymptotics can be determined in $\tilde{O}(hD^4)$ bit operations. Each contribution has the form

$$A_k = \left(T^{-k} k^{(1-n)/2} \cdot (2\pi)^{(1-n)/2}\right)(C + O(1/k))$$

and can be found to $2^{-\kappa}$ precision in $\tilde{O}(h(dD)^3 + D\kappa)$ bit ops.

The genericity assumptions heavily restrict the form of the asymptotic growth. Removing more of these assumptions is ongoing work.

Example 1

Example (Apéry)

$$F(a,b,c,z) = \frac{1}{1 - z(1+a)(1+b)(a+c)(1+b+c+bc+abc)}$$

Here a Kronecker representation is

$$P(u) = u^2 - 366u - 17711$$

$$a = \frac{2u - 1006}{P'(u)}, \qquad b = c = \frac{-320}{P'(u)}, \qquad z = \frac{-164u - 7108}{P'(u)}$$

There are two real critical points, and one is positive. After testing minimality, one has proven asymptotics

```
> A, U, PRINT := DiagonalAsymptotics(numer(F),denom(F),[a,b,c,z],u,k, useFGb):
  A, U;
```

$$\frac{1}{4}\,\frac{\left(\frac{2\,u-366}{34\,u+1458}\right)^{k}\sqrt{2}\,\sqrt{\frac{2\,u-366}{-96\,u-4192}}}{k^{3/2}\,\pi^{3/2}},\ \left[RootOf\left(\_Z^2 - 366\,\_Z - 17711,\ -43.27416997969\right.\right.$$

Example 2

Example (Restricted Words in Factors)

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

```
> ASM, U := DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
  ASM;
```

$$\frac{1}{2}\Bigg(\Bigg(\frac{84u^{20} + 240u^{19} - 285u^{18} - 1548u^{17} - 2125u^{16} - 1408u^{15} + 255u^{14} + 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7}}{-12u^{20} + 30u^{19} + 258u^{18} + 500u^{17} + 440u^{16} - 102u^{15} - 378u^{14} - 1544u^{13} - 2142u^{12} - 550u^{11} - 2222u^{10} - 1644u^{9} + 2860u^{8} - 1848u^{7} + 12}$$

$$\sqrt{\frac{84u^{20} + 240u^{19} - 285u^{18} - 1548u^{17} - 2125u^{16} - 1408u^{15} + 255u^{14} + 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7} +}{-162u^{18} - 612u^{17} - 902u^{16} - 616u^{15} + 254u^{14} + 548u^{13} + 2054u^{12} + 2156u^{11} + 898u^{10} + 2268u^{9} + 2462u^{8} - 2088u^{7} + 1312u^{6}}}$$

$$- 255u^{16} - 190u^{15} - 19u^{14} + 46u^{13} + 461u^{12} + 628u^{11} + 133u^{10} + 374u^{9} + 161u^{8} - 384u^{7} + 146u^{6} - 138u^{5} - 285u^{4} - 40u^{3} + 91u^{2} - 30u +$$

$$+ 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7} + 924u^{6} - 492u^{5} - 675u^{4} + 632u^{3} - 249u^{2} + 24u + 16\Bigg)\Bigg)$$

```
> U;
```

$$[RootOf(4\_Z^{21} + 12\_Z^{20} - 15\_Z^{19} - 86\_Z^{18} - 125\_Z^{17} - 88\_Z^{16} + 17\_Z^{15} + 54\_Z^{14} + 193\_Z^{13} + 238\_Z^{12} + 5$$

0.2557418387670495142539197612059706034569501897187817325149891776278724062164321975380157608214281
6507895675196348205194331680572580787580891704416439705170142841771704333646705442481154976482193
0559798252115981422944236560602741008070812421670189339690141212064002448936239379373317765301730
9973437435585018972425842775861808925373839579447363050379209371594603834274703556502530247272512

Conclusion

We

- Give the first complexity bounds for methods in analytic combinatorics in several variables

- Combine strong symbolic results on the Kronecker representation with fast algorithms on univariate polynomials

Work in progress: extend beyond some of the assumptions.