Certified Numerics in the Asymptotics of Rational Function Coefficients

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Asymptotics of Multiple Binomial Sums

Input: Multiple Binomial Sum

$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

easy

Generating function is a diagonal

$$S(z) = \sum_{n\geq 0} S_n z^n = \operatorname{Diag}\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

Griffiths-Dwork reduction

analytic combinatorics in several variables

$$z^2(4z+1)\left(16z-1\right)S^{(3)}(z)+\cdots+2\,(30z+1)\,S(z)=0$$

analytic combinatorics

Output: Asymptotic behaviour

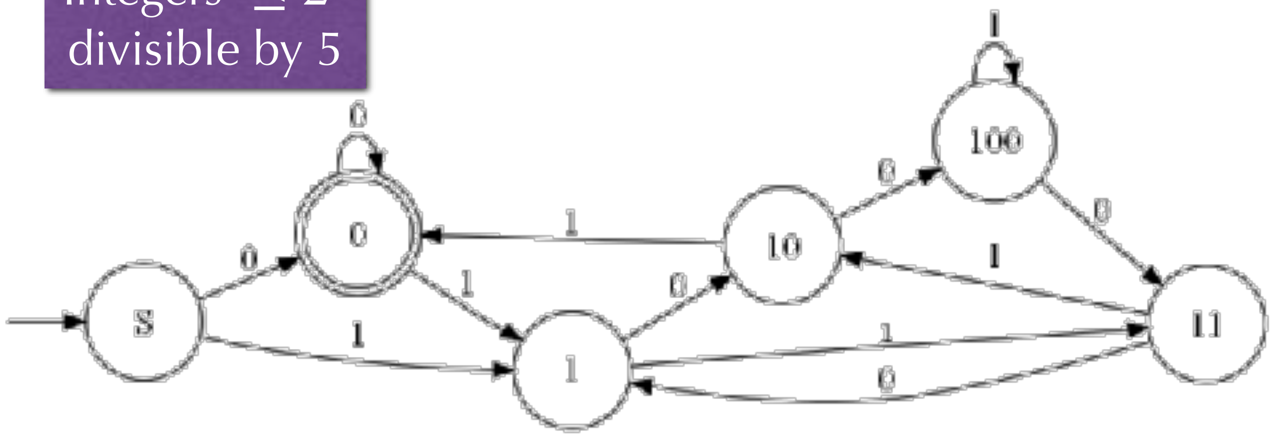
$$S_n = 16^n n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1-\frac{9}{16n}+O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty$$

Initial motivation: compare these approaches

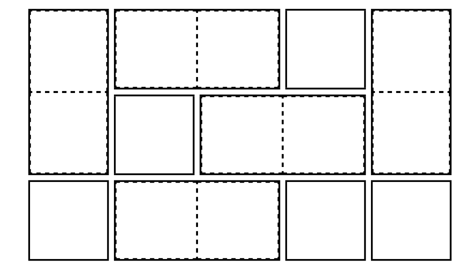
I. Univariate Rational Functions, Linear Recurrent Sequences

Linear Recurrent Sequences

Integers $\leq 2^n$ divisible by 5



Tilings of rectangles of bounded height by dominos and monominos



$$u_{n+k} = a_0 u_n + \cdots + a_{k-1} u_{n+k-1} \text{ with initial conditions } u_0, \ldots, u_{k-1}$$

very well understood

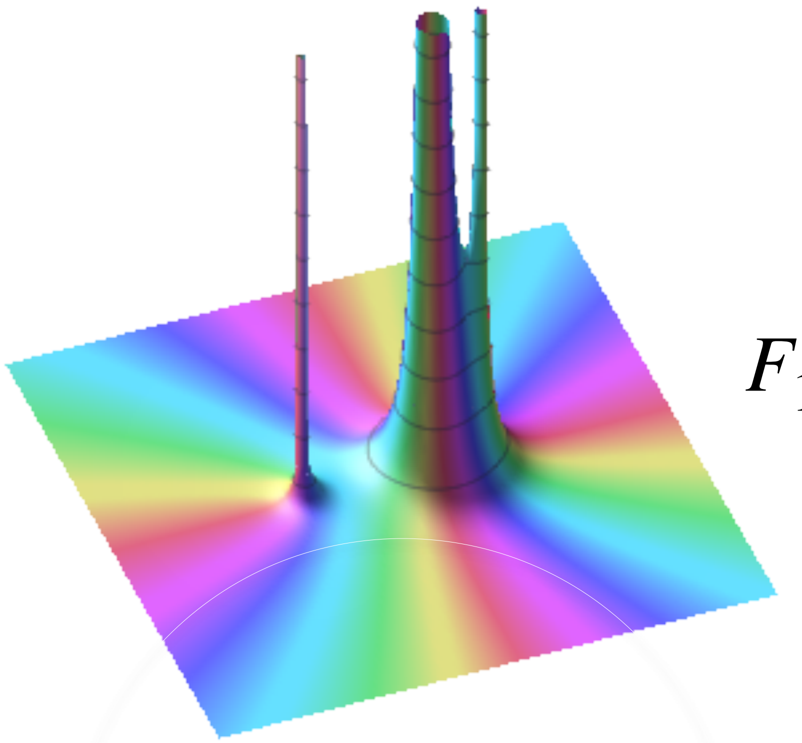
(u_n) is a LRS $\iff$ its generating series $U(z) := \sum_{n=0}^{\infty} u_n z^n$ is rational

Ex. Fibonacci: $F_{n+2} = F_{n+1} + F_n, \quad F_0 = F_1 = 1$

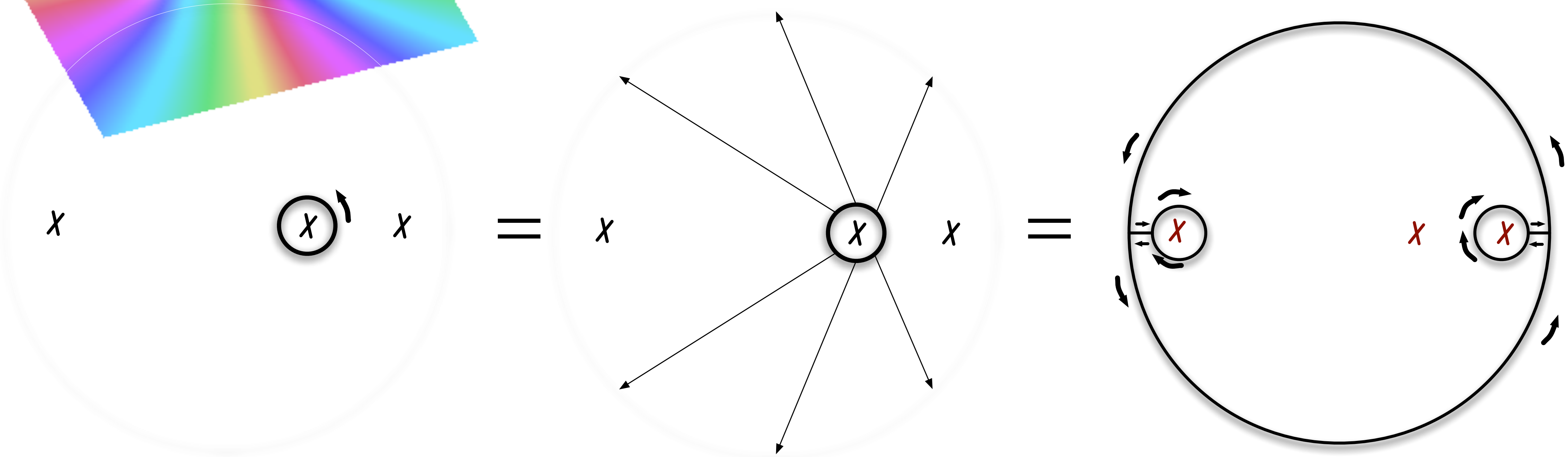
$$F(z) = \frac{z}{1 - z - z^2} = \frac{(2\phi - 1)/5}{1 - z\phi} - \frac{(2\phi - 1)/5}{1 + z/\phi}$$

Fibonacci Numbers

$$F_n = \frac{1}{2\pi i} \oint \frac{F(z)}{z^{n+1}}\, dz$$



$$F_1 = 1 = \frac{1}{2\pi i} \oint \frac{1}{1 - z - z^2} \frac{dz}{z^2}$$



As n increases, the smallest singularities dominate.

$$F_n = \frac{\phi^{-n-1}}{1 + 2\phi} + \frac{\overline{\phi}^{-n-1}}{1 + 2\overline{\phi}}$$

Conway's sequence

1,11,21,1211,111221,...

Generating function for lengths:

$$f(z) = P(z)/Q(z)$$

with $\deg Q = 72$.

Smallest singularity

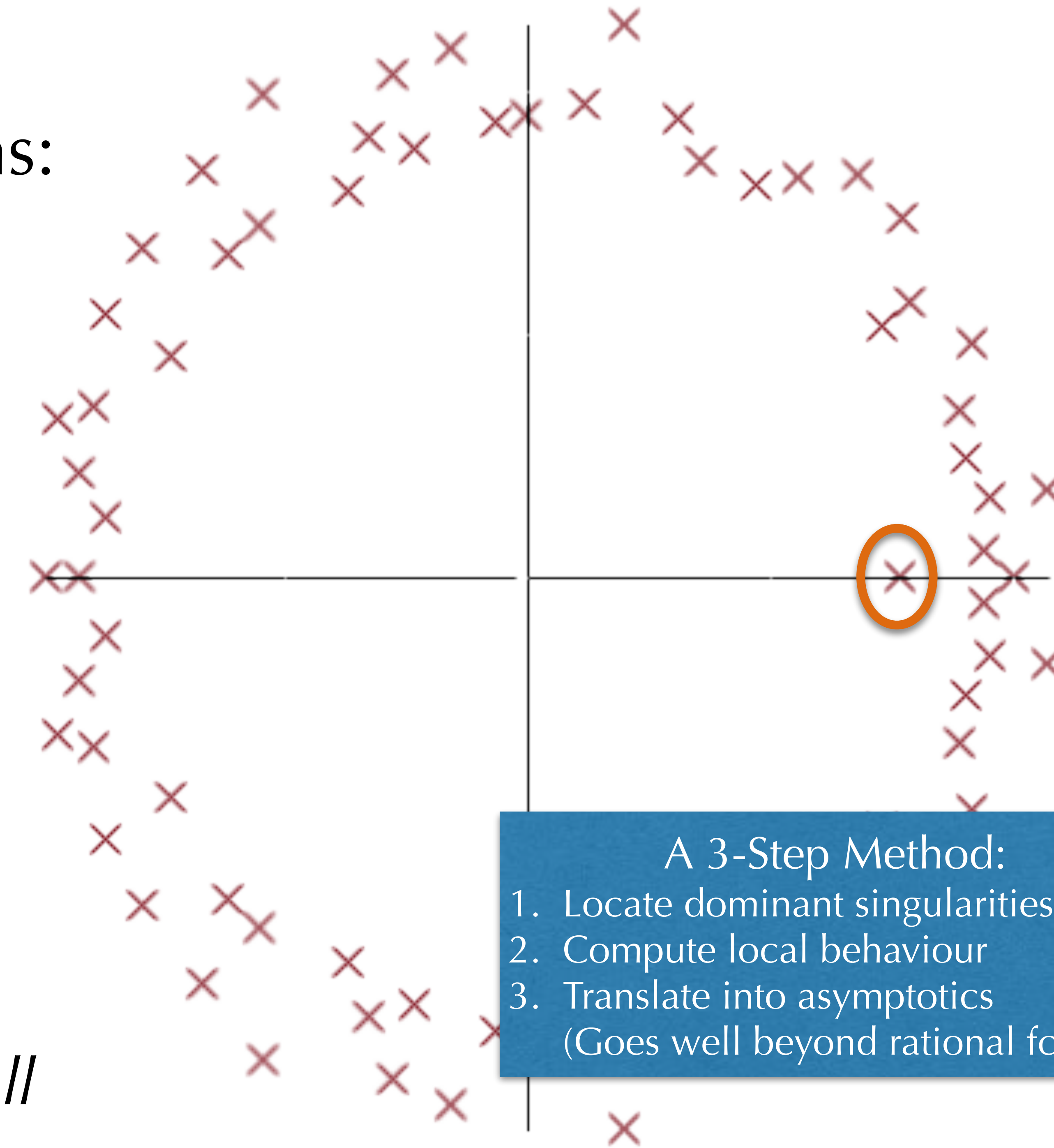
$\rho \approx 0.7671198507$

$$\boxed{\ell_n \simeq 2.04216\,\rho^{-n}}$$

$c = \rho^{-1}\,\mathrm{Res}(f, z=\rho)$

algebraic

remainder exponentially small



A 3-Step Method:

1. Locate dominant singularities
2. Compute local behaviour
3. Translate into asymptotics
   (Goes well beyond rational fcns)

[Conway 1987]

How many digits do we need?

Certified Roots of Polynomials

Def. Separation

$$\mathrm{sep}(P) := \min_{\substack{P(\alpha) = P(\beta) = 0, \\ \alpha \neq \beta}} |\alpha - \beta| \, .$$

Def. Height

$$H\left(\sum_{i=0}^{d} a_i X^i\right) := \max_i |a_i| \, .$$

Mahler's thm. If $P \in \mathbb{Z}[X]$ has degree d,

$$\mathrm{sep}(P) > \kappa(d) H(P)^{-d+1} \, .$$

explicit function of d

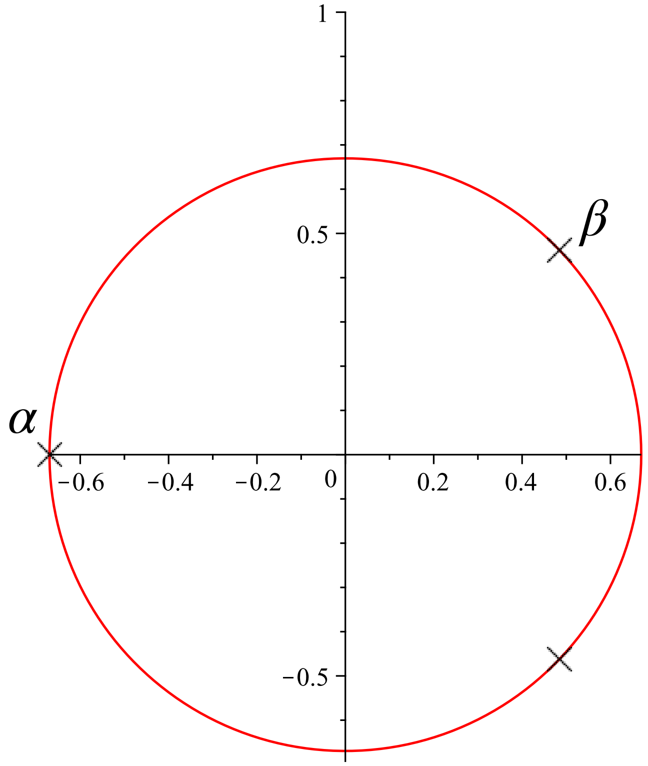
not known to be tight (except for $d = 3$)
worst known family gives $-(2d-1)/3$.

Isolating disks of radius ε for all roots can be computed in time $\tilde{O}(d^3 + d^2 \log H(P) - d \log \varepsilon)$.

[Mahler 64; BugeaudDujella 14; Mehlhorn Sagraloff 15-16]

II. Absolute Separation

Absolute Separation



$P = 10X^3 - 3X^2 - 2X + 3$

$|\beta| - |\alpha| \simeq 5 \cdot 10^{-4}$

Def.

$$\operatorname{abssep}(P) := \min_{\substack{P(\alpha) = P(\beta) = 0, \\ |\alpha| \neq |\beta|}} \Big|\, |\alpha| - |\beta| \,\Big| \,.$$

not the same as before

Thm: $\operatorname{abssep}(P) > \kappa(d)\, H(P)^{-e(d)}$

$$e(d) \leq (d-1)(d-2)(d-3)/2 = d^3/2 - 3d^2 + \cdots \qquad (d \geq 6)\,.$$

No good lower bound known

Sufficient to separate roots by absolute value in polynomial complexity

$e(d) \ll d^3$ would be nice

[Gourdon-S. 96; Dubicka-Sha 15; Sha 19; Bugeaud-Dujella-Fang-Pejkovic-S. 19]

Auxiliary Polynomials

From $P(X) = \sum_{i=0}^{d} a_i X^i = a_d \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X]$ of height $H(P)$

construct

$$M(X) = a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i - \alpha_j)^2) \in \mathbb{Z}[X]$$

and lower bound its nonzero roots.

Prop. 1 [Cauchy] If $\alpha \neq 0$,

$$P(\alpha) = 0 \Rightarrow |\alpha| \geq \frac{1}{1 + H(P)}.$$

Prop. 2 [Symmetric fcns]

$G \in \mathbb{Z}[X_1, \ldots, X_d]$ symmetric

with $\deg_{X_i} G \leq k$ for all i

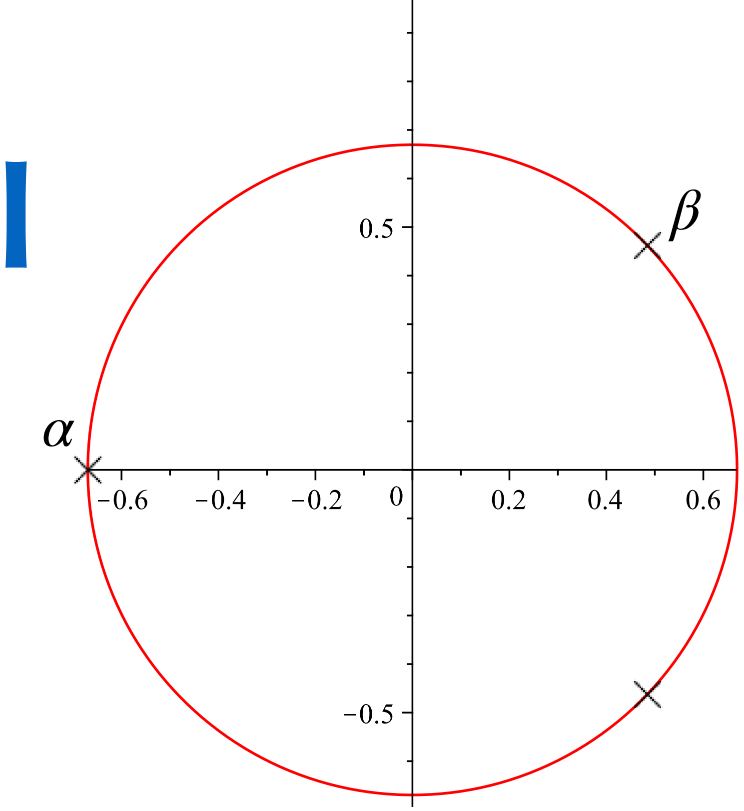
$\Rightarrow a_d^k G(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}[a_0, \ldots, a_d]$

of total degree $\leq k$.

Application to $M \to |\alpha_i - \alpha_j|^2 > \kappa H^{-2(d-1)}$.

Recovers Mahler's exponent

A Bigger Polynomial



$$a_d^{(d-1)(d-2)(d-3)} \prod_{\substack{i<j,\\ k<\ell,\\ \{i,j\}\cap\{k,\ell\}=\varnothing}} \left(X^{1/2} - (\alpha_i\alpha_j - \alpha_k\alpha_\ell) \right)$$

$$\Rightarrow (|\alpha|^2 - |\beta|^2)^2 \gg H^{-(d-1)(d-2)(d-3)}$$

gives exponent $(d-1)(d-2)(d-3)/2$ for the general case

Other Useful Polynomials

$$a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i + \alpha_j)^2)$$

$$\alpha_j, \alpha_k \text{ real} \Rightarrow \Big| |\alpha_j| - |\alpha_k| \Big| > \kappa H^{-(d-1)}$$

optimal

$$a_d^{2(d-1)(d-2)} \prod_{i<j, k \notin \{i,j\}} (X - (\alpha_k^2 - \alpha_i \alpha_j))$$

4 is optimal when $d = 3$

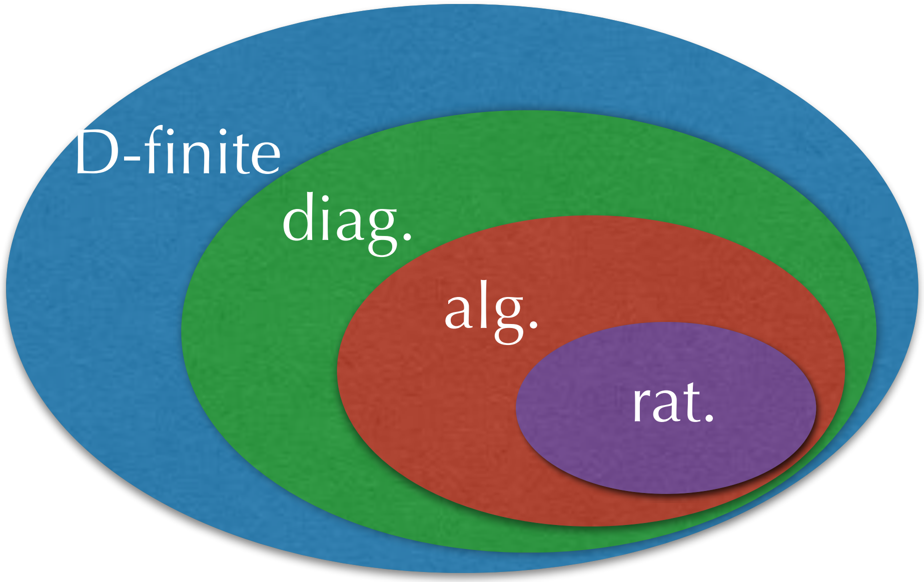
$$\alpha_k \text{ real} \Rightarrow \Big| |\alpha_j| - |\alpha_k| \Big| > \kappa H^{-2(d-1)(d-2)}$$

Application: if $0 < r_1 \leq \cdots \leq r_k$ are the real positive roots of P, all roots of modulus exactly the r_i can be computed in time $\tilde{O}(d^3 \log H(P))$.

Used in the combinatorial case later

[Bugeaud-Dujella-Pejkovic-S. 17;Bugeaud-Dujella-Fang-Pejkovic-S. 19]

III. Multiple Binomial Sums, Diagonals and Multiple Integrals

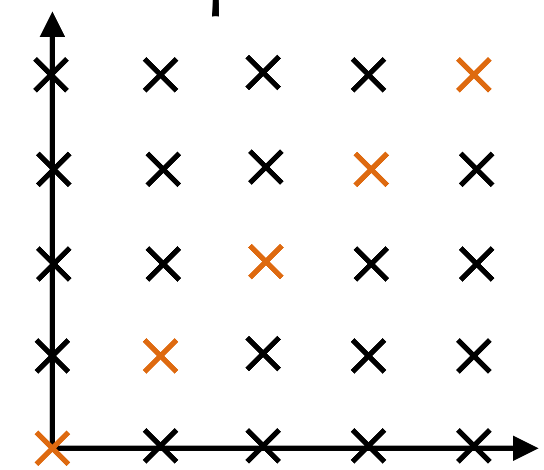


Diagonals

in this talk

If $F(\underline{z}) = \dfrac{G(\underline{z})}{H(\underline{z})}$ is a multivariate rational function with Taylor expansion

$$F(\underline{z}) = \sum_{\underline{i} \in \mathbb{N}^n} c_{\underline{i}} \underline{z}^{\underline{i}},$$



its **diagonal** is $\operatorname{Diag} F = \sum_{k \geq 0} c_{k,\dots,k} z^k$.

$\binom{2k}{k}$: $\qquad \dfrac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$

$\dfrac{1}{k+1}\binom{2k}{k}$: $\qquad \dfrac{1-2x}{(1-x-y)(1-x)} = 1 + y + 1xy - x^2 + y^2 + \cdots + 2x^2y^2 + \cdots$

Apéry's a_k : $= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ $\qquad \dfrac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$

Multiple Binomial Sums

over a field $\mathbb{K}$

Sequences constructed from

$$\text{the binomial sequence } (n,k) \mapsto \binom{n}{k};$$

$$\text{geometric sequences } n \mapsto C^n, C \in \mathbb{K};$$

$$\text{Kronecker's } \delta : n \mapsto \delta_n$$

using algebra operations and

$$\text{affine changes of indices } (u_{\underline{n}}) \mapsto (u_{\lambda(\underline{n})});$$

$$\text{indefinite summation } (u_{\underline{n},k}) \mapsto \left(\sum_{k=0}^{m} u_{\underline{n},k}\right).$$

Diagonals & Multiple Binomial Sums

$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$\boxed{\textbf{Thm.}\text{ Diagonals} \equiv \text{binomial sums with 1 free index.}}$

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> BinomSums[sumtores](S,u): (…)
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$$\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

has for diagonal the generating function of S_n

[Bostan-Lairez-S.17]

From Sum to Residue to Diagonal

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \left(= \binom{2n}{n} \right)$$

$$\binom{n}{k} := [x^k](1+x)^n = \frac{1}{2\pi i} \oint (1+x)^n \frac{dx}{x^{k+1}}$$

$$\binom{n}{k}^2 = \frac{1}{(2\pi i)^2} \oint (1+x_1)^n (1+x_2)^n \frac{dx_1 dx_2}{x_1^{k+1} x_2^{k+1}}$$

Geometric sums

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{1}{(2\pi i)^2} \oint (1+x_1)^n (1+x_2)^n \frac{1 - 1/(x_1 x_2)^{n+1}}{x_1 x_2 - 1} dx_1 dx_2$$

$$\sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 z^n = \frac{1}{(2\pi i)^2} \oint \left(\frac{1}{x_1 x_2 - z(1+x_1)(1+x_2)} + \frac{1}{1 - z(1+x_1)(1+x_2)} \right) \frac{dx_1 dx_2}{1 - x_1 x_2}$$

$\times\, x_1 x_2$ and $z \mapsto z x_1 x_2$

$$= \mathrm{Diag} \left(\left(\frac{1}{1 - z(1+x_1)(1+x_2)} + \frac{1}{1 - z x_1 x_2 (1+x_1)(1+x_2)} \right) \frac{1}{1 - x_1 x_2} \right)$$

Can be turned into a general algorithm

IV. Analytic Combinatorics in several variables

2013

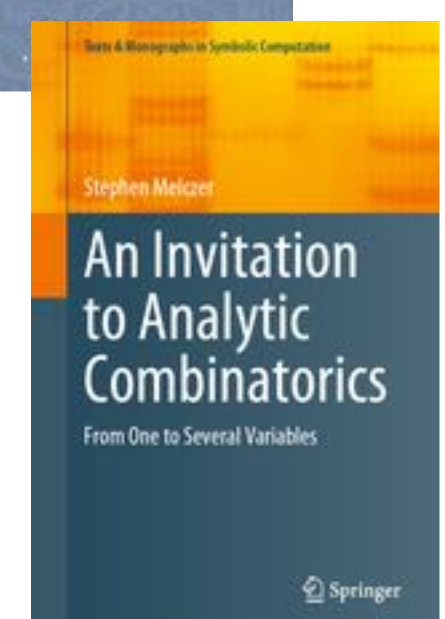
$$S_n = \sum_{r\ge 0}\sum_{s\ge 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$$S(z) = \sum_{n\ge 0} S_n z^n = \operatorname{Diag}\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

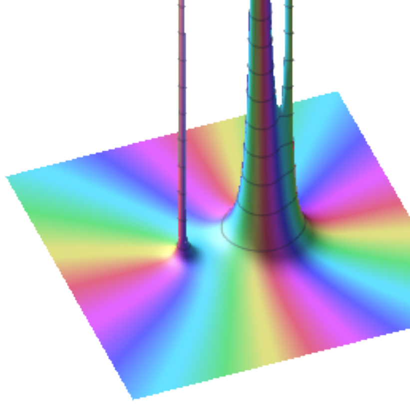
$$z-1)\,S^{(3)}(z) + \cdots + 2\,(30z+1)\,S(z) = 0$$

$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1-\frac{9}{16n}+O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty$$

2020



Coefficients of Diagonals



$$F(\underline{z}) = \frac{G(\underline{z})}{H(\underline{z})} \qquad c_{k,\ldots,k} = \left(\frac{1}{2\pi i}\right)^n \int_T \frac{G(\underline{z})}{H(\underline{z})} \frac{dz_1 \cdots dz_n}{(z_1 \cdots z_n)^{k+1}}$$

Critical points: extrema of $|z_1 \cdots z_n|$ on $\mathscr{V} := \{\underline{z} \mid H(\underline{z}) = 0\}$.

$$\operatorname{rank}\begin{pmatrix} \frac{\partial H}{\partial z_1} & \cdots & \frac{\partial H}{\partial z_n} \\ \frac{\partial (z_1 \cdots z_n)}{\partial z_1} & \cdots & \frac{\partial (z_1 \cdots z_n)}{\partial z_n} \end{pmatrix} \leq 1 \quad \text{i.e.} \quad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

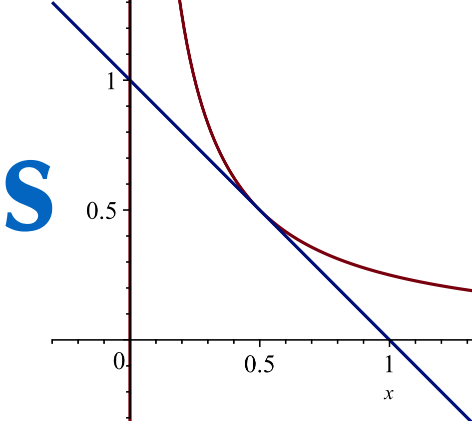
Ideal also in the Griffiths-Dwork method

Minimal ones: on the boundary of the domain of convergence.

A 3-step method

1a. locate the critical points (**algebraic** condition);
1b. find the minimal ones (**semi-algebraic** condition);
2. translate (easy in simple cases).

Ex.: Central Binomial Coefficients



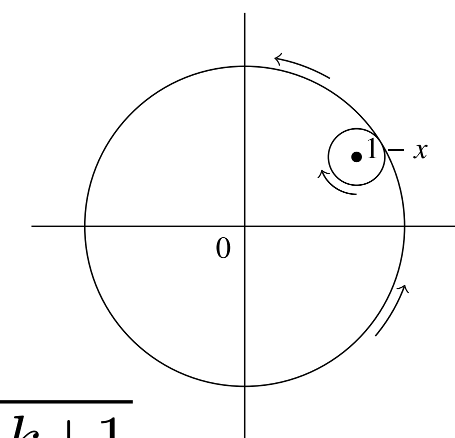
$$\binom{2k}{k}: \qquad \frac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$$

(1). Critical points: $1 - x - y = 0, x = y \Longrightarrow x = y = 1/2.$

(2). Minimal ones. Easy.

In general, this is the difficult step.

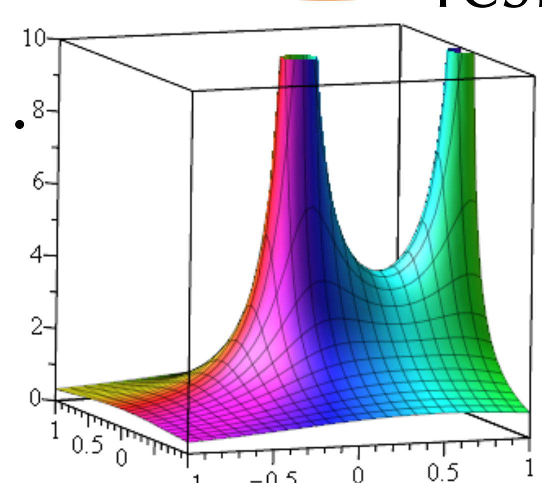
(3). Analysis close to the minimal critical point:



$$a_k = \frac{1}{(2\pi i)^2} \iint \frac{1}{1-x-y} \frac{dx\,dy}{(xy)^{k+1}} \approx \frac{1}{2\pi i} \int \frac{dx}{(x(1-x))^{k+1}}$$

residue

$$\approx \frac{4^{k+1}}{2\pi i} \int \exp(4(k+1)(x-1/2)^2)\,dx \approx \frac{4^k}{\sqrt{k\pi}}.$$



saddle-point approx

More Generally, Smooth Minimal Critical Point

$\underline{\zeta}$ smooth:
$\nabla H(\underline{\zeta}) \neq 0$

Satisfied generically

$$\text{Wlog } \partial H/\partial z_n(\underline{\zeta}) \neq 0$$

Implicit function theorem

$$g \text{ s.t. } H(\hat{z}, g(\hat{z})) = 0, \quad \hat{z} = (z_1, \ldots, z_{n-1})$$

Step 1. Residue

$$c_{\underline{k}} = \left(\frac{1}{2\pi i}\right)^{n-1} \oint \frac{G(\hat{z}, g(\hat{z}))}{\partial_n H(\hat{z}, g(\hat{z}))} \frac{\hat{\partial} \hat{z}}{\psi(\hat{z})^{k+1}}, \qquad \psi(\hat{z}) := z_1 \ldots z_{n-1} g(\hat{z})$$

Step 2. Saddle-point analysis

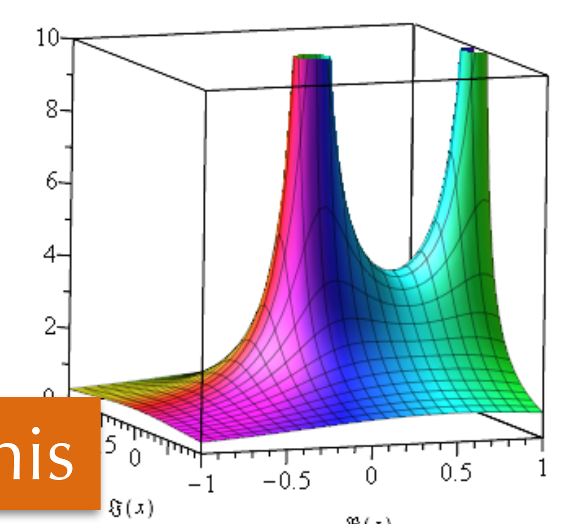
$\underline{\zeta}$ critical

Hessian matrix

$$\psi(\hat{z}) = \zeta_1 \cdots \zeta_n + 0 \cdot (\hat{z} - \hat{\zeta}) + \frac{1}{2}(\hat{z} - \hat{\zeta})^T \mathscr{H}(\zeta)(\hat{z} - \hat{\zeta}) + O(|\hat{z} - \hat{\zeta}|^3)$$

Thm. Under mild conditions,

$$c_{\underline{k}} = \underline{\zeta}^{-k} k^{\frac{1-n}{2}} \left(\frac{(2\pi)^{(1-n)/2}}{\sqrt{(\underline{\zeta}^{3-n}/\zeta_n^2)\,|\mathscr{H}(\underline{\zeta})|}} \cdot \frac{-G(\underline{\zeta})}{\zeta_n \partial H/\partial z_n(\underline{\zeta})} + O(k^{-1}) \right)$$



Aim: automate this

[PemantleWilson13;GaoRichmond92]

Locating the Critical Points

Algebraic part: ``compute'' the solutions of the system

$$H(\underline{z}) = 0 \qquad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

$$\deg H = d,\ \max|\mathrm{coeff}(H)| \leq 2^h, D := d^n,$$

Prop. Under genericity assumptions, a probabilistic algorithm finds

$$\left.\begin{aligned} P(u) &= 0 \\ P'(u)z_1 - Q_1(u) &= 0 \\ &\vdots \\ P'(u)z_n - Q_n(u) &= 0 \end{aligned}\right\} \quad \begin{aligned} &\deg \leq nD, \\ &\text{height} = \tilde{O}(D(d+h)) \end{aligned}$$

in $\tilde{O}(D^3(d+h))$ bit ops.

Kronecker representation

System reduced to a univariate polynomial

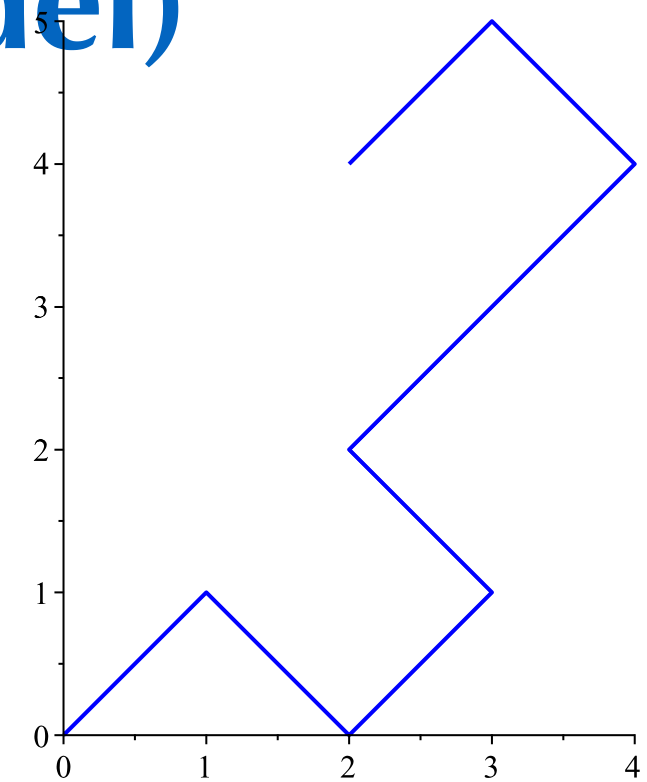
History and Background: see Castro, Pardo, Hägele, and Morais (2001)

[Giusti-Lecerf-S.01;Schost02;SafeySchost16]

Example (Lattice Path Model)

The number of walks from the origin taking steps *{NW,NE,SE,SW}* and staying in the first quadrant is

$$\mathrm{Diag}\,\frac{(1+x)(1+y)}{1-t(1+x^2+y^2+x^2+y^2)}$$



Kronecker representation of the critical points:

$$P(u) = 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920$$
$$Q_x(u) = 336u^2 + 344u - 105898$$
$$Q_y(u) = -160u^2 + 2824u - 48982$$
$$Q_t(u) = 4u^3 + 39u^2 - 4339u/2 + 4669/2$$

ie, they are given by:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Next: which one(s) of these 4 is minimal?

V. Certified Solutions of Polynomial Systems

Numerical Kronecker Representation

$$\left.\begin{array}{c} P(u) = 0 \\ P'(u)z_1 - Q_1(u) = 0 \\ \vdots \\ P'(u)z_n - Q_n(u) = 0 \end{array}\right\} \quad + \quad \text{isolating intervals/disks for the real/complex roots of } P$$

degree $\mathscr{D}$, height $\mathscr{H}$ — $\tilde{O}(\mathscr{D}^2(\mathscr{D} + \mathscr{H}))$

all z_i at precision $2^{-\kappa}$ — $\tilde{O}(\mathscr{D}^3 + n(\mathscr{D}^2\mathscr{H} + \mathscr{D}\kappa))$

(Technical) bounds on the complexity to
- decide whether a polynomial $Q(\underline{z})$
  . vanishes at some of the solutions,
  . is >0 or <1 at some of the real solutions;
- group solutions that have the same $|z_i|, i = 1,\ldots,n$.

Complexity uses bounds on the absolute separation

Also in a multi-degree and/or a straight-line program setting.

[Melczer-S.21]

VI. Minimal Critical Points in the Combinatorial Case

Semi-Algebraic Problem

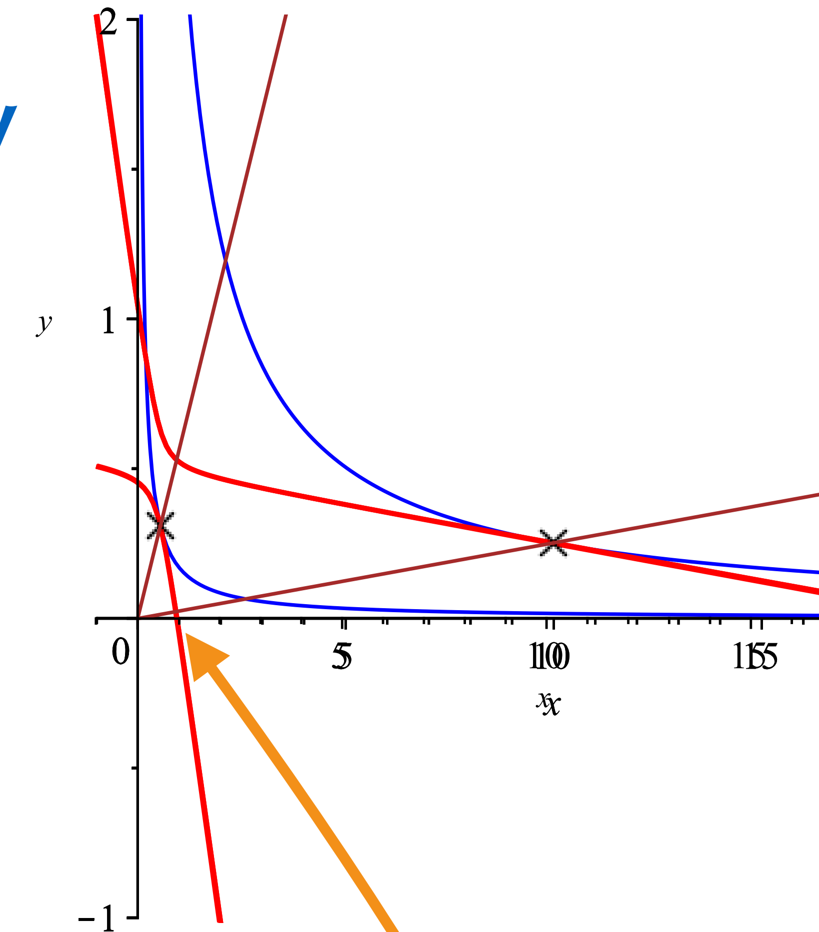
Combinatorial Generating Functions

Def. $F(z_1, \ldots, z_n)$ is **combinatorial** if every coefficient is ≥ 0.

Prop. [PemantleWilson] In the combinatorial case, one of the minimal critical points has positive real coordinates.

1. Use this criterion to find minimal critical points
2. Find all minimal critical points with the same $|z_1|, \ldots, |z_n|$
3. Add asymptotic contributions from each of them

Testing Minimality



$$F = \frac{1}{H} = \frac{1}{(1-x-y)(20-x-40y)-1}$$

Critical point equation $x\frac{\partial H}{\partial x} = y\frac{\partial H}{\partial y}$:

$$x(2x+41y-21) = y(41x+80y-60)$$

→ 4 critical points, 2 of which are real:

$$(x_1, y_1) = (9.9971, 0.2528), \quad (x_2, y_2) = (0.54823, 0.30998)$$

Add $H(tx, ty) = 0$ and compute a Kronecker representation:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Solve numerically and keep the real positive sols:

(0.55,0.31,0.99), (0.55,0.31,1.71), (9.99,0.25,0.09), (9.99,0.25,0.99)

(x_1, y_1) is not minimal, (x_2, y_2) is.

Algorithm and Complexity

Thm. If $F(\underline{z})$ is combinatorial, then under regularity conditions, the points contributing to dominant diagonal asymptotics can be determined in $\tilde{O}(D^4(d+h))$ bit operations. Each contribution has the form

$$A_k = \left(T^{-k} k^{(1-n)/2} (2\pi)^{(1-n)/2}\right) (C + O(1/k))$$

T, C can be found to precision $2^{-\kappa}$ in $\tilde{O}(D^3 d^3 h^3 + D\kappa)$ bit ops.

explicit algebraic numbers

half-integer

This result covers the easiest cases.
All conditions hold generically and can be checked within the same complexity, except combinatoriality.

[Melczer-S. 21]

Example: Apéry's sequence

$$\frac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$$

Kronecker representation of the critical points:

$$P(u) = u^2 - 366u - 17711$$

$$x = \frac{2u-1006}{P'(u)}, \quad y = z = -\frac{320}{P'(u)}, \quad t = -\frac{164u+7108}{P'(u)}$$

There are two real critical points, and one is positive. After testing minimality, one has proved asymptotics

```
> A, U := DiagonalAsymptotics(numer(F),denom(F),[t,x,y,z], u, k):
> evala(allvalues(subs(u=U[1],A)));
```

$$\frac{\left(17+12\sqrt{2}\right)^k \sqrt{2}\sqrt{24+17\sqrt{2}}}{8\,k^{3/2}\,\pi^{3/2}}$$

Example: Restricted Words in Factors

$$[x^i y^j]\frac{1-x^3y^6+x^3y^4+x^2y^4+x^2y^3}{1-x-y+x^2y^3-x^3y^3-x^4y^4-x^3y^6+x^4y^6}$$

= #words over {0,1} with i 0 and j 1 and without 10101101 or 1110101

```
> A,U:=DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
> A;
```

$$\left(\left(\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}-605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-12u^{20}+30u^{19}+258u^{18}+500u^{17}+440u^{16}-102u^{15}-378u^{14}-1544u^{13}-2142u^{12}-550u^{11}-2222u^{10}-1644u^{9}+2860u^{8}-1848u^{7}+1230u^{6}+2160u^{5}-2686u^{4}+1494u^{3}-228u^{2}-320u+84}\right)^k\right.$$

$$\sqrt{2}\sqrt{\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}+605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-162u^{18}-612u^{17}-902u^{16}-616u^{15}+254u^{14}+548u^{13}+2054u^{12}+2156u^{11}+898u^{10}-2268u^{9}+2462u^{8}-2088u^{7}+1312u^{6}-540u^{5}-1410u^{4}+1188u^{3}-290u^{2}+32}}$$

$$\left.(12u^{20}+36u^{19}-21u^{18}-170u^{17}-255u^{16}-190u^{15}-19u^{14}+46u^{13}+461u^{12}-628u^{11}-133u^{10}+374u^{9}-161u^{8}-384u^{7}+146u^{6}-138u^{5}-285u^{4}-40u^{3}+91u^{2}-30u+32)\right)\Big/\Big(2\sqrt{k}\sqrt{\pi}\,(84u^{20}$$

$$+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}-2856u^{11}-605u^{10}+2020u^{9}-1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16)\Big)$$

```
> U;
```

$$[RootOf(4\_Z^{21}+12\_Z^{20}-15\_Z^{19}-86\_Z^{18}-125\_Z^{17}-88\_Z^{16}+17\_Z^{15}+54\_Z^{14}+193\_Z^{13}+238\_Z^{12}+55\_Z^{11}+202\_Z^{10}-137\_Z^{9}-220\_Z^{8}+132\_Z^{7}-82\_Z^{6}-135\_Z^{5}+158\_Z^{4}-83\_Z^{3}+12\_Z^{2}+16\_Z-4,\ 0.25574184)]$$

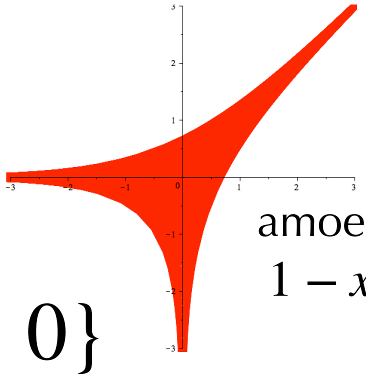
```
> evalf(subs(u=U[1],A));
```

$$\frac{0.60294599\ 3.9101932^k}{\sqrt{k}}$$

VII. Minimal Critical Points
General Case
Semi-Algebraic Problem

Minimal Critical Points

The connected components of the complement of amoebas are convex



$$\mathrm{amoeba}(H) := \{(\log|z_1|, \ldots, \log|z_n|) \mid \underline{z} \in \mathbb{C}^{*n}, H(\underline{z}) = 0\}$$

Consequence: With $\mathscr{D}$ the domain of convergence of F, $\underline{u} \notin \mathscr{D} \Rightarrow \exists t \in (0,1), \underline{z} \in \partial\mathscr{D}$ s.t. $|z_j| = t|u_j|, j = 1,\ldots,n$.

—> Criterion in the non-combinatorial case

Split into Real & Imaginary Parts

$f(\underline{z}) \in \mathbb{C}[\underline{z}]$ splits into $f(\underline{x} + i\underline{y}) = f^{(R)}(\underline{x}, \underline{y}) + i f^{(I)}(\underline{x}, \underline{y})$

$f^{(R)}, f^{(I)}$ in $\mathbb{R}[\underline{x}, \underline{y}]$

—> $2n+2$ critical point equations in $2n+2$ real unknowns

Cauchy-Riemann

$$\begin{cases} H^{(R)}(\underline{a},\underline{b}) = H^{(I)}(\underline{a},\underline{b}) & = 0 & \\ a_j\left(\partial H^{(R)}/\partial x_j\right)(\underline{a},\underline{b}) + b_j\left(\partial H^{(R)}/\partial y_j\right)(\underline{a},\underline{b}) - \lambda_R & = 0, & j = 1,\dots,n \\ a_j\left(\partial H^{(I)}/\partial x_j\right)(\underline{a},\underline{b}) + b_j\left(\partial H^{(I)}/\partial y_j\right)(\underline{a},\underline{b}) - \lambda_I & = 0, & j = 1,\dots,n \end{cases}$$

Minimal Critical Points

Needed: no real zero of $H(\underline{x} + i\underline{y})$ with

$$|x_j + iy_j| = t\,|a_j + ib_j|, \quad j = 1,\ldots,n$$

with $0 < t < 1$.

Add new equations:

$$H^{(R)}(t\underline{x}, t\underline{y}) = H^{(I)}(t\underline{x}, t\underline{y}) = 0,$$
$$x_j^2 + y_j^2 = t(a_j^2 + b_j^2), \quad j = 1,\ldots,n$$

$n + 2$ eqns in
$2n + 1$ unknowns

And setup a (structured) system for the critical points of

$$\pi_t : (\underline{a}, \underline{b}, \underline{x}, \underline{y}, \lambda_R, \lambda_I, t) \mapsto t\,.$$

$4n + 4$ eqns in
$4n + 4$ unknowns

Bit complexity for min crit pt selection: $\tilde{O}(2^{3n}D^9d^5h)\,.$

Rest as before

Conclusion

Comparison of Approaches