

# The Dynamic Dictionary of Mathematical Functions

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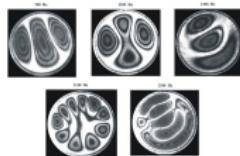
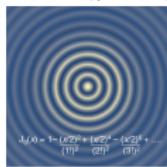
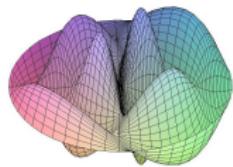
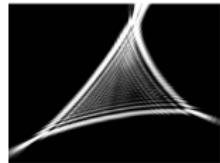
Joint work with: Alexandre Benoit, Alin Bostan, Frédéric Chyzak,  
Alexis Darrasse, Stefan Gerhold, Marc Mezzarobba

## I Motivation

# Mathematical Functions



$\exp, \log, \sin, \cos, \tan,$   
 $\sinh, \cosh, \tanh, \arcsin, \arccos, \arctan,$   
 $\text{arcsinh}, \text{arccosh}, \text{arctanh},$   
 Bessel  $J_\nu, I_\nu, Y_\nu, K_\nu$ , Airy  $Ai$  and  $Bi$ ,  
 hypergeometric, generalized hypergeo-  
 metric, classical orthogonal polynomials,  
 Struve  $H_\nu, L_\nu, M_\nu, K_\nu$ , Lommel  $s_{\mu,\nu}$  and  $S_{\mu,\nu}$ ,  
 Anger  $J_\nu$ , Weber  $E_\nu$ , Whittaker  $W_{\kappa,\mu}, M_{\kappa,\mu} \dots$



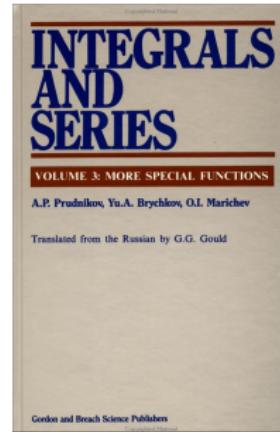
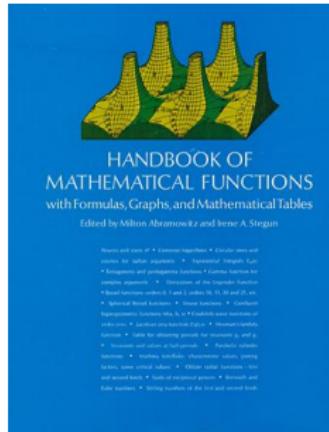
## Special functions

Functions that have been met sufficiently often to deserve a name.

Scientists need help with these functions

# Dictionaries

Among the most cited documents in the scientific literature.



Thousands of **useful** mathematical formulas,  
computed, compiled and edited by hand.

Started between 60 and 30 years ago.

# Progress in the Past 30 Years

Two important changes in the way we work:

- **Symbolic Computation.** Several million users.



Mathematical functions implemented from these dictionaries.

- **The Web**



New kinds of interaction with documents.

# First Step: the NIST DLMF

## Navigation, Exports, Search Engine

**NIST Digital Library of Mathematical Functions**  
companion to the *NIST Handbook of Mathematical Functions*

**Project News**

- 2010-05-11 [Handbook published and DLMF goes public](#)
- 2010-05-06 [Firefox 3.6 now on Windows](#)
- [More news](#)

**Preface**  
Mathematical Introduction  
1 Algebraic and Analytic Methods  
2 Asymptotic Approximations  
3 Numerical Methods  
4 Elementary Functions  
5 Gamma Function  
6 Exponential, Logarithmic, Sine, and Cosine Integrals  
7 Error Functions, Dawson's and Fresnel Integrals  
8 Incomplete Gamma and Related Functions  
9 Airy and Related Functions  
10 Bessel Functions  
11 Struve and Related Functions  
12 Parabolic Cylinder Functions  
13 Confluent Hypergeometric Functions  
14 Legendre and Related Functions  
15 Hypergeometric Function  
16 Generalized Hypergeometric Functions and Meijer  $G$ -Function  
17  $q$ -Hypergeometric and Related Functions  
18 Orthogonal Polynomials

19 Elliptic Integrals  
20 Theta Functions  
21 Multidimensional Theta Functions  
22 Jacobian Elliptic Functions  
23 Weierstrass Elliptic and Modular Functions  
24 Bernoulli and Euler Polynomials  
25 Bernhard Related Functions  
26 Combinatorial Analysis  
27 Functions of Number Theory  
28 Mathieu Functions and Hill's Equation  
29 Lamé Functions  
30 Spheroidal Wave Functions  
31 Heun Functions  
32 Painlevé Transcendents  
33 Coulomb Functions  
34  $\mathcal{J}_1$ ,  $\mathcal{G}_1$ ,  $\mathcal{H}_1$ : Symbols  
35 Functions of Matrix Argument  
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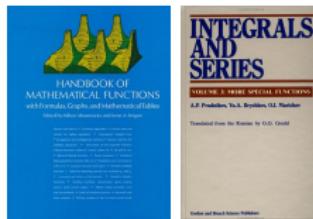
Still computed, compiled and edited by hand.

# The Dynamic Dictionary of Mathematical Functions

## Aim of the project

**DDMF** = Mathematical Handbooks + Computer Algebra + Web

- ① Computer algebra algorithms to **generate** the formulas;
- ② Web-like interaction with the document **and the computation**.



## Building Blocks:

- ① Linear differential equations as a data-structure;
- ② New language for maths on the web.  
(compatible with browsers!).

## II Demonstration

# Demo

<http://ddmf.msr-inria.inria.fr>

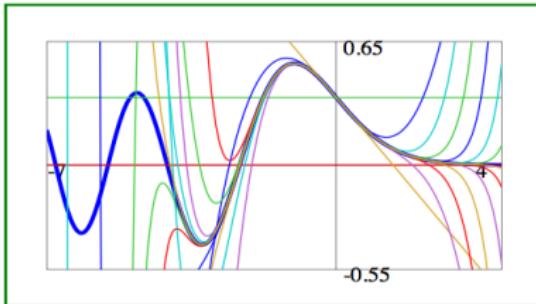
## Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on [Mathematical Functions](#), with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

### Select a special function from the list

- [Help](#) on selecting and configuring the mathematical rendering
- DDMF [developers](#) list
- [Motivation](#) of the project
- List of [related projects](#)
- Release history

The DDMF project (2008–2010) is hosted and supported by the [Microsoft Research – INRIA Joint Centre](#).



Generated on 2010-06-25 17:03:34 using commit b9253b3..., Fri Jun 25 16:21:12 2010 +0200 (modified).  
Powered by [DynaMoW](#).

### Contents

rendering [link](#)

- The [inverse cosecant](#)  $\text{arccsc}(x)$
- The [inverse cosine](#)  $\text{arccos}(x)$
- The [inverse cotangent](#)  $\text{arccot}(x)$
- The [inverse hyperbolic cosecant](#)  $\text{arccsch}(x)$
- The [Airy function of the first kind](#)  $\text{Ai}(x)$
- The [inverse secant](#)  $\text{arcsec}(x)$
- The [inverse sine](#)  $\text{arcsin}(x)$
- The [inverse tangent](#)  $\text{arctan}(x)$
- The [Airy function of the second kind](#)  $\text{Bi}(x)$
- The [hyperbolic cosine integral](#)  $\text{Chi}(x)$
- The [cosine integral](#)  $\text{Ci}(x)$
- The [cosine](#)  $\text{cos}(x)$
- The [exponential integral](#)  $\text{Ei}(x)$
- The [error function](#)  $\text{erf}(x)$
- The [complementary error function](#)  $\text{erfc}(x)$
- The [imaginary error function](#)  $\text{erfi}(x)$
- The [inverse hyperbolic cosine](#)  $\text{arccosh}(x)$
- The [inverse hyperbolic cotangent](#)  $\text{arccoth}(x)$
- The [inverse hyperbolic secant](#)  $\text{arcsech}(x)$
- The [inverse hyperbolic sine](#)  $\text{arsinh}(x)$
- The [inverse hyperbolic tangent](#)  $\text{arctanh}(x)$
- The [hyperbolic cosine](#)  $\text{cosh}(x)$
- The [hyperbolic sine](#)  $\text{sinh}(x)$
- The [dilogarithm](#)  $\text{dilog}(x)$
- The [hyperbolic sine integral](#)  $\text{Shi}(x)$
- The [sine integral](#)  $\text{Si}(x)$
- The [sine](#)  $\text{sin}(x)$
- The [exponential](#)  $\text{e}^x$
- The [logarithm](#)  $\ln(x)$

### III DynaMoW: Dynamic Mathematics on the Web

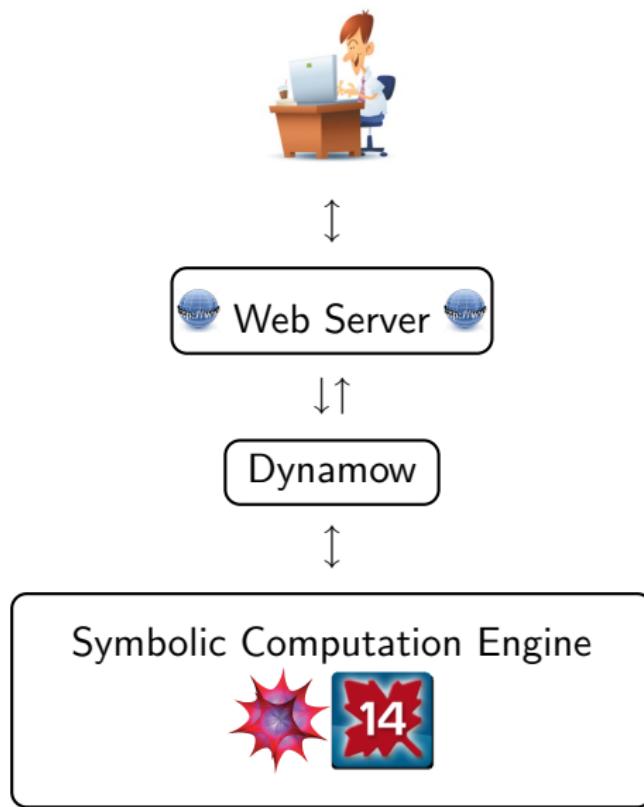
## Principle

The document being generated by the computer algebra system is an object of the language.

## Consequences:

- ① the structure of the document may depend on values that have been computed;
- ② intermediate steps can be turned into a mathematical proof in natural language;
- ③ easy to write demo code.

# Architecture



# DynaMoW, an OCaml Library

DynaMoW = ocaml + quotations + antiquotations

Symbolic result, converted to LaTeX, put into a paragraph

```
let res = <:symb< symbolic expression >> in
  <:par< some text <:imath< some latex $(symb:res) >>
>>
```

Using ocaml values in symbolic computations

```
let n = 23 and s = "foo" in
  <:symb< f($int:n), $(str:s) >>
```

Symbolic objects cast to ocaml types

```
let n = 23 + <:int< symbolic expression >> in ...
if <:bool< symbolic expression >> then ... else ...
<:unit< f := symbolic expression >>
```

# Example

The proof that a function is odd

equation into a homogeneous one.

The homogeneous differential equation and the fact that

$$\frac{d^i}{dx^i}y(-x) = (-1)^i \frac{d^i}{dx^i}y(x), \quad \text{for } i \in \mathbb{N},$$

imply that the function  $y_1(x) = -y(-x)$  satisfies the differential equation

$$(-1 - x^2) \frac{d^2}{dx^2}y_1(x) - 2x \frac{d}{dx}y_1(x) = 0$$

with initial conditions

metadata

$$y_1(0) = 0, (y'_1)(0) = 1.$$

metadata

The functions  $\arctan(x)$  and  $-\arctan(-x)$  thus satisfy the same differential equation, and their derivatives at  $x = 0$  agree up to order 1. Since  $x = 0$  is an ordinary point of the equation, these functions are analytic and equal in a neighborhood of 0:

$$\arctan(x) = -\arctan(-x).$$

This identity extends to the whole common domain of definition of these functions by uniqueness of the analytic continuation.

Sample Implementation with DynaMoW

```
let ending = Wording.ending_of_seq <:symb< inicondsAlt >> in
let body =
  <:par< The $str:homStr differential equation and the fact that
    <:dmath< <:symb< diff(y(-x),[x$1]) = (-1)^i*diff(y(x),[x$1])
      \quad \text{for } i \in \mathbb{N}, >>
    imply that the function <:imath< y_1(x) = $str:sign y(-x) >>
  satisfies the differential equation
  <:dmath< add(op(i+2,altDiffeq) * diff(y[1](x),[x$1]),
    i=seq(nops(altDiffeq)-2..0,-1)) = -op(1,altDiffeq)
  with initial condition $str:ending
  <:dmath< <:symb< op(inicondsAlt) >> . >> >> :: body in

let sf_name_as_math =
  if sf_name = "" then "y" else <:isymb< $str:sf_name >> in

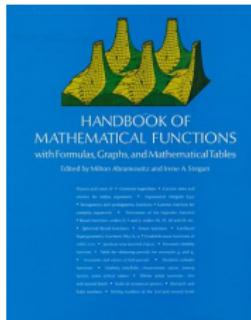
let body =
  let ordinary_point = DC.Text (Glossary.g "ordinary point") in
  <:par< The functions <:imath< $str:sf_name_as_math(x) >>
    and <:imath< $str:sign($str:sf_name_as_math)(-x) >>
    thus satisfy the same differential
    equation, and their derivatives at <:imath< x = 0 >> agree up
    order <:isymb< nops(iniconds)-1 >>. Since <:imath< x = 0 >>
    is an $par_entity:ordinary_point of the equation, these fun
    are analytic and equal in a neighborhood of 0:
    <:dmath< $str:sf_name_as_math(x) =
      $str:sign($str:sf_name_as_math)(-x) . >>
    This identity extends to the whole common domain of definitio
    of these functions by uniqueness of the analytic continuation
  >> :: body
in
  ((title req_params, List.rev body), ret)
```

jsMath

## IV Symbolic Computation

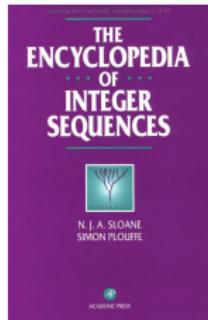
## Defining a Mathematical Function by an Equation

- Classical:  
polynomials represent their roots better than radicals.  
**Algorithms:** Euclidean division and algorithm, Gröbner bases.
  - More Recent:  
same for linear differential or recurrence equations.  
**Algorithms:** non-commutative analogues & gen. func.



About 25% of Sloane's encyclopedia,  
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure

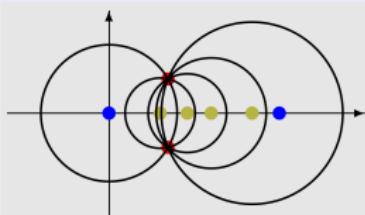


# Guaranteed Numerical Evaluation

## 1. Inside the disk of convergence

- ① Effective majorant-series analysis;
- ② Efficient evaluation of truncated series;
- ③ Time complexity quasi-linear wrt precision.

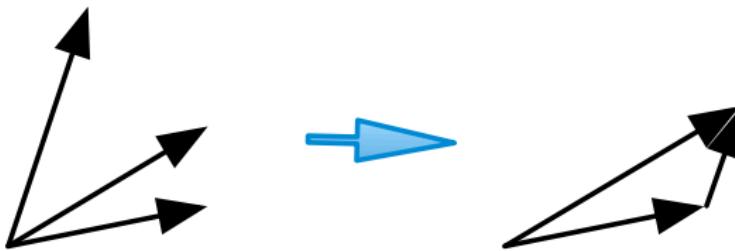
## 2. Effective analytic continuation



Path:  $0 \rightarrow 1.5 \rightarrow 2.3 \rightarrow 3 \rightarrow 4.22 \rightarrow 5.$

[MezzarobbaSalvy2010]

# Computation of Identities by Confinement



## Starting Point

$k + 1$  vectors in dimension  $k \rightarrow$  an identity.

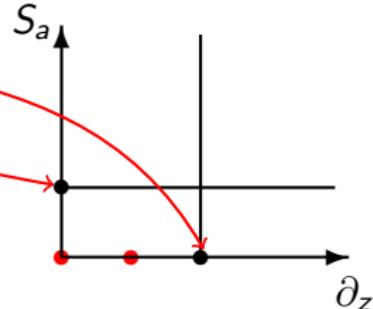
# First Example: Contiguity of Hypergeometric Series

$$F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1).$$

$$\begin{cases} z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0. \\ S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F. \end{cases}$$

Gauss 1812: contiguity relation.

$\dim=2 \Rightarrow S_a^2 F, S_a F, F$  linearly dependent.



# Second Example: Mehler's Identity by Confinement

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

- ① Definition of Hermite polynomials  $H_n(t)$ : recurrence of order 2  $\leftrightarrow$  vector space of dimension 2 over  $\mathbb{Q}(t, n)$ ;
- ② Product: vector space over  $\mathbb{Q}(x, y, n)$  generated by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order at most 4; (confinement)

- ③ Translate into differential equation (and solve).



## I. Definition

```

> R1 := {H(n+2)=(-2 n - 2) H(n) + 2 H(n + 1) x, H(0)=1, H(1)=2 x} :
=> R2 := subs(H=H2, x=y, R1);
      R2 := [H2(0)=1, H2(n + 2)=(-2 n - 2) H2(n) + 2 H2(n + 1) y, H2(1)=2 y]

```

## II. Product

```

> R3 := gfun :- poltorec(H(n)· H2(n)· v(n), [R1, R2, {v(n + 1) · (n + 1)=v(n), v(1)=1}], [H(n), H2(n), v(n)], c(n));
R3 := {c(0)=1, c(1)=4 x y, c(2)=8 x2 y2 + 2 - 4 y2 - 4 x2, c(3)= $\frac{32}{3}$  x3 y3 + 24 x y - 16 x y3 - 16 x3 y, (16 n
+ 16) c(n) - 16 x y c(n + 1) + (-8 n - 20 + 8 y2 + 8 x2) c(n + 2) - 4 x c(n + 3) y + (n + 4) c(n + 4)}

```

## III. Differential Equation

```

> gfun :- rectodiffeq(R3, c(n), f(u));
      { (16 u3 - 16 u2 y x - 4 u + 8 u y2 + 8 u x2 - 4 x y) f(u) + (16 u4 - 8 u2 + 1) ( $\frac{d}{du}$  f(u)), f(0)=1}

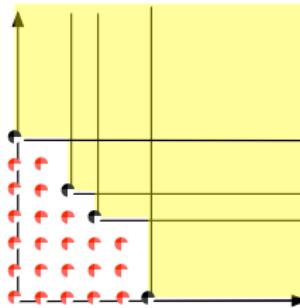
```

```
> dsolve(% , f(u));
```

$$f(u) = \frac{\text{Ie}^{\left(\frac{-4 x y u + y^2 + x^2}{(2 u - 1) (2 u + 1)}\right)}}{\text{e}^{(-y^2 - x^2)} \sqrt{2 u + 1} \sqrt{2 u - 1}}$$

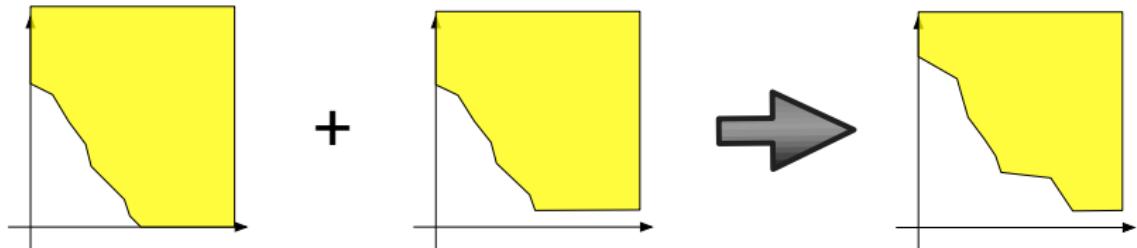
# Data-Structure

- ① Univariate case: linear differential/recurrence equation  
+ ini. cond;
- ② Multivariate case: operators (a Gröbner basis of them)  
+ ini. cond.



D-finiteness  $\equiv$  finite dimension

# Confinement: Beyond Finite Dimension



Use the Hilbert dimension to drive the computation by increasing degrees.

# Diff. under $\int$ + Integration by Parts $\rightarrow$ Algorithm?

Ex.:  $\int_0^1 \underbrace{\frac{\cos zt}{\sqrt{1-t^2}}}_{f(z,t)} dt = \frac{\pi}{2} J_0(z), \quad (zJ_0'' + J_0' + zJ_0 = 0, J_0(0) = 1).$

Proof: initial conditions and

$$z \frac{\partial^2}{\partial z^2} f(z, t) + \frac{\partial}{\partial z} f(z, t) + zf(z, t) = \frac{\partial}{\partial t} \left( \frac{t^2 - 1}{t} \frac{\partial}{\partial z} f(z, t) \right)$$

## Creative Telescoping [Zeilberger]

**Input:** (a basis of) linear operators that annihilate  $f$ ;

**Output:**  $A$  free of  $t, \partial/\partial t$ , certificate  $B$ , such that

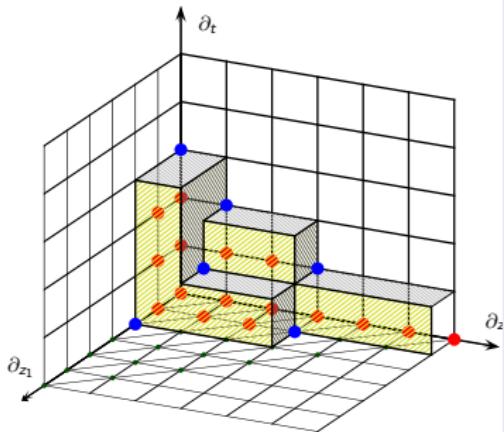
$$A(f) = \frac{\partial}{\partial t} B(f).$$

**Algorithm:** sometimes. (Why would they exist?)

+ variants for multiple sum/int.

# Algorithms

- ➊ Hypergeometric case [Zeilberger 1990];
- ➋ D-finite case: [Chyzak 2000];
- ➌ Non-D-finite:



**Algorithm** [ChyzakKauersSalvy2009]

for  $s = 0, 1, 2, \dots$ , until  $\dim_H \mathcal{J} \leq \text{bound}$ :

- ➊ reduce  $A - \partial_t B$  with

$$A := \sum_{|\alpha| \leq s} \eta_\alpha(z) \partial_z^\alpha, \quad B := \sum_{\beta \in M_s(\mathcal{I})} \phi_\beta(z, t) \partial_t^\beta,$$

for **undetermined** rational  $\eta_\alpha(z)$ ,  $\phi_\beta(z, t)$ .

- ➋ extract coeffs of  $M_{s+1}(\mathcal{I})$  to form a linear system of first order w.r.t.  $\partial_t$
- ➌ solve and set  $\mathcal{J}$  to the ideal of the  $A$ 's  
return the pairs  $(A, B)$ .

# Typical Examples

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy - u(x^2 + y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

[Mehler1866]

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2 - k)/2}}{(q; q)_{n-k} (q; q)_{n+k}}$$

[Andrews1974]

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

[GlasserMontaldi1994]

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$

[Doetsch1930]

# Examples of Non-D-Finite Identities

$$\sum_{k=0}^n \binom{n}{k} i(k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n \quad [\text{Abel 1826}]$$

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle \quad [\text{Frobenius 1910}]$$

$$\sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^n \binom{n}{k} B_{m+k} \quad [\text{Gessel 2003}]$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x) dx = \beta^{-k} B(k, n - k) \zeta(n - k, \alpha)$$

$$\int_0^\infty x^{\alpha-1} \text{Li}_n(-xy) dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy) dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

# Special Cases with Specific Algorithms

- ① Recurrence for the Taylor coefficients:

$$\frac{1}{2\pi i} \oint f(z) \frac{dz}{z^{n+1}};$$

- ② Recurrence for the Chebyshev coefficients: [BenoitSalvy2009]

$$\frac{2}{\pi} \int_{-1}^1 \frac{f(t) T_n(t)}{\sqrt{1-t^2}} dt;$$

- ③ Laplace transform:

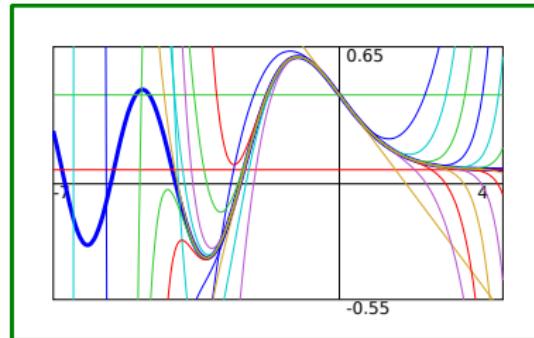
$$\int_0^{+\infty} e^{-zt} f(t) dt.$$

# Future Work

- More expansions on bases of functions;
- more integral transforms;
- families of functions or functions with parameters;
- automatic generation of numerical code;
- information on the zeros of functions;
- user-defined functions.

Summary: you want to bookmark

<http://ddmf.msr-inria.inria.fr>



# THE END