Special Function Identities

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I Introduction
Aim of the project

DDMF = Mathematical Handbooks + Computer Algebra + Web

1. Develop and use computer algebra algorithms to generate the formulas;
2. Provide web-like interaction with the document and the computation.

http://ddmf.msr-inria.inria.fr/
Equations Are a Good Data Structure

- **Classical:**
  polynomials represent their roots better than radicals.
  **Algorithms:** Euclidean division and algorithm, Gröbner bases.

- **Recent:**
  same for linear differential or recurrence equations.
  **Algorithms:** non-commutative analogues.

About 25% of Sloane’s encyclopedia,
60% of Abramowitz & Stegun.

\[
\text{eqn+ini. cond.} = \text{data structure}
\]
Examples of Identities

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3
\]

[Strehl92]

\[
\int_{0}^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1 - a^4)}{2\pi a^2}
\]

[GlMo94]

\[
\frac{1}{2\pi i} \oint \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1 + 4y^2)^{3/2}} \, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor!}
\]

[Doetsch30]

\[
\sum_{k=0}^{n} \frac{q^{k^2}}{(q; q)_k(q; q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}}
\]

[Andrews74]

\[
\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j}(q; q)_i(q; q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k}(q; q)_{n-k}}
\]

[Paule85].
More Identities

\[ \sum_{k=0}^{n} \binom{n}{k} i(k + i)^{k-1}(n - k + j)^{n-k} = (n + i + j)^n \quad [\text{Abel1826}] \]

\[ \sum_{k=0}^{n} (-1)^{m-k} k!(\binom{n-k}{m-k}) \{n+1\} = \langle n \rangle, \quad [\text{Frobenius1910}] \]

\[ \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \quad [\text{Gessel03}] \]

\[ \int_{0}^{\infty} x^{k-1} \zeta(n, \alpha + \beta x) \, dx = \beta^{-k} B(k, n - k) \zeta(n - k, \alpha), \]

\[ \int_{0}^{\infty} x^{\alpha-1} \text{Li}_{n}(-xy) \, dx = \frac{\pi(-\alpha)^{n}y^{-\alpha}}{\sin(\alpha \pi)}, \]

\[ \int_{0}^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy) \, dx = \frac{\pi y^{-s}}{\sin((a + s) \pi)} \frac{\Gamma(s)}{\Gamma(1 - a)} \]
Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

- 1st slide: Zeilberger’s algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: recent generalization of previous ones (with Chyzak & Kauers).

Ideas

Confinement in finite dimension + Creative telescoping.
II Confinement in Finite Dimension
Confinement Provokes Identities

Idea: confine a function and all its derivatives.

Obvious

\(k + 1\) vectors in dimension \(k\) \(\rightarrow\) an identity.
First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

> series(sin(x)^2+cos(x)^2-1,x,4);

\[ O(x^4) \]

Why is this a proof?

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: \( y' = 0 \);
4. \( \rightarrow \) \( \sin^2 + \cos^2 - 1 \) satisfies a LDE of order at most 4;
5. Cauchy’s theorem concludes.

Second algorithmic proof (same idea): 
\[ F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \]

> for n to 5 do
>   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
Third Proof: Contiguity of Hypergeometric Series

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1). \]

\[
\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \quad \Rightarrow \quad z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,
\]

\[
\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a+1} \quad \Rightarrow \quad S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.
\]

Gauss 1812: contiguity relation.

\[ \text{dim}=2 \Rightarrow S_a^2 F, S_a F, F \text{ linearly dependent:} \]

(Coordinates in \( \mathbb{Q}(a, b, c, z) \).)

\[
(a+1)(z-1)S_a^2 F + (b-a-1)z + 2 - c + 2a)S_a F + (c-a-1)F = 0.
\]
1 Monomial ordering: order on \( \mathbb{N}^k \), compatible with \(+\), 0 minimal.

2 Gröbner basis of a (left) ideal \( \mathcal{I} \): corners of stairs.

3 Quotient \( \mod \mathcal{I} \): basis below the stairs (\( \text{Vect}\{\partial^\alpha f\} \)).

4 Reduction of \( P \): Rewrite \( P \mod \mathcal{I} \) on this basis.

5 Dimension of \( \mathcal{I} \): “size” of the quotient infinitely far.

6 D-finiteness: \( \dim = 0 \).

→ An access to (finite dimensional) vector spaces
Examples

Binomial coeffs $\binom{n}{k}$ wrt $S_n, S_k$

Hypergeometric sequences

Stirling nbs wrt $S_n, S_k$

Bessel $J_\nu(x)$ wrt $S_\nu, \partial_x$

Orthogonal pols wrt $S_n, \partial_x$

Abel type wrt $S_m, S_r, S_k, S_s$

$hgm(m, k)(k + r)^k(m - k + s)^{m-k}\frac{r}{k+r}$

dim = 2 in space of dim 4.
Closure Properties

Proposition

\[ \dim \text{ann}(f + g) \leq \max(\dim \text{ann} f, \dim \text{ann} g), \]
\[ \dim \text{ann}(fg) \leq \dim \text{ann} f + \dim \text{ann} g, \]
\[ \dim \text{ann} \partial f \leq \dim \text{ann} f. \]

Algorithms by linear algebra.
Fourth Algorithmic Proof: Mehler’s Identity for Hermite Polynomials

\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \exp \left( \frac{4u(x-y)(x^2+y^2)}{1-4u^2} \right) \frac{1}{\sqrt{1-4u^2}} \]

1. **Definition of Hermite polynomials (D-finite over \( \mathbb{Q}(x) \)):** recurrence of order 2;
2. **Product by linear algebra:** \( H_{n+k}(x)H_{n+k}(y)/(n+k)! \), \( k \in \mathbb{N} \) generated over \( \mathbb{Q}(x, n) \) by
   \[
   \frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}
   \]
   \( \to \) recurrence of order at most 4;
3. **Translate into differential equation.**
\section{I. Definition}

\begin{align*}
  &R_1 := \{ H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0) = 1, H(1) = 2x \} : \\
  &R_2 := \text{subs}\{H=H_2, x=y, R_1\}; \\
  &R_2 := \{ H_2(0) = 1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1) = 2y \}
\end{align*}

\section{II. Product}

\begin{align*}
  &R_3 := \text{gfun} : \text{poltorec}\{H(n) \cdot H_2(n) \cdot v(n), [R_1, R_2, \{v(n+1) \cdot (n+1) = v(n), v(1) = 1\}], [H(n), H_2(n), v(n)], c(n)\}; \\
  &R_3 := \left\{ c(0) = 1, c(1) = 4xy, c(2) = 8x^2y^2 + 2 - 4y^2 - 4x^2, c(3) = \frac{32}{3}x^3y^3 + 24xy - 16xy^3 - 16x^3y, (16n + 16)c(n) - 16xyc(n+1) + (-8n - 20 + 8y^2 + 8x^2)c(n+2) - 4xc(n+3)y + (n+4)c(n+4) \right\}
\end{align*}

\section{III. Differential Equation}

\begin{align*}
  &\text{gfun} : \text{rectodiffeq}\{R_3, c(n), f(u)\}; \\
  &\left\{ (16u^3 - 16u^2yx - 4u + 8uy^2 + 8ux^2 - 4xy) f(u) + (16u^4 - 8u^2 + 1) \left( \frac{d}{du} f(u) \right), f(0) = 1 \right\} \\
  &\text{dsolve}\{\%_1, f(u)\}; \\
  &f(u) = \frac{1}{e^{(-y^2-x^2)}} \sqrt{2u+1} \sqrt{2u-1} \left( \frac{-4xyu + y^2 + x^2}{(2u-1)(2u+1)} \right)
\end{align*}
III Creative Telescoping
Summation by Creative Telescoping

\[ I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

IF one knows Pascal’s triangle:

\[
\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} = 2\binom{n}{k} + \binom{n}{k+1} - \binom{n}{k},
\]

then summing over \( k \) gives

\[ I_{n+1} = 2I_n. \]

The initial condition \( I_0 = 1 \) concludes the proof.
Creative Telescoping (Zeilberger 90)

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum “telescopes”, leading to $A(n, S_n) \cdot F_n = 0$. 

$$F_n = \sum_k u_{n,k} = ?$$
Creative Telescoping (Zeilberger 90)

$I(x) = \int_{\Omega} u(x, y) \, dy = ?$

**IF** one knows $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ such that

$$(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,$$

then the integral “telescopes”, leading to $A(x, \partial_x) \cdot I(x) = 0$.

Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Creative telescoping = “differentiation” under integral + “integration” by parts
Ex.: $\int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = \frac{\pi}{2} J_0(z), \quad (zJ''_0 + J'_0 + zJ_0 = 0, \ J_0(0) = 1)$. 

$I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt, \quad I'(z) = \int_0^1 -t \frac{\sin zt}{\sqrt{1 - t^2}} \, dt,$

$I''(z) = \int_0^1 -t^2 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = -I(z) + \int_0^1 \sqrt{1 - t^2} \cos zt \, dt,$

$I''(z) + I(z) = \left[ \sqrt{1 - t^2} \frac{\sin zt}{z} \right]_0^1 + \int_0^1 \frac{t}{\sqrt{1 - t^2}} \frac{\sin zt}{z} \, dt = -\frac{I'(z)}{z}.$

ann $\frac{\cos zt}{\sqrt{1 - t^2}} \equiv A(z, \partial_z) - \partial_t \frac{t^2 - 1}{t} \partial_z$
Diff. under $\int +$ Integration by Parts $\rightarrow$ Algorithm?

Ex.: $\int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = \frac{\pi}{2} J_0(z), \quad (zJ_0'' + J_0' + zJ_0 = 0, \, J_0(0) = 1).

$$\text{ann} \frac{\cos zt}{\sqrt{1 - t^2}} \ni A(z, \partial_z) - \partial_t \frac{t^2 - 1}{t} \partial_z$$

Creative Telescoping

Input: generators of (a subideal of) $\text{ann} f$;
Output: $A, B$ such that $A - \partial_t B \in \text{ann} f$, $A$ free of $t, \partial_t$.
Algorithm: sometimes. (Why would they exist?)

Telescoping of $\mathcal{I}$ wrt $t$:

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathcal{Q}(z, t)\langle \partial_z, \partial_t \rangle) \cap \mathcal{Q}(z)\langle \partial_z \rangle.$$
Example: Pascal’s Triangle Again

\[(S_nS_k - S_k - 1) \cdot \left(\begin{array}{c} n \\ k \end{array}\right) = 0 = \left(S_n - 2S_n - 2 + (S_k - 1)(S_n - 1)\right) \cdot \left(\begin{array}{c} n \\ k \end{array}\right).
\]

Sum over \(k \Rightarrow (S_n - 2) \sum_k \left(\begin{array}{c} n \\ k \end{array}\right) = 0.

Reduce all monomials of degree \(\leq s = 2\):

\[
\begin{align*}
1 & \rightarrow 1, \\
S_n & \rightarrow \frac{n + 1}{n + 1 - k} 1, \\
S_k & \rightarrow \frac{n - k}{k + 1} 1 \\
S_n^2 & \rightarrow \frac{(n + 2)(n + 1)}{(n + 2 - k)(n + 1 - k)} 1, \\
S_k^2 & \rightarrow \frac{(n - k - 1)(n - k)}{(k + 2)(k + 1)} 1, \\
S_nS_k & \rightarrow \frac{n + 1}{k + 1} 1.
\end{align*}
\]

Common denominator: \(D_2 = (k + 1)(k + 2)(n + 1 - k)(n + 2 - k)\).

\(D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k\) confined in

\[\text{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41).\]

This has to happen for some degree: \(\text{deg } D_s = O(s)\).
Polynomial Growth

**Definition (Polynomial Growth $p$)**

There exists a sequence of polynomials $P_s$, s.t. for all $(a_1, \ldots, a_k)$ with $a_1 + \cdots + a_k \leq s$, $P_s \partial_1^{a_1} \cdots \partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

**Theorem (ChyzakKauersSalvy2009)**

$$\dim T_t(I) \leq \max(\dim I + p - 1, 0).$$

**Proof.** Same as above. Set $q := \dim I + p$.

- In degree $s$, $\dim O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \ldots, \partial_{i_q}$: $O(s^{q+1})$; \Rightarrow any $q$ variables linearly dependent $\Rightarrow \dim \leq q - 1$.

This proof gives an algorithm. Also, bounds available.
Examples (all with $p = 1$)

- Proper hypergeometric [Wilf & Zeilberger 1992]:

$$Q(n, k) \xi^k \prod_{i=1}^{u} (a_i n + b_i k + c_i)! \prod_{i=1}^{v} (u_i n + v_i k + w_i)!,$$

$Q$ polynomial, $\xi \in \mathbb{C}$, $a_i, b_i, u_i, v_i$ integers.

- Differential D-finite (definite integration);

- Stirling: ok for $n \geq 3$, e.g., Frobenius:

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ n+1 \right\}_{k+1} = \langle n \rangle. $$

- Abel type: dim = 2 $\rightarrow$ ok for $n \geq 4$, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k + i)^{k-1}(n - k + j)^{n-k} = (n + i + j)^n.$$
IV Conclusion
Conclusion

**Summary:**
- Linear differential/recurrence equations as a data structure;
- Confinement in vector spaces + creative telescoping → identities.

**Also:**
- Fast algorithms: Zeilberger 1990 (hypergeom); Chyzak 2000 (D-finite)
  Us 2009 (non-D-finite).
- Bounds → identities;
- Fast algorithms for special classes;
- Efficient numerical evaluation.

**Open questions:**
- Replace polynomial growth by something intrinsic;
- Exploit symmetries;
- Structured Padé-Hermite approximants;
- Understand non-minimality.