Composition of Power Series, Change of Basis and Orthogonal Polynomials

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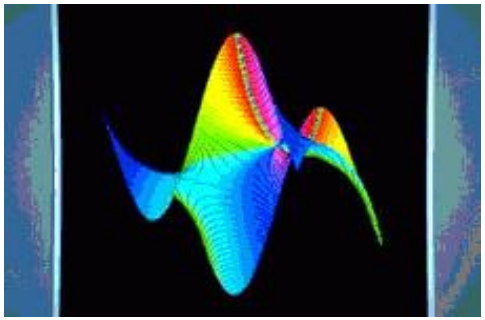

June 2nd, 2008
Joint work with Alin Bostan and Éric Schost
arXiv:0804.2337 (ISSAC'08) and arXiv:0804.2373.

I Introduction

Univariate Composition

Problem (Composition of Power Series)

Input: two power series f and g at precision N, with $g(0) = 0$.
Output: $f(g)$ at precision N.

Measure of complexity: number of arithmetic operations.
Hypothesis for the talk: characteristic 0.

General algorithms:

- Naïve: $\mathcal{O}(N^3)$;
- Naïve with fast multiplication: $\mathcal{O}(N\,\mathrm{M}(N))$;
- Brent & Kung (1978): $\mathcal{O}(\sqrt{N \log N}\,\mathrm{M}(N))$.

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Fast Multiplication of Polynomials of degree N

- Naïve: $\mathrm{M}(N) = \mathcal{O}(N^2)$;
- Karatsuba (1963): $\mathrm{M}(N) = \mathcal{O}(N^{1.59})$;
- FFT: $\mathrm{M}(N) = \mathcal{O}(N \log N)$ (Schönhage & Strassen 71).

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Nothing better known in general $\longrightarrow$ exploit structure
Special f's by Newton iteration: $\mathcal{O}(\mathrm{M}(N))$ for inverse (Sieveking 72, Kung 74), log, exp, power (Brent 75);

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Special g's by Divide-and-Conquer: $\mathcal{O}(\mathrm{M}(N) \log N)$;

- polynomials (Brent & Kung 78);
- algebraic series (van der Hoeven 02);

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f a polynomial, special g's:

- $x + a$: $\mathcal{O}(\mathrm{M}(N))$ (Aho, Steiglitz, Ullman 75);
- $(ax + b)/(cx + d)$: $\mathcal{O}(\mathrm{M}(N))$ (Pan 98);
- NEW: several other functions in $\mathcal{O}(\mathrm{M}(N))$;
- NEW: exp and log in $\mathcal{O}(\mathrm{M}(N) \log N)$.

Bivariate Evaluation

Problem (Bivariate Evaluation)

$$F(x,t) = \sum_{i,j} F_{i,j} x^i t^j = \sum_j G_j(x) t^j.$$

Input: $a_0, \ldots, a_N$;
Output: $\sum_j a_j G_j(x) \bmod x^{N+1}$.

Examples:

- $G_j = g^j$: univariate composition;
- G_j polynomial of degree j: change of basis from (G_j) to (x^j).

Our Result

Good complexity for this map and its inverse for "nice" F.

Special Cases with Fast Change of Basis

Very fast ($\mathcal{O}(\mathsf{M}(N))$)

- Orthogonal: Jacobi, Legendre, Chebyshev U, T, Gegenbauer, Laguerre, Hermite.
- Other: Fibonacci, Bernoulli, Euler, Mott, Spread, Bessel,...

Fast ($\mathcal{O}(\mathsf{M}(N)\log N)$)

- Orthogonal: Meixner, Meixner-Pollaczek, Krawtchouk, Charlier;
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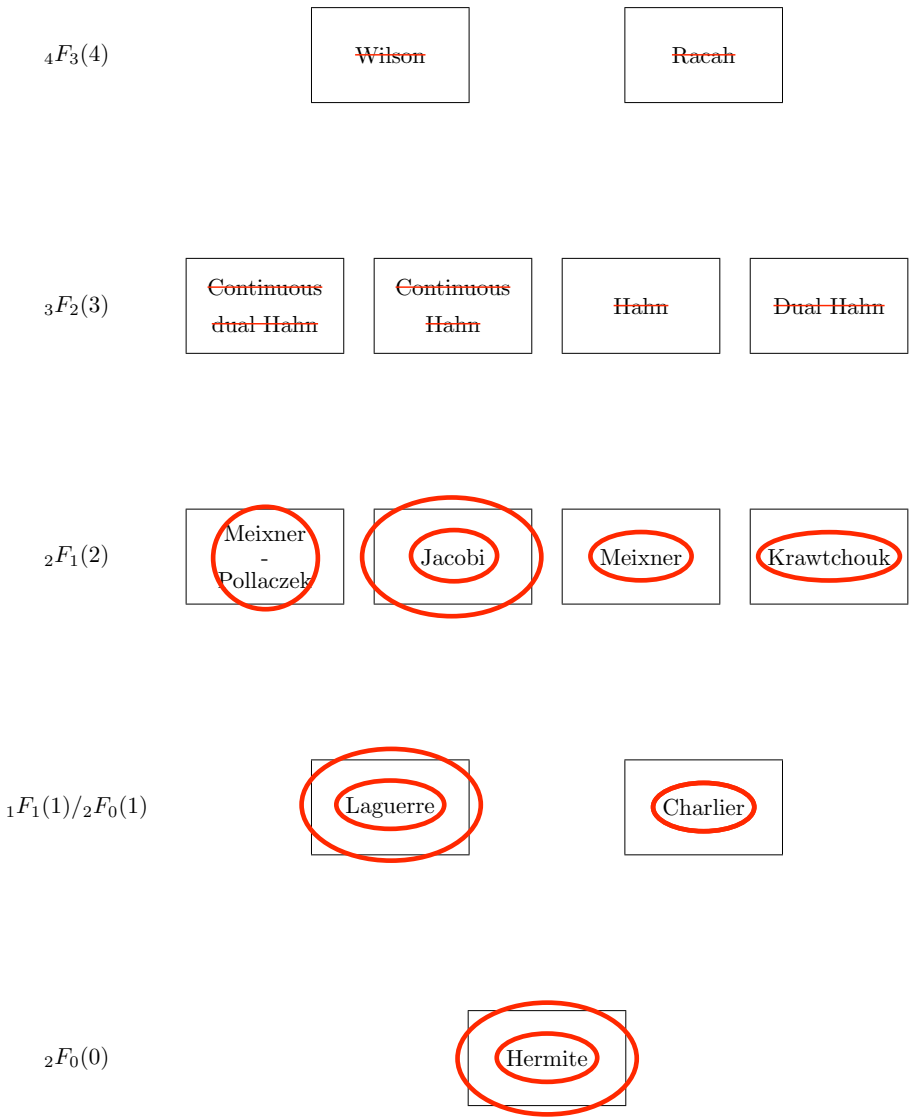

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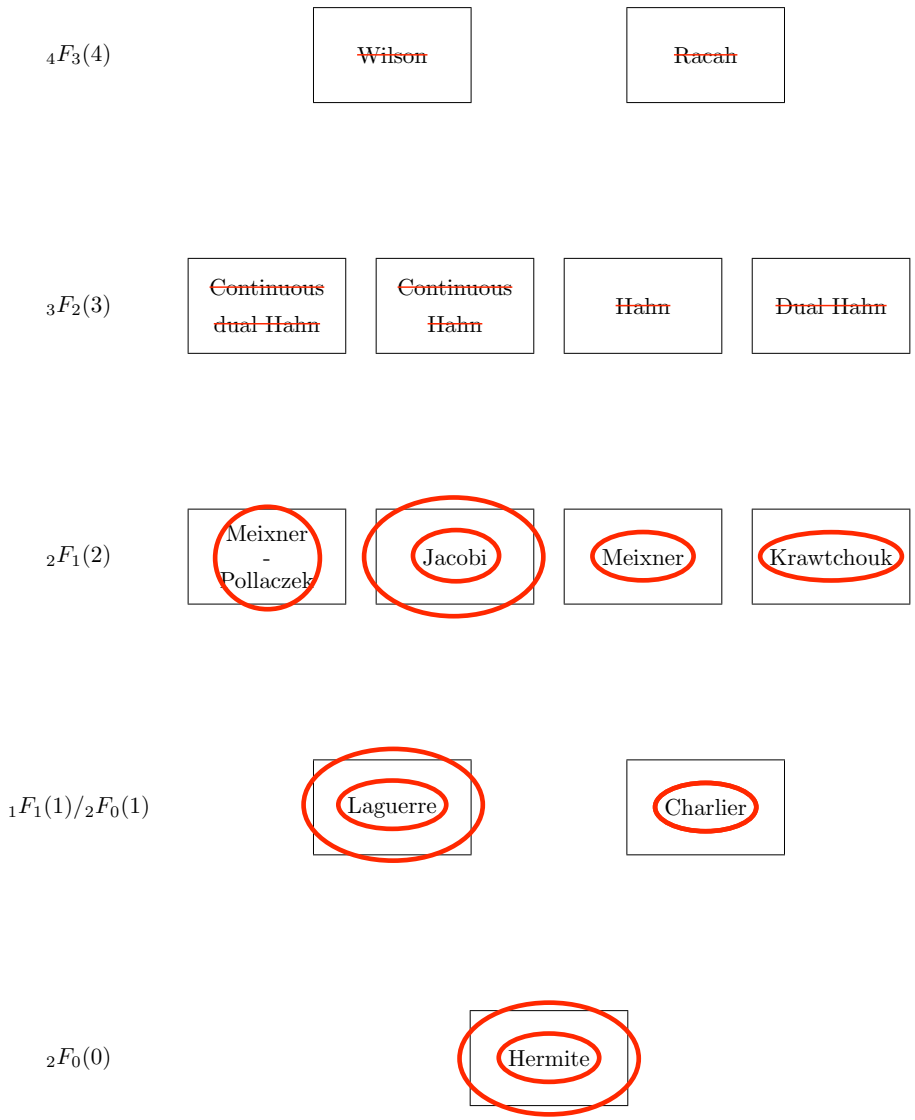

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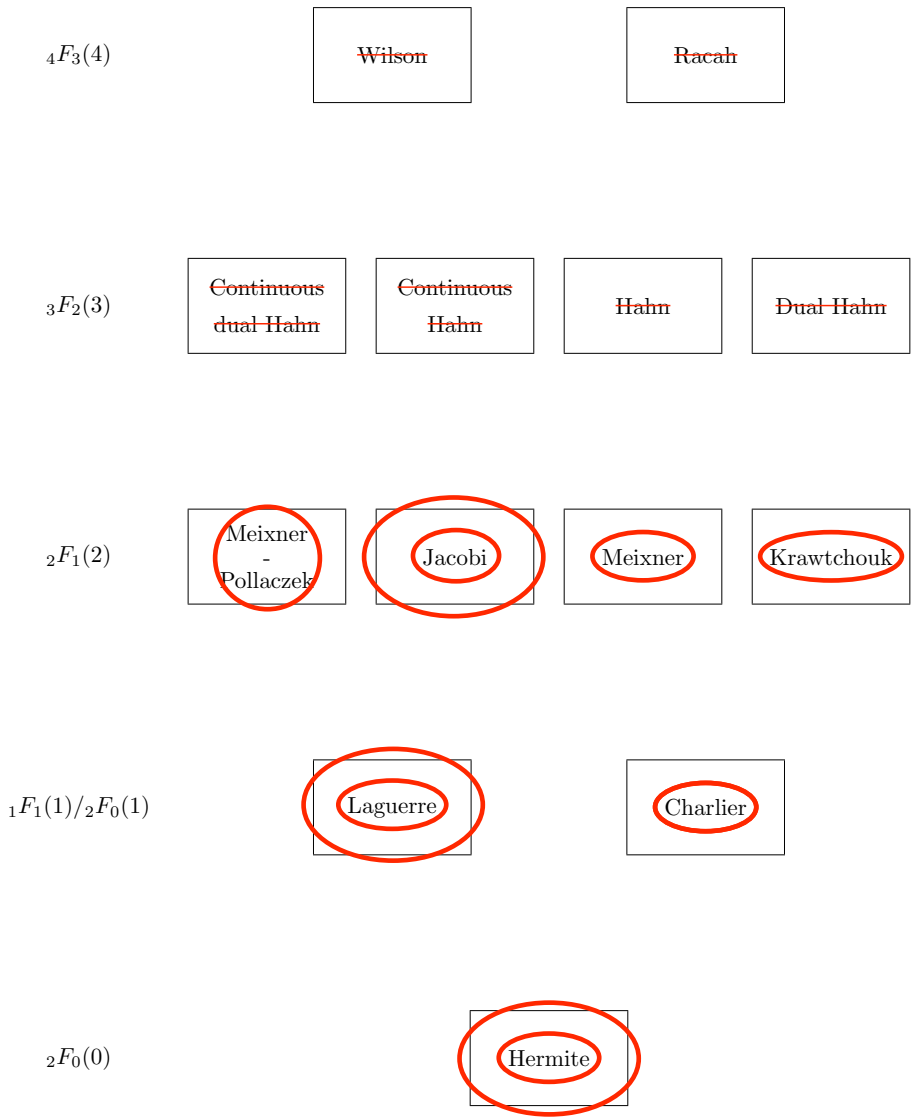

Other algorithm giving all orthogonal pols in $\mathcal{O}(\mathsf{M}(N)\log N)$

II Very Fast Composition ($\mathcal{O}(M(N))$ operations)

Shift is Cheap (Aho, Steiglitz, Ullman 1975)

Problem (Polynomial Shift)

Input: $P(x)$ polynomial of degree N, a a point.
Output: $P(x+a)$.

Horner: $\mathcal{O}(N^2)$ operations
Taylor:

$$P^{(N-i)}(a) = \sum_{j=0}^{i} P^{(N-i+j)}(0)\frac{a^j}{j!}.$$

Consequence: generating series

$$\sum_{i=0}^{N} \underbrace{P^{(N-i)}(a)}_{\text{coeffs of } P(x+a)} x^i + \cdots = \sum_{k=0}^{n} \underbrace{P^{(k)}(0)}_{\text{coeffs of } P(x)} x^{N-k} \times \sum_{j\geq 0} \frac{a^j x^j}{j!}.$$

$\mathrm{M}(N) + \mathcal{O}(N)$ operations.

Warm-up: Euler Transform for 3 Multiplications

Problem (Euler Transform)

Input: $P(x)$ polynomial of degree N (or truncated series);

Output: First N coefficients of $\frac{1}{1-x} P\left(\frac{x}{1-x}\right)$.

Algorithm:

$S_1(x) := P(x-1)$;

$S_2(x) := x^N S_1(1/x)$;

$S_3(x) := S_2(x+1)$;

$S_4(x) := S_3(-x)$;

return $(1-x)^{-N-1} S_4(x)$.

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$$S_3(x) := S_2(x+1); \quad = (1+x)^N P\left(\frac{-x}{1+x}\right)$$

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return $(1-x)^{-N-1} S_4(x)$. $\mathsf{M}(N) + \mathcal{O}(N)$ ops.

More Systematically

Collection of operators on series...

$$A_a : g \mapsto a + g$$
$$M_\lambda : g \mapsto \lambda g$$
$$\mathrm{Inv} : g \mapsto 1/g$$
$$P_k : g \mapsto g^k$$
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...and associativity rules

$$f(A_a(g)) = f|_{x \mapsto x+a} \circ g$$
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Theorem

For any well-defined g composition of $\{A_a, M_\lambda, \mathrm{Inv}, P_k, R_k\}$ and any polynomial f, $f \circ g \bmod x^N$ computed in $\mathcal{O}(\mathrm{M}(N))$ ops.

Proof. Fast right and left compositions.

What can we construct with $\{A_a, M_\lambda, \mathrm{Inv}, P_k, R_k\}$?

Polynomials & Rational Functions

Proposition (Degree 2)

All polynomials and rational functions of degree 2, and their (compositional) inverses.

1. $ax + b$: easy;
2. $ax^2 + bx + c = (b - a^2/4) + (x + a/2)^2$;
3. $(ax + b)/(cx + d) = a/c + (b - ad/c)/(cx + d)$ (Pan, 98);
4. $x/(x^2 - d) = \frac{1}{2\sqrt{d}} \sqrt{\left(\frac{2d}{d - x^2} - 1\right)^2 - 1}$;
5. $(x^2 - d)/x = 4^{-1}$;
6. $(x^2 + ax - d)/x = a + 5$;
7. $(x^2 + ax + b)/(x + c) = 6|_{x \mapsto x+c, d \mapsto ac-b}$;
8. $(x + c)/(x^2 + ax + b) = 7^{-1}$;
9. $(x^2 + dx + e)/(x^2 + ax + b) = 1 + (d - a) \cdot 8|_{c \mapsto (e-b)/(d-a)}$;
10. $(x^2 + ax + e)/(x^2 + ax + b) = 1 + (e - b) \cdot 2|_{a \mapsto 1, b \mapsto a, c \mapsto b}$.

Sequence Transformations

Problem (Binomial Convolution)

Input: sequence (a_n), $n = 0, \ldots, N$;
Output: sequence (b_n), $n = 0, \ldots, N$, where

$$b_n = \sum_{k \geq 0} \binom{n + pk}{m + qk} a_k.$$

Generating series

$$\sum_n b_n z^n = \frac{z^m}{(1-z)^{m+1}} A\left(\frac{z^{p-q}}{(1-z)^p}\right).$$

Algorithm in $\mathcal{O}(\mathrm{M}(N))$ for

- $p = 1, q = 0$ (Euler);
- $p = 2$ and $q = 0$ or $q = 1$.

III Fast Composition ($\mathcal{O}(M(n)\log n)$ operations)

Result

Theorem

For any well-defined g composition of $\{A_a, M_\lambda, \mathrm{Inv}, P_k, R_k,$ polynomial, algebraic function, $\exp, \log\}$ and any polynomial f, $f \circ g \bmod x^N$ computed in $\mathcal{O}(\mathrm{M}(N)\log N)$ ops.

Proof.

- Fast left compositions (Newton, Brent & Kung);
- Fast right compositions:
  - polynomials: Brent & Kung (78);
  - algebraic functions: van der Hoeven (02);
  - exp, log: next screens.

Exponential

Problem (Right Composition with Exp)

Input: $(a_0, \ldots, a_N)$
Output: coefficients of $\sum_j a_j e^{jx} \bmod x^{N+1}$.

Laplace: $\mathcal{L}(x^i) := i! x^i \Rightarrow \sum_{j=0}^{N} a_j e^{jx} = \mathcal{L}^{-1} \underbrace{\sum_{j=0}^{N} \frac{a_j}{1-jx}}_{:=S_0^N}$.

Transposed Multipoint Evaluation (Bostan, Lecerf, Schost 03)

1. Divide and conquer: numerator and denominator of S_0^N from those of $S_0^{N/2}$ and $S_{N/2+1}^N$;
2. Compute the series expansion of the quotient

3. Multiply the ith coeff by $1/i!$, $i = 0, \ldots, N$

Exponential

Problem (Right Composition with Exp)

Input: $(a_0, \ldots, a_N)$
Output: coefficients of $\sum_j a_j e^{jx} \bmod x^{N+1}$.

Laplace: $\mathcal{L}(x^i) := i! x^i \Rightarrow \sum_{j=0}^{N} a_j e^{jx} = \mathcal{L}^{-1} \underbrace{\sum_{j=0}^{N} \frac{a_j}{1-jx}}_{:=S_0^N}$.

Transposed Multipoint Evaluation (Bostan, Lecerf, Schost 03)

1. Divide and conquer: numerator and denominator of S_0^N from those of $S_0^{N/2}$ and $S_{N/2+1}^N$; $\frac{3}{2}\mathrm{M}(N)\log N + \mathcal{O}(\mathrm{M}(N))$ ops
2. Compute the series expansion of the quotient $\mathcal{O}(\mathrm{M}(N))$ ops
3. Multiply the ith coeff by $1/i!$, $i = 0, \ldots, N$ $\mathcal{O}(N)$ ops

Matrices Acting on the Vector of Coefficients

Free (0 op)

- Rec: permutation $i \mapsto N - i$;

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Very fast ($\mathcal{O}(\mathrm{M}(N))$):

- Multiplication by a fixed P: Mul(P);
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Check

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- Multipoint evaluation: $V(a_0, \ldots, a_N)$ (Vandermonde);
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- Exp: $\mathcal{L}^{-1} \cdot V(0, \ldots, N)^t$; Check

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Slow ($\mathcal{O}(\sqrt{N \log N}\, \mathrm{M}(N))$)

- Composition with g: $(g_{i,k})$, with $g^k = \sum g_{i,k} x^i \bmod x^N$.

$\log(1+x)$ and Transposed Interpolation

$$P(x) \xrightarrow{A_{-1}} \sum_{j=0}^{N} a_j x^j \xrightarrow{\mathcal{L}} \sum_{j=0}^{N} \frac{b_j}{1-jx} \xrightarrow{\mathrm{MEval}^t} \sum_{j=0}^{N} c_j x^j \xrightarrow{\mathcal{L}^{-1}} P(e^x-1)+O(x^{N+1}).$$

Problem: Transposed Interpolation

Input: $C(x)$ polynomial of degree N

Output: $(b_0, \ldots, b_N)$ such that $C(x) = \sum_{j=0}^{N} \frac{b_j}{1-jx} + O(x^{N+1})$.

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Let $P(x) := \prod_{j=0}^{N}(X-j)$, then

$$\frac{1}{x}C(\frac{1}{x}) = \sum_{j=0}^{N} \frac{b_j}{x-j} = \frac{Q(x)}{P(x)} + O(x^{-N-2}) \Rightarrow b_j = \frac{Q(j)}{P'(j)}.$$

Algorithm (Bostan, Lecerf, Schost 03)

1. Expand P by divide-and-conquer;
2. Compute $C \times \mathrm{Rec}(P) = \mathrm{Rec}(Q) + O(x^{N+1})$;
3. Evaluate Q and P' at $0, \ldots, N$;
4. Return $(Q(0)/P'(0), \ldots, Q(N)/P'(N))$.

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Algorithm (Bostan, Lecerf, Schost 03) with linear rec for P'

1. Expand P by divide-and-conquer; $\frac{1}{2}\mathrm{M}(N)\log N + \mathcal{O}(N)$
2. Compute $C \times \mathrm{Rec}(P) = \mathrm{Rec}(Q) + O(x^{N+1})$; $\mathrm{M}(N) + \mathcal{O}(N)$
3. Evaluate Q and P' at $0, \ldots, N$; $\mathrm{M}(N)\log N + \mathcal{O}(N)$
4. Return $(Q(0)/P'(0), \ldots, Q(N)/P'(N))$. $\mathcal{O}(N)$.

IV Change of Basis

Composition Sequences

Definition (Composition Sequence)

A map $F = o_1 \circ \cdots \circ o_m$ with o_i's in $\{A_a, M_\lambda, P_k, R_k, \mathrm{Inv}, \exp, \log\}$.
It *defines* $F(x)$ if all intermediates $o_i \circ \cdots \circ o_m(x)$ are power series.

Summary so far

P a polynomial, F a composition sequence defining $F(x)$, then

- $F(x) \bmod x^N$;
- $P(F(x)) \bmod x^N$

can be computed in a number of operations

$$T_F(N) = \begin{cases} \mathcal{O}(\mathrm{M}(N)) & \text{if no } o_i \text{ is exp or log,} \\ \mathcal{O}(\mathrm{M}(N)\log N) & \text{otherwise.} \end{cases}$$

Main Theorem

Bivariate Evaluation with $F(x,t) = \sum F_{ij} x^i t^j = \sum G_j(x) t^j$

Input: $(a_0, \ldots, a_N)$;
Output: $\sum_{i=0}^{N} b_i x^i = \sum a_j G_j(x) \bmod x^{N+1}$.

Main Theorem of (Bostan, S., Schost, 08)

Generating function $F(x,t) = u(x)\, f\big(g(x) \times h(t)\big)\, v(t)$ with

- g and h given by composition sequences;
- u, v, f given by their coefficients $\bmod x^{N+1}$;
- well-defined: $g(0)h(0) = 0$;

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- g and h given by composition sequences;
- u, v, f given by their coefficients $\bmod x^{N+1}$;
- well-defined: $g(0)h(0) = 0$;
- $g'(0)h'(0)u(0)v(0) \neq 0$, all coefficients of $f \neq 0$,

then $(a_i) \mapsto (b_i)$ in $\mathcal{O}(T_g(N) + T_h(N) + \mathrm{M}(N))$ ops
then $(b_i) \mapsto (a_i)$ in $\mathcal{O}(T_g(N) + T_h(N) + \mathrm{M}(N))$ ops.

Examples in $\mathcal{O}(\mathrm{M}(N))$

Sheffer Sequences: $F = \exp(xh(t))v(t)$

Name	Notation	h	v
Laguerre	L_n^α	$t(1-t)^{-1}$	$(1-t)^{-1-\alpha}$
Hermite	H_n	$2t$	$\exp(-t^2)$
Euler	E_n^α	t	$2^\alpha(1+e^t)^{-\alpha}$
Bernoulli	B_n^α	t	$t^\alpha(1+e^t)^{-\alpha}$
Mott	M_n	$(\sqrt{1-t^2}-1)/t$	1
Bessel	p_n	$1-\sqrt{1-2t}$	1

Examples in $\mathcal{O}(\mathrm{M}(N))$

Non-Sheffer:

- Fibonacci

$$\sum_{n\geq 0} F_n(x)t^n = \frac{1}{1-t^2}\frac{1}{1-\frac{xt}{1-t^2}};$$

- Chebyshev

$$-\frac{1}{2} + \sum_{n\geq 0} T_n(x)t^n = \frac{1-t^2}{2(1+t^2)}\frac{1}{1-\frac{2xt}{1+t^2}};$$

- Jacobi with $(\alpha+\beta+1)\neq 0$

$$\sum_{n\geq 0} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} P_n^{(\alpha,\beta)}(x)t^n =$$

$$(1+t)^{-\alpha-\beta-1}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+1}{2}\\ \beta+1\end{matrix}\,\middle|\,\frac{2t(1+x)}{(1+t)^2}\right).$$

Other Sheffer Sequences, but in $\mathcal{O}(\mathrm{M}(N)\log N)$

Name	Notation	h	v
Falling factorial	$(x)_n$	$\log(1+t)$	1
Bell	ϕ_n	$\exp(t)-1$	1
Bernoulli, 2nd kind	b_n	$\log(1+t)$	$t/\log(1+t)$
Poisson-Charlier	$c_n(x;a)$	$\log(1+t/a)$	$\exp(-t)$
Actuarial	$a_n^{(\beta)}$	$1-\exp(t)$	$\exp(\beta t)$
Narumi	$N_n^{(a)}$	$\log(1+t)$	$t^a\log(1+t)^{-a}$
Peters	$P_n^{(\lambda,\mu)}$	$\log(1+t)$	$(1+(1+t)^\lambda)^{-\mu}$
Meixner-Pollaczek	$P_n^{(\lambda)}(x;\phi)$	$i\log(\frac{1-te^{i\phi}}{1-te^{-i\phi}})$	$(1+t^2-2t\cos\phi)^{-\lambda}$
Meixner	$m_n(x;\beta,c)$	$\log(\frac{1-t/c}{1-t})$	$(1-t)^{-\beta}$
Krawtchouk	$K_n(x;p,N)$	$\log(\frac{p-(1-p)t}{p(1+t)})$	$(1+t)^N$

Falling factorial known (Gerhard 00);
Last 3 known in direct sense $(P_i)\mapsto(x^i)$
(Driscoll, Healy, Rockmore 97; Pott, Steidl, Tasche 98).

Algorithm

Bivariate Eval, $f(g(x)h(t)) = \sum F_{ij}x^i t^j = \sum G_j(x)t^j$

Input: $(a_0, \ldots, a_N)$; composition seq. for g and h; series for f
Output: $\sum_{i=0}^{N} b_i x^i = \sum a_j G_j(x) \bmod x^{N+1}$.

$$[x^i t^j] f(g(x)h(t)) = F_{ij} = \sum_k f_k g_{ik} h_{jk}, \quad \text{where} \begin{cases} g^k &= \sum_i g_{ik} x^i, \\ h^k &= \sum_j h_{jk} x^j. \end{cases}$$

In matrix form,

$$(b) = (F_{ij})(a) = (g_{ik})\,\mathrm{Diag}(f)(h_{jk})^t(a).$$

We know how to multiply by (g_{ik}) (fast right composition) if we have fast multiplication by $(h_{jk})^t$, we're done.

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Basic Transpositions

Free (0 op) Transpose

- Rec: permutation $i \mapsto N - i$;

Super fast ($\mathcal{O}(N)$):

- Diag(u), e.g., $M_\lambda, \mathcal{L}, \mathcal{L}^{-1}$;

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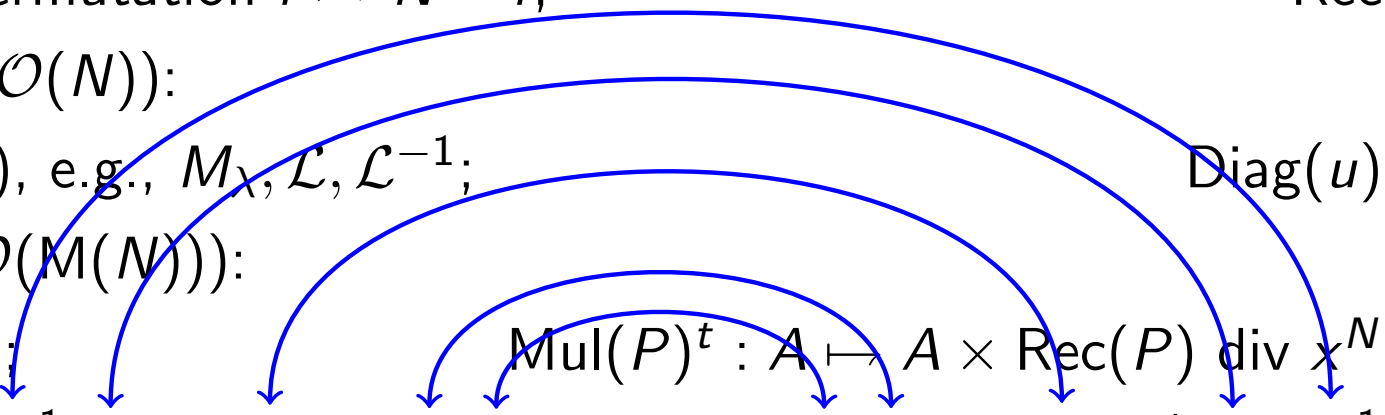

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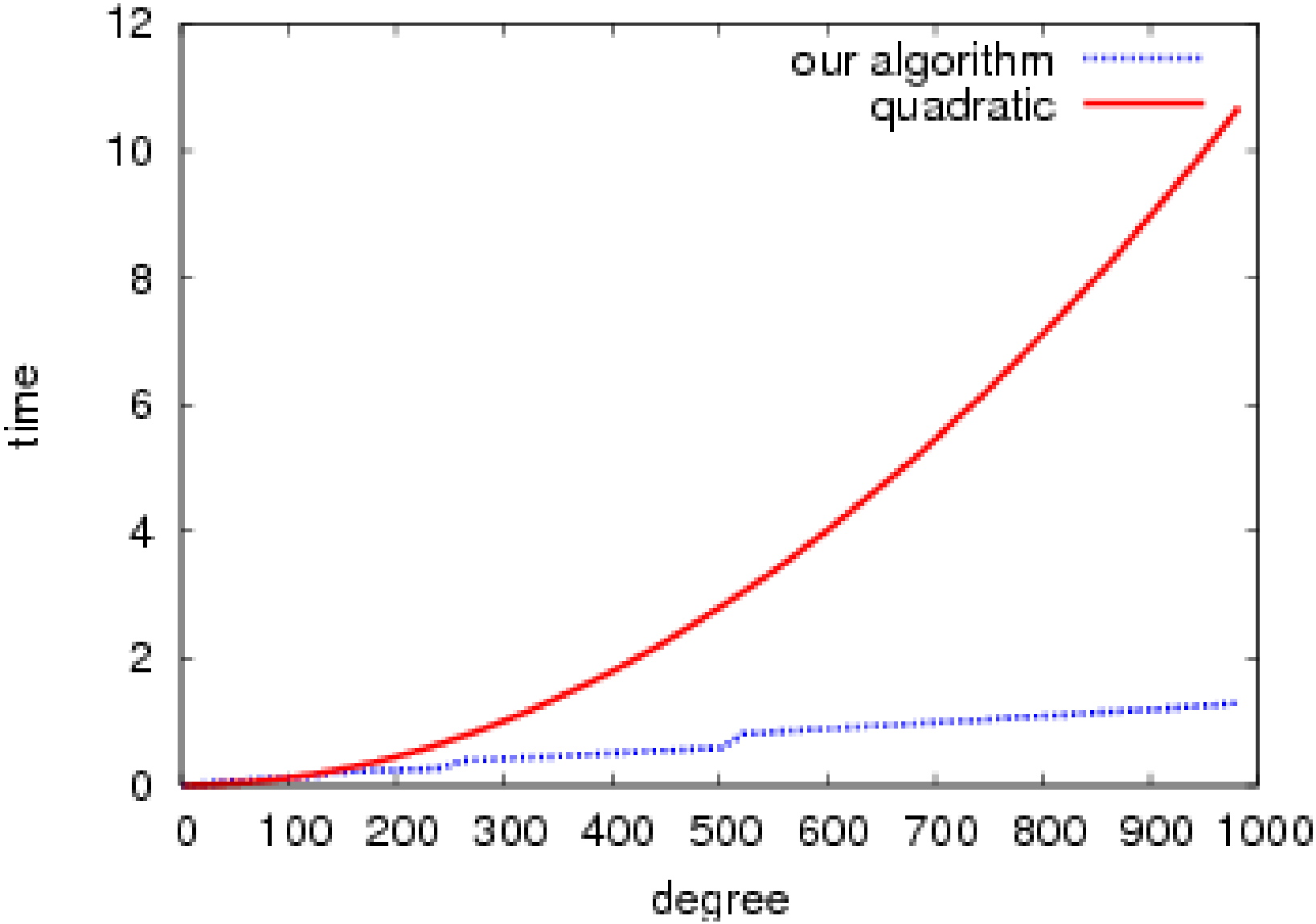
End of Proof

Example: Conversion to Jacobi



V General Orthogonal Polynomials

Basic Facts on Orthogonal Polynomials

1. A non-decreasing function μ on (a, b).
2. A scalar product $\langle f, g \rangle = \ell(fg) = \int_a^b fg \, d\mu$ s.t. $\forall n, \ell(x^n) < \infty$.
3. Gram-Schmidt for (x^n) produces (P_n) s.t. $\deg P_n = n$ and if $\deg q < n$, then $\langle P_n, q \rangle = 0$.
4. Therefore $\langle xP_n | P_i \rangle = \langle P_n | xP_i \rangle = 0$ for $i < n - 1$, and

$$\boxed{P_{n+1}(x) = (a_n x + b_n) P_n(x) + c_n P_{n-1}(x), \quad a_n c_n \neq 0.} \qquad (R)$$

Starting from $P_{-1} = 0$, $P_0 = 1$, this defines the (P_n).

Direct Conversion in $\mathcal{O}(\mathsf{M}(N)\log N)$ ops

$$\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_n x + b & c_n \\ 1 & 0 \end{pmatrix}}_{:=M^{(n-1,n)}} \begin{pmatrix} P_n \\ P_{n-1} \end{pmatrix} = M^{(n-1,n)} \cdots M^{(0,1)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix}.$$

Aim: $$\sum_{i=0}^{2n+1} \alpha_i P_i = \sum_{i=0}^{n} \begin{pmatrix} \alpha_{2i} & \alpha_{2i+1} \end{pmatrix} M^{(0,2i)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} =: S^{(0,2n)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix}.$$

Algorithm Direct Conversion

Input: $(\alpha_{2m}, \ldots, \alpha_{2p+1})$
Output: $M^{(2m,2p)}$ and $S^{(2m,2p)}$

1. $q := \lceil \frac{m+p}{2} \rceil$;
2. Compute recursively $M^{(2m,2q)}, S^{(2m,2q)}, M^{(2q+2,2p)}, S^{(2q+2,2p)}$;
3. Return $M^{(2q+2,2p)} M^{(2m,2q)}$ and $S^{(2m,2q)} + S^{(2q+2,2p)} M^{(2m,2q)}$.

Note: This gives fast multiplication by $\mathbf{P} := ([x^i]P_j)$.

Inverse Conversion in $\mathcal{O}(\mathrm{M}(N)\log N)$ operations

$$\boxed{P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x), \quad a_n c_n \neq 0.} \qquad (R)$$

Starting from $P_{-1} = 0$, $P_0 = 1$, this defines the (P_n).

Theorem (Favard (35); Shohat (36))

Conversely, given $(a_n), (b_n), (c_n)$ and (R), there exists ℓ such that

- $\ell(P_j P_i) = 0$, $j < i$; $d_i := \ell(P_i^2) = (-1)^i c_2 \cdots c_{i+1}/a_{i+1} \neq 0$;
- $L(x) := \sum_{i \geq 0} \ell(x^n)x^n = \dfrac{Q_{n-1}}{\mathrm{Rec}\,(P_n)} + \mathcal{O}(x^{2n})$, *where*

  $Q_n = (a_{n+1} + b_{n+1}x)Q_{n-1} + c_{n+1}x^2 Q_{n-2}$, $Q_{-1} = 0$, $Q_0 = 1$.

Inverse Conversion in $\mathcal{O}(\mathsf{M}(N)\log N)$ operations

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$$\mathbf{P}^t \cdot (\ell(x^{i+j})) \cdot \mathbf{P} = \mathrm{Diag}(d_i) \Rightarrow \mathbf{P}^{-1} = \mathrm{Diag}(d_i)^{-1} \cdot \mathbf{P}^t \cdot (\ell(x^{i+j})).$$

Algorithm $(a) \mapsto \mathbf{P}^{-1}(a)$ (Bostan, S., Schost 08b)

1. Compute (d_i);
2. Compute $L(x)$ mod x^{2n} using Direct Conversion;
3. $(a) \mapsto (\ell(x^{i+j})) \cdot (a)$ is a transposed multiplication by $L(x)$;
4. Algorithm $(a) \mapsto \mathbf{P}^t(a)$ obtained by transposing the code for Direct Conversion.

VI Conclusion

Conclusion, Further Work, Work in Progress

- Summary: transposition helps conversions to and from polynomials and power series with generating function $u(x)v(t)f(g(x)h(t))$.
- General orthogonal polynomials in $\mathcal{O}(\mathrm{M}(N)\log N)$ by a fast effective Shohat-Favard theorem and again transposition;
- k-term recurrences with $k > 3$?
- D-finite series decompose on orthogonal polynomials in $O(N)$ ops: A. Benoit, extending (Pazkowski, Lewanowicz, Rebillard,...);
- Numerically stable algorithms?
- $\mathcal{O}(\mathrm{M}(N))$: How far can one go?