Examples

I. Enumeration

```
> with(NewtonGF):
> NewtonGF:-version();
```

$$2.2 \tag{1.1}$$

Check with the web page whether you have the latest version:

http://perso.ens-lyon.fr/bruno.salvy/software/the-newtongf-package/

First example: binary trees

```
> bintrees:={B=Union(Epsilon,Prod(Z,B,B))}:
> GFSeries(bintrees,labelled,z,30);
```

$$[Z = z + O(z^{30}), B = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9 + 16796z^{10} + 58786z^{11} + 208012z^{12} + 742900z^{13} + 2674440z^{14} + 9694845z^{15} + 35357670z^{16} + 129644790z^{17} + 477638700z^{18} + 1767263190z^{19} + 6564120420z^{20} + 24466267020z^{21} + 91482563640z^{22} + 343059613650z^{23} + 1289904147324z^{24} + 4861946401452z^{25} + 18367353072152z^{26} + 69533550916004z^{27} + 263747951750360z^{28} + 1002242216651368z^{29} + O(z^{30})] \tag{1.1.1}$$

Try it with higher precision:

```
> GFSeries(bintrees,labelled,z,100);
```

$$[Z = z + O(z^{100}), B = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9 + 16796z^{10} + 58786z^{11} + 208012z^{12} + 742900z^{13} + 2674440z^{14} + 9694845z^{15} + 35357670z^{16} + 129644790z^{17} + 477638700z^{18} + 1767263190z^{19} + 6564120420z^{20} + 24466267020z^{21} + 91482563640z^{22} + 343059613650z^{23} + 1289904147324z^{24} + 4861946401452z^{25} + 18367353072152z^{26} + 69533550916004z^{27} + 263747951750360z^{28} + 1002242216651368z^{29} + 3814986502092304z^{30} + 14544636039226909z^{31} + 55534064877048198z^{32} + 212336130412243110z^{33} + 812944042149730764z^{34} + 3116285494907301262z^{35} + 11959798385860453492z^{36} + 45950804324621742364z^{37} + 176733862787006701400z^{38} + 680425371729975800390z^{39} + 2622127042276492108820z^{40} + 10113918591637898134020z^{41} + 39044429911904443959240z^{42} + 150853479205085351660700z^{43} + 583300119592996693088040z^{44} + 2257117854077248073253720z^{45} + 8740328711533173390046320z^{46} + 33868773757191046886429490z^{47} + 131327898242169365477991900z^{48} \tag{1.1.2}$$

$+ 509552245179617138054608572 z^{49}$
$+ 1978261657756160653623774456 z^{50}$
$+ 7684785670514316385230816156 z^{51}$
$+ 29869166945772625950142417512 z^{52}$
$+ 116157871455782434250553845880 z^{53}$
$+ 451959718027953471447609509424 z^{54}$
$+ 1759414616608818870992479875972 z^{55}$
$+ 6852456927844873497549658464312 z^{56}$
$+ 26700952856774851904245220912664 z^{57}$
$+ 104088460289122304033498318812080 z^{58}$
$+ 405944995127576985730643443367112 z^{59}$
$+ 1583850964596120042686772779038896 z^{60}$
$+ 6182127958584855650487080847216336 z^{61}$
$+ 24139737743045626825711458546273312 z^{62}$
$+ 94295850558771979787935384946380125 z^{63}$
$+ 368479169875816659479009042713546950 z^{64}$
$+ 1440418573150919668872489894243865350 z^{65}$
$+ 5632681584560312734993915705849145100 z^{66}$
$+ 22033725021956517463358552614056949950 z^{67}$
$+ 86218923998960285726185640663701108500 z^{68}$
$+ 337485502510215975556783793455058624700 z^{69}$
$+ 1321422108420282270489942177190229544600 z^{70}$
$+ 5175569924646105559418940193995065716350 z^{71}$
$+ 20276890389709399862928998568254641025700 z^{72}$
$+ 79463489365077377841208237632349268884500 z^{73}$
$+ 311496878311103321137536291518809134027240 z^{74}$
$+ 1221395654430378811828760722007962130791020 z^{75}$
$+ 4790408930363303911328386208394864461024520 z^{76}$
$+ 18793142726809884575211361279087545193250040 z^{77}$
$+ 73745243611532458459690151854647329239335600 z^{78}$
$+ 289450081175264899454283846029490767264392230 z^{79}$
$+ 1136359577947336271931632877004667456667613940 z^{80}$
$+ 4462290049988320482463241297506133183499654740 z^{81}$
$+ 17526585015616776834735140517915655636396234280 z^{82}$
$+ 68854441132780194707888052034668647142985206100 z^{83}$

$+ 2705574510393951180286424632891685664206712804 40\, z^{84}$

$+ 1063353702922273835973036658043476458723103404520\, z^{85}$

$+ 4180080073556524734514695828170907458428751314320\, z^{86}$

$+ 16435314834665426797069144960762886143367590394940\, z^{87}$

$+ 64633260585762914370496637486146181462681535261000\, z^{88}$

$+ 254224158304000796523953440778841647086547372026600\, z^{89}$

$+ 1000134600800354781929399250536541864362461089950800\, z^{90}$

$+ 3935312233584004685417853572763349509774031680023800\, z^{91}$

$+ 15487357822491889407128326963778343232013931127835600\, z^{92}$

$+ 60960876535340415751462563580829648891969728907438000\, z^{93}$

$+ 239993345518077005168915776623476723006280827488229600\, z^{94}$

$+ 944973797977428207852605870454939596837230758234904050\, z^{95}$

$+ 3721443204405954385563870541379246659709506697378694300\, z^{96}$

$+ 14657929356129575437016877846657032761712954950899755100\, z^{97}$

$+ 57743358069601357782187700608042856334020731624756611000\, z^{98}$

$+ 227508830794229349661819540395688853956041682601541047340\, z^{99} + O(z^{100})]$

Labelled and unlabelled rooted trees

```
> trees:={T=Prod(Z,Set(T))}:
> GFSeries(trees,labelled,z,20);
```

$$\left[Z = z + O(z^{20}),\ T = z + z^2 + \frac{3}{2} z^3 + \frac{8}{3} z^4 + \frac{125}{24} z^5 + \frac{54}{5} z^6 + \frac{16807}{720} z^7 + \frac{16384}{315} z^8 + \frac{531441}{4480} z^9 + \frac{156250}{567} z^{10} + \frac{2357947691}{3628800} z^{11} + \frac{2985984}{1925} z^{12} + \frac{1792160394037}{479001600} z^{13} + \frac{7909306972}{868725} z^{14} + \frac{320361328125}{14350336} z^{15} + \frac{35184372088832}{638512875} z^{16} + \frac{2862423051509815793}{20922789888000} z^{17} + \frac{5083731656658}{14889875} z^{18} + \frac{5480386857784802185939}{6402373705728000} z^{19} + O(z^{20})\right] \quad \mathbf{(1.2.1)}$$

```
> coeff(subs(%,T),z,19)*19!;
```

$$1041273502979112415328 41 \quad \mathbf{(1.2.2)}$$

```
> GFSeries(trees,unlabelled,z,20);
```

$$[Z = z + O(z^{20}),\ T = z + z^2 + 2 z^3 + 4 z^4 + 9 z^5 + 20 z^6 + 48 z^7 + 115 z^8 + 286 z^9 + 719 z^{10} + 1842 z^{11} + 4766 z^{12} + 12486 z^{13} + 32973 z^{14} + 87811 z^{15} + 235381 z^{16} + 634847 z^{17} + 1721159 z^{18} + 4688676 z^{19} + O(z^{20})] \quad \mathbf{(1.2.3)}$$

This one is also fast:

```
> GFSeries(trees,unlabelled,z,100);
```

$$\left[Z=z+\mathrm{O}\left(z^{100}\right), T=z+z^{2}+2 z^{3}+4 z^{4}+9 z^{5}+20 z^{6}+48 z^{7}+115 z^{8} \quad (1.2.4)$$

$$+286 z^{9}+719 z^{10}+1842 z^{11}+4766 z^{12}+12486 z^{13}+32973 z^{14}$$

$$+87811 z^{15}+235381 z^{16}+634847 z^{17}+1721159 z^{18}+4688676 z^{19}$$

$$+12826228 z^{20}+35221832 z^{21}+97055181 z^{22}+268282855 z^{23}$$

$$+743724984 z^{24}+2067174645 z^{25}+5759636510 z^{26}+16083734329 z^{27}$$

$$+45007066269 z^{28}+126186554308 z^{29}+354426847597 z^{30}$$

$$+997171512998 z^{31}+2809934352700 z^{32}+7929819784355 z^{33}$$

$$+22409533673568 z^{34}+63411730258053 z^{35}+179655930440464 z^{36}$$

$$+509588049810620 z^{37}+1447023384581029 z^{38}+4113254119923150 z^{39}$$

$$+11703780079612453 z^{40}+33333125878283632 z^{41}$$

$$+95020085893954917 z^{42}+271097737169671824 z^{43}$$

$$+774088023431472074 z^{44}+2212039245722726118 z^{45}$$

$$+6325843306177425928 z^{46}+18103111141539779470 z^{47}$$

$$+51842285219378800562 z^{48}+148558992149369434381 z^{49}$$

$$+425976989835141038353 z^{50}+1222179262369751914558 z^{51}$$

$$+3508609802706585591648 z^{52}+10078062032127180323468 z^{53}$$

$$+28963544938490115587690 z^{54}+83281891024323882188934 z^{55}$$

$$+239588251950971630070883 z^{56}+689586695750027771528858 z^{57}$$

$$+1985698827814122851389544 z^{58}+5720475695410470698034352 z^{59}$$

$$+16486885726043465205200778 z^{60}+47536435298225838513777689 z^{61}$$

$$+137116646299836640013582158 z^{62}+395661426200172120893172166 z^{63}$$

$$+1142146565612503377367247619 z^{64}$$

$$+3298218689025396468807928287 z^{65}$$

$$+9527778769277367435762139714 z^{66}$$

$$+27533018688066675122704256503 z^{67}$$

$$+79590547737981375530085744985 z^{68}$$

$$+230149693903609741713900061706 z^{69}$$

$$+665727478405857651614359412994 z^{70}$$

$$+1926264145029683789201902595481 z^{71}$$

$$+5575255070241458769777337977216 z^{72}$$

$$+16141340753693289446826761342965 z^{73}$$

$$+46745197819192522341803684458539 z^{74}$$

$$+135410903503191503503384705970501 z^{75}$$

$$+392361462360303212219218458046293 z^{76}$$

$$
\begin{aligned}
&+ 1137187531620128717310046733793377\, z^{77} \\
&+ 3296764685809404767411359346950251\, z^{78} \\
&+ 9559850833672724865816863589301148\, z^{79} \\
&+ 27728021718947447790064072086211720\, z^{80} \\
&+ 80443073708065346515907947187548704\, z^{81} \\
&+ 233430673259657258534788317265918255\, z^{82} \\
&+ 677523352768124889000758550090536896\, z^{83} \\
&+ 1966913982276181876912481643993713829\, z^{84} \\
&+ 5711352604881678383437989565512293584\, z^{85} \\
&+ 16587575765193708421040000378631758598\, z^{86} \\
&+ 48185360814676586544379947795827489489\, z^{87} \\
&+ 140001774817536903942967946008051857304\, z^{88} \\
&+ 406851804555194134855599602163662891898\, z^{89} \\
&+ 1182554992433068847214862126968409557746\, z^{90} \\
&+ 3437850734230350035768412388914235959988\, z^{91} \\
&+ 9996120786153335036227014784756370016600\, z^{92} \\
&+ 29070538491469576103323086044968128987841\, z^{93} \\
&+ 84557105785656201807648937916789795631644\, z^{94} \\
&+ 245992006962808524149265893171336435342958\, z^{95} \\
&+ 715754664594286649942853831332902539636094\, z^{96} \\
&+ 2082946825910124637547671590789256847274970\, z^{97} \\
&+ 6062636513439329551550570219718718950653609\, z^{98} \\
&+ 17648704032743316047867540848170516567878799\, z^{99} + \mathrm{O}\left(z^{100}\right)]
\end{aligned}
$$

Series-parallel graphs

```
> spgraphs:={Graph=Union(Z,Series,Parallel),Series=Sequence
  (Union(Z,Parallel),card>=2),Parallel=Set(Union(Z,Series),
  card>=2)}:
> subs(GFSeries(spgraphs,labelled,z,20),Graph);
```

$$
\begin{aligned}
&z + \frac{3}{2} z^2 + \frac{19}{6} z^3 + \frac{65}{8} z^4 + \frac{2791}{120} z^5 + \frac{17101}{240} z^6 + \frac{1152019}{5040} z^7 \qquad \textbf{(1.3.1)} \\
&+ \frac{2037605}{2688} z^8 + \frac{935494831}{362880} z^9 + \frac{10815911381}{1209600} z^{10} \\
&+ \frac{1257770533339}{39916800} z^{11} + \frac{513183875243}{4561920} z^{12} + \frac{2528224238464471}{6227020800} z^{13} \\
&+ \frac{42978132697166941}{29059430400} z^{14} + \frac{7101273378743303779}{1307674368000} z^{15}
\end{aligned}
$$

$$+ \frac{2154248190413524297}{107296358400} z^{16} + \frac{2414845703016939977501}{32335220736000} z^{17}$$

$$+ \frac{85145513111617353034483}{304874938368000} z^{18} + \frac{127652707703771090396080939}{121645100408832000} z^{19}$$

$$+ \mathrm{O}\left(z^{20}\right)$$

Again, the complexity wrt size is good:

```
> subs(GFSeries(spgraphs,labelled,z,40),Graph);
```

$$z + \frac{3}{2} z^{2} + \frac{19}{6} z^{3} + \frac{65}{8} z^{4} + \frac{2791}{120} z^{5} + \frac{17101}{240} z^{6} + \frac{1152019}{5040} z^{7} \qquad \textbf{(1.3.2)}$$

$$+ \frac{2037605}{2688} z^{8} + \frac{935494831}{362880} z^{9} + \frac{10815911381}{1209600} z^{10}$$

$$+ \frac{1257770533339}{39916800} z^{11} + \frac{513183875243}{4561920} z^{12} + \frac{2528224238464471}{6227020800} z^{13}$$

$$+ \frac{42978132697166941}{29059430400} z^{14} + \frac{7101273378743303779}{1307674368000} z^{15}$$

$$+ \frac{2154248190413524297}{107296358400} z^{16} + \frac{2414845703016939977501}{32335220736000} z^{17}$$

$$+ \frac{85145513111617353034483}{304874938368000} z^{18} + \frac{127652707703771090396080939}{121645100408832000} z^{19}$$

$$+ \frac{11677645222677726521937131}{2948972131123200} z^{20}$$

$$+ \frac{403361811421003310215787216 29}{26889969564057 60000} z^{21}$$

$$+ \frac{21364022867981502697751948848301}{374666909259202560000} z^{22}$$

$$+ \frac{562137620301761921158196997 6775219}{25852016738884976640000} z^{23}$$

$$+ \frac{37799892078938885368971313424 0543}{4545409536507248640 00} z^{24}$$

$$+ \frac{49463801860289824862432783268081462991}{1551121004333098598400 0000} z^{25}$$

$$+ \frac{1647886593356028445966797297822214557301}{134430487042201878528000000} z^{26}$$

$$+ \frac{467508823423318881574212394692518961045 29}{98989722276530474188800 0000} z^{27}$$

$$+ \frac{6708068016944160900607873318619934548 09}{3678894052630031499264000} z^{28}$$

$$+ \frac{6236563217003078093533444609380041534645943031}{884176199373970195454361600 0000} z^{29}$$

$$+ \frac{3138838406092202102504585608640422694687408753}{114828077840775350059008 0000000} z^{30}$$

$$+ \frac{8725551890447354544046175693837732644366547298 0739}{822283865417792281772556288 0000000} z^{31}$$

$$+ \frac{425737394305602655617598769965005304141454227911 73}{103188563503409227516556083200000} z^{32}$$

$$+ \frac{13950221879812033253678569490794508028154640304466166 71}{868331761881188649551819440128000000} z^{33}$$

$$+ \frac{32447228980517025622032896917637944725556315677063368 39}{517952279016849369908102823936000000} z^{34}$$

$$+ \frac{252746256778648715360664863248003095610582975168450450167 79}{1033314796638614492966665133752320000000} z^{35}$$

$$+ \frac{1371527565492761675349081484453332706970903530790638723 49}{14343293880466597935916693585920000000} z^{36}$$

$$+ \frac{46828794445858838833502651199990925813000767590216108573664370 1}{125125028102057682239236178014371840000000} z^{37}$$

$$+ 2556603487516955873862726885920158947239917553695979561374567 2781 / 174340872488867037253335741366691430400000000 z^{38}$$

$$+ 617515728876796289352864663575404618818951509123962987515842902081 / 1073572741115654913612646407363310387200000000 z^{39} + O(z^{40})$$

```
> subs(GFSeries(spgraphs,unlabelled,z,20),Graph);
```

$$z + 2 z^2 + 5 z^3 + 15 z^4 + 48 z^5 + 167 z^6 + 602 z^7 + 2256 z^8 + 8660 z^9 + 33958 z^{10} + 135292 z^{11} + 546422 z^{12} + 2231462 z^{13} + 9199869 z^{14} + 38237213 z^{15} + 160047496 z^{16} + 674034147 z^{17} + 2854137769 z^{18} + 12144094756 z^{19} + O(z^{20}) \quad \textbf{(1.3.3)}$$

II. Asymptotics

Utilities

```
> libname:="/Users/salvy/lib/maple/Algolib/current/algolib",
  libname:
> singularity_analysis:=proc(f,z,n,k) map(simplify,equivalent
  (args)) assuming n::posint end:
```

Catalan numbers

```
> bintrees;
```

$$\{B = Union(E, Prod(Z, B, B))\} \quad \textbf{(2.2.1)}$$

```
> combstruct[gfsolve](bintrees,labelled,z);
```

$$\left\{B(z) = -\frac{-1 + \sqrt{1 - 4 z}}{2 z}, Z(z) = z\right\} \quad \textbf{(2.2.2)}$$

```
> solB:=subs(%,B(z));
```

$$solB := -\frac{-1 + \sqrt{1 - 4 z}}{2 z} \quad \textbf{(2.2.3)}$$

```
> as_B:=singularity_analysis(solB,z,n,2);
```

(2.2.4)

$$as\_B := \frac{4^n}{\sqrt{\pi}\, n^{3/2}} - \frac{9\, 4^n}{8\sqrt{\pi}\, n^{5/2}} + O\left(\frac{4^n}{n^{7/2}}\right) \quad \textbf{(2.2.4)}$$

Root with an only child

```
> B1:=2*z*solB;
```

$$B1 := 1 - \sqrt{1-4z} \quad \textbf{(2.3.1)}$$

```
> as_B1:=singularity_analysis(B1,z,n,2);
```

$$as\_B1 := \frac{4^n}{2\sqrt{\pi}\, n^{3/2}} + \frac{3\, 4^n}{16\sqrt{\pi}\, n^{5/2}} + O\left(\frac{4^n}{n^{7/2}}\right) \quad \textbf{(2.3.2)}$$

```
> asympt(as_B1/as_B,n);
```

$$\frac{1}{2} + \frac{3}{4n} + O\left(\frac{1}{n^2}\right) \quad \textbf{(2.3.3)}$$

Path length

```
> bintrees;
```

$$\{B = Union(E, Prod(Z, B, B))\} \quad \textbf{(2.4.1)}$$

```
> pl_trees:={path(B)=Union(0,Prod(0,size(B)+path(B),size(B)+
path(B)))}:
> combstruct[agfeqns](bintrees,pl_trees,labelled,z,[[u,path]])
;
```

$$[B(z,u) = 1 + z\, B(z\,u, u)^2, Z(z,u) = z\,u] \quad \textbf{(2.4.2)}$$

```
> op(1,%);
```

$$B(z,u) = 1 + z\, B(z\,u, u)^2 \quad \textbf{(2.4.3)}$$

We want d/du at u=1:

```
> convert(diff(%,u),D);
```

$$D_2(B)(z,u) = 2\,z\,B(z\,u,u)\,\left(D_1(B)(z\,u,u)\,z + D_2(B)(z\,u,u)\right) \quad \textbf{(2.4.4)}$$

```
> eval(%,u=1);
```

$$D_2(B)(z,1) = 2\,z\,B(z,1)\,\left(D_1(B)(z,1)\,z + D_2(B)(z,1)\right) \quad \textbf{(2.4.5)}$$

```
> isolate(%,D[2](B)(z,1));
```

$$D_2(B)(z,1) = \frac{2\,B(z,1)\,D_1(B)(z,1)\,z^2}{-2\,B(z,1)\,z + 1} \quad \textbf{(2.4.6)}$$

```
> P:=subs(B(z,1)=solB,D[1](B)(z,1)=diff(solB,z),op(2,%));
```

$$P := -\frac{\left(-1 + \sqrt{1-4z}\right) z \left(\frac{1}{\sqrt{1-4z}\; z} + \frac{-1+\sqrt{1-4z}}{2\,z^2}\right)}{\sqrt{1-4z}} \quad \textbf{(2.4.7)}$$

```
> singularity_analysis(P,z,n,2);
```

$$4^n - \frac{3\, 4^n}{\sqrt{\pi}\,\sqrt{n}} + O\left(\frac{4^n}{n^{3/2}}\right) \quad \textbf{(2.4.8)}$$

```
> asympt(%/as_B/n,n);
```

(2.4.9)

$$\frac{\sqrt{\pi}}{\sqrt{\frac{1}{n}}} - 3 + \frac{9\sqrt{\pi}\sqrt{\frac{1}{n}}}{8} + \mathrm{O}\left(\frac{1}{n}\right) \qquad \textbf{(2.4.9)}$$