

```
[> read "alvalues.mpl";
```

## Simple minimization

```
> deq:={z^2*(z-3)*diff(y(z),z,z)+(4*z^7+z^2+3*z-9)*diff(y(z),z)+4*z^5*(5*z+3)*y(z),y(0)=1};
```

$$deq := \left\{ z^2 (z - 3) \left( \frac{d^2}{dz^2} y(z) \right) + (4z^7 + z^2 + 3z - 9) \left( \frac{d}{dz} y(z) \right) + 4z^5 (5z + 3) y(z), y(0) = 1 \right\} \quad (1.1)$$

```
> gfun:-minimizediffeq(deq,y(z));
```

$$\left\{ 4z^5 y(z) + (z - 3) \left( \frac{d}{dz} y(z) \right), y(0) = 1 \right\} \quad (1.2)$$

## Exceptional value from [Lorch-Muldoon 95]

```
> F:='diff'(BesselJ(0,z),z$4);
```

$$F := \frac{d^4}{dz^4} \text{BesselJ}(0, z) \quad (2.1)$$

```
> deq:=gfun:-holexprtodiffeq(F,y(z));
```

$$deq := \left\{ (z^{11} - 12z^9 + 54z^7 + 36z^5 + 945z^3 + 6480z) y(z) + (6z^{10} - 99z^8 + 576z^6 - 2754z^4 + 4050z^2 + 3645) \left( \frac{d}{dz} y(z) \right) \right. \quad (2.2)$$

$$+ (2z^{11} - 21z^9 + 72z^7 + 378z^5 - 2754z^3 + 4131z) \left( \frac{d^2}{dz^2} y(z) \right) + (6z^{10} - 96z^8 + 540z^6 - 1296z^4 + 1134z^2) \left( \frac{d^3}{dz^3} y(z) \right)$$

$$\left. + (z^{11} - 12z^9 + 54z^7 - 108z^5 + 81z^3) \left( \frac{d^4}{dz^4} y(z) \right), y(0) = \frac{3}{8} \right\}$$

```
> alvalues(deq,y(z));
```

$$\{y(\text{RootOf}(\_Z^2 - 3)) = 0\} \quad (2.3)$$

```
> map(allvalues,%);
```

$$\{y(\sqrt{3}) = 0, y(-\sqrt{3}) = 0\} \quad (2.4)$$

> gfun:-minimizediffeq(deq,y(z));

$$\left\{ (z^5 - 10z^3 + 45z)y(z) + (z^4 - 18z^2 + 45) \left( \frac{d}{dz} y(z) \right) + (z^5 - 6z^3 + 9z) \left( \frac{d^2}{dz^2} y(z) \right), y(0) = \frac{3}{8} \right\} \quad (2.5)$$

> eval(%,z=sqrt(3));

$$\left\{ 24\sqrt{3}y(\sqrt{3}), y(0) = \frac{3}{8} \right\} \quad (2.6)$$

## ▼ A 1F1

> F:=hypergeom([4],[1/7],z);

$$F := \text{hypergeom}\left([4], \left[\frac{1}{7}\right], z\right) \quad (3.1)$$

> deq:=gfun:-holxprtodiffeq(F,y(z));

$$\text{deq} := \left\{ -28y(z) + (-7z + 1) \left( \frac{d}{dz} y(z) \right) + 7 \left( \frac{d^2}{dz^2} y(z) \right) z, y(0) = 1 \right\} \quad (3.2)$$

> algvalues(deq,y(z));

$$\left\{ y\left(-\frac{6}{7}\right) = -\frac{65}{49}, y(\text{RootOf}(49\_Z^2 + 525\_Z + 1170)) = \frac{104}{49(7\text{RootOf}(49\_Z^2 + 525\_Z + 1170) + 26)} \right\} \quad (3.3)$$

> map(allvalues,%);

$$\left\{ y\left(-\frac{6}{7}\right) = -\frac{65}{49}, y\left(-\frac{75}{14} - \frac{3\sqrt{105}}{14}\right) = \frac{104}{49\left(-\frac{23}{2} - \frac{3\sqrt{105}}{2}\right)}, y\left(-\frac{75}{14} + \frac{3\sqrt{105}}{14}\right) = \frac{104}{49\left(-\frac{23}{2} + \frac{3\sqrt{105}}{2}\right)} \right\} \quad (3.4)$$

## ▼ Another 1F1

> F:=hypergeom([5],[-7/5],z);

$$F := \text{hypergeom}\left([5], \left[-\frac{7}{5}\right], z\right) \quad (4.1)$$

> deq:=gfun:-holxprtodiffeq(F,y(z));

$$\text{deq} := \left\{ -25 y(z) + (-5z - 7) \left( \frac{d}{dz} y(z) \right) + 5 \left( \frac{d^2}{dz^2} y(z) \right) z, y(0) = 1 \right\} \quad (4.2)$$

**> algvalues(deq,y(z));**

$$\left\{ y(\text{RootOf}(25\_Z^2 + 220\_Z + 264)) = \frac{1377}{25 (5 \text{RootOf}(25\_Z^2 + 220\_Z + 264) - 3)}, y(\text{RootOf}(25\_Z^2 + 420\_Z + 1224)) = \right. \\ \left. - \frac{187}{25 (5 \text{RootOf}(25\_Z^2 + 420\_Z + 1224) + 17)} \right\} \quad (4.3)$$

**> map(allvalues,%);**

$$\left\{ y\left(-\frac{42}{5} - \frac{6\sqrt{15}}{5}\right) = -\frac{187}{25 (-25 - 6\sqrt{15})}, y\left(-\frac{42}{5} + \frac{6\sqrt{15}}{5}\right) = -\frac{187}{25 (-25 + 6\sqrt{15})}, y\left(-\frac{22}{5} - \frac{2\sqrt{55}}{5}\right) \right. \\ \left. = \frac{1377}{25 (-25 - 2\sqrt{55})}, y\left(-\frac{22}{5} + \frac{2\sqrt{55}}{5}\right) = \frac{1377}{25 (-25 + 2\sqrt{55})} \right\} \quad (4.4)$$

## An example of minimization

**> T:=n!\*(n+k)!/k!^4/(n-k)!^3;**

$$T := \frac{n! (n+k)!}{k!^4 (n-k)!^3} \quad (5.1)$$

Aim:

**> Sum(Sum(T,k=0..n)\*z^n,n=0..infinity);**

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{n! (n+k)!}{k!^4 (n-k)!^3} \right) z^n \quad (5.2)$$

**> SumTools[Hypergeometric][Zeilberger](T,n,k,Sn)[1];**

$$\begin{aligned} & (-29412 n^{10} - 871416 n^9 - 11480487 n^8 - 88507038 n^7 - 441858488 n^6 - 1491520538 n^5 - 3444891165 n^4 - 5371293584 n^3 \\ & - 5406327008 n^2 - 3169244928 n - 820935936) S n^4 + (470592 n^9 + 11354400 n^8 + 120533388 n^7 + 738392622 n^6 \\ & + 2874782186 n^5 + 7371364948 n^4 + 12439443776 n^3 + 13312054230 n^2 + 8191131786 n + 2206141704) S n^3 + (3294144 n^8 \end{aligned} \quad (5.3)$$

$$\begin{aligned}
& + 64657152 n^7 + 549637656 n^6 + 2641270728 n^5 + 7841655446 n^4 + 14715627332 n^3 + 17029441376 n^2 + 11099435178 n \\
& + 3116435176) S n^2 + (8353008 n^7 + 134716536 n^6 + 914453220 n^5 + 3384151662 n^4 + 7370502898 n^3 + 9445876552 n^2 \\
& + 6597664136 n + 1938794528) S n - 117648 n^6 - 1603296 n^5 - 8783484 n^4 - 24580704 n^3 - 36734684 n^2 - 27550600 n \\
& - 8098784
\end{aligned}$$

**> rec:=add(coeff(% ,Sn,i)\*u(n+i),i=0..4);**

$$\begin{aligned}
rec := & (-117648 n^6 - 1603296 n^5 - 8783484 n^4 - 24580704 n^3 - 36734684 n^2 - 27550600 n - 8098784) u(n) + (8353008 n^7 \\
& + 134716536 n^6 + 914453220 n^5 + 3384151662 n^4 + 7370502898 n^3 + 9445876552 n^2 + 6597664136 n + 1938794528) u(n + 1) \\
& + (3294144 n^8 + 64657152 n^7 + 549637656 n^6 + 2641270728 n^5 + 7841655446 n^4 + 14715627332 n^3 + 17029441376 n^2 \\
& + 11099435178 n + 3116435176) u(n + 2) + (470592 n^9 + 11354400 n^8 + 120533388 n^7 + 738392622 n^6 + 2874782186 n^5 \\
& + 7371364948 n^4 + 12439443776 n^3 + 13312054230 n^2 + 8191131786 n + 2206141704) u(n + 3) + (-29412 n^{10} - 871416 n^9 \\
& - 11480487 n^8 - 88507038 n^7 - 441858488 n^6 - 1491520538 n^5 - 3444891165 n^4 - 5371293584 n^3 - 5406327008 n^2 \\
& - 3169244928 n - 820935936) u(n + 4)
\end{aligned} \tag{5.4}$$

**> ini:={seq(u(i)=value(eval(Sum(T,k=0..n),n=i)),i=0..3)};**

$$ini := \left\{ u(0) = 1, u(1) = 3, u(2) = \frac{31}{2}, u(3) = \frac{133}{2} \right\} \tag{5.5}$$

**> deq:=gfun:-rectodiffeq({rec} union ini,u(n),y(z));**

$$\begin{aligned}
deq := & \left\{ (-8098784 z^2 + 12566016 z + 130900) y(z) + (-99370416 z^3 + 1926228512 z^2 + 19342508 z - 8500) \left( \frac{d}{dz} y(z) \right) + \right. \\
& -199657712 z^4 + 12964744750 z^3 + 1538809630 z^2 + 6481588 z - 3400) \left( \frac{d^2}{dz^2} y(z) \right) + (-127952328 z^5 + 20233650850 z^4 \\
& + 7961739664 z^3 + 361212946 z^2 - 677432 z) \left( \frac{d^3}{dz^3} y(z) \right) + (-32463564 z^6 + 11235459042 z^5 + 10659020750 z^4 \\
& + 1604950578 z^3 - 33526532 z^2) \left( \frac{d^4}{dz^4} y(z) \right) + (-3368016 z^7 + 2594670492 z^6 + 5333737872 z^5 + 1939363976 z^4
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
& - 134858852 z^3) \left( \frac{d^5}{dz^5} y(z) \right) + (-117648 z^8 + 251658648 z^7 + 1140591768 z^6 + 897354222 z^5 - 150521960 z^4) \left( \frac{d^6}{dz^6} y(z) \right) \\
& + (8353008 z^8 + 104186880 z^7 + 180068748 z^6 - 65198130 z^5) \left( \frac{d^7}{dz^7} y(z) \right) + (3294144 z^8 + 15589728 z^7 \\
& - 12362847 z^6) \left( \frac{d^8}{dz^8} y(z) \right) + (470592 z^8 - 1018476 z^7) \left( \frac{d^9}{dz^9} y(z) \right) - 29412 z^8 \left( \frac{d^{10}}{dz^{10}} y(z) \right), y(0) = 1, D(y)(0) = 3 \}
\end{aligned}$$

This prints more detail of the intermediate computations

```
> infolevel[gfun:-minimizediffeq]:=2;
```

$infolevel_{minimizediffeq} := 2$

(5.7)

```
> st:=time():gfun:-minimizediffeq(deq,y(z));time()-st;
```

```
minfact: looking for a factor of order 9
minfact: proved no factor of order 9
minfact: looking for a factor of order 8
minfact: proved no factor of order 8
minfact: looking for a factor of order 7
minfact: proved no factor of order 7
minfact: looking for a factor of order 6
minfact: bound on the degrees of the coefficients at order 6: [102, 101, 100, 98, 98, 98, 98]
minfact: found a right factor of order 6
minfact: looking for a factor of order 5
minfact: bound on the degrees of the coefficients at order 5: [15, 14, 13, 13, 13, 13]
minfact: looking for a factor of order 4
minfact: bound on the degrees of the coefficients at order 4 not larger
minfact: looking for a factor of order 3
minfact: bound on the degrees of the coefficients at order 3 not larger
minfact: looking for a factor of order 2
minfact: bound on the degrees of the coefficients at order 2 not larger
minfact: looking for a factor of order 1
minfact: bound on the degrees of the coefficients at order 1 not larger
```

$$\left\{ \begin{aligned}
& (7529472 z^6 - 82003392 z^5 + 314044752 z^4 - 369874128 z^3 - 13835038 z^2 - 6817782 z - 83776) y(z) + (22588416 z^7 \\
& - 1771198848 z^6 + 3372156432 z^5 - 3161430264 z^4 - 847969794 z^3 - 56798636 z^2 + 88774 z + 5440) \left( \frac{d}{dz} y(z) \right) \\
& + (7529472 z^8 - 2414492640 z^7 + 2098630848 z^6 - 658222092 z^5 - 2601270010 z^4 - 350298046 z^3 - 22001542 z^2 + 24956 z \\
& + 2176) \left( \frac{d^2}{dz^2} y(z) \right) + (-534592512 z^8 - 638281440 z^7 + 933574800 z^6 - 994536060 z^5 - 647392936 z^4 - 7335568 z^3 \\
& + 1806848 z^2 + 12512 z) \left( \frac{d^3}{dz^3} y(z) \right) + (-210825216 z^8 + 21253248 z^7 + 138725376 z^6 - 315822072 z^5 + 37815874 z^4 \\
& + 2480442 z^3 + 12512 z^2) \left( \frac{d^4}{dz^4} y(z) \right) + (-30117888 z^8 + 52246656 z^7 - 49321200 z^6 + 17653020 z^5 + 809428 z^4 \\
& + 3536 z^3) \left( \frac{d^5}{dz^5} y(z) \right) + (1882368 z^8 - 2206584 z^7 + 1703460 z^6 + 67815 z^5 + 272 z^4) \left( \frac{d^6}{dz^6} y(z) \right), y(0) = 1, D(y)(0) = 3 \}
\end{aligned} \right.$$

0.945 (5.8)

### More details on the first minimization

```

> deq:={z^2*(z-3)*diff(y(z),z,z)+(4*z^7+z^2+3*z-9)*diff(y(z),z)+4*z^5*(5*z+3)*y(z),y(0)=1};

```

$$deq := \left\{ z^2 (z - 3) \left( \frac{d^2}{dz^2} y(z) \right) + (4 z^7 + z^2 + 3 z - 9) \left( \frac{d}{dz} y(z) \right) + 4 z^5 (5 z + 3) y(z), y(0) = 1 \right\}$$

(6.1)

```

> infolevel[gfun:-minimizediffeq]:=3:
> gfun:-minimizediffeq(deq,y(z));
minfact: looking for a factor of order 1
bound_deg_coeffs: # apparent singularities <= 967
minfact: bound on the degrees of the coefficients at order 1: [970, 974]
minfact: computing 10 modular coefficients

```

```
minfact: try guessing mod p
minfact: found modular operator of order 1
minfact: modular gcd of order 1 and degree 5
minfact: computing 19 rational coefficients
minfact: try guessing
minfact: checking equation of order 1
minfact: found a right factor of order 1
```

$$\left\{ 4z^5 y(z) + (z-3) \left( \frac{d}{dz} y(z) \right), y(0) = 1 \right\}$$

**(6.2)**