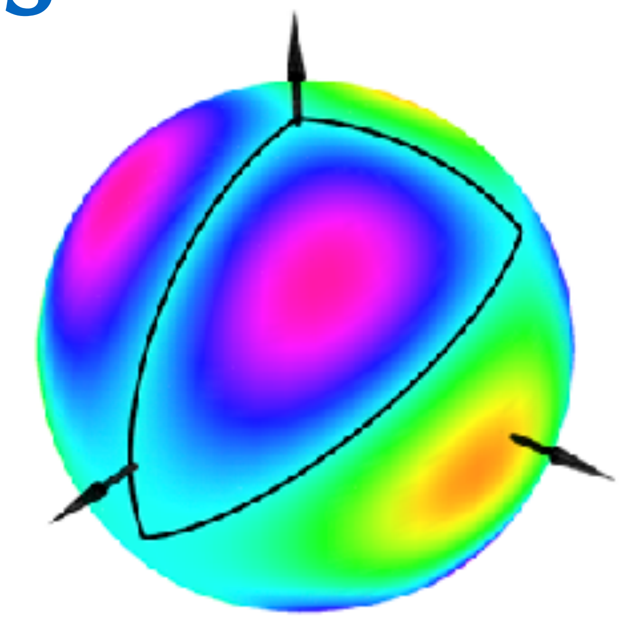
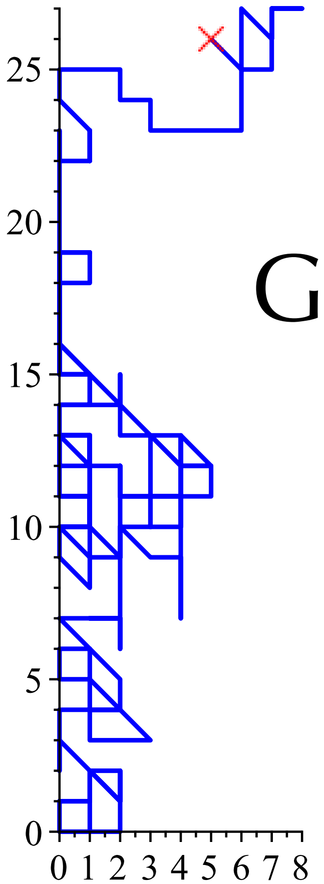
Walks in Cones and Tight Enclosures of Laplacian Eigenvalues



Bruno Salvy

AriC, Inria, ENS de Lyon



Groupe de Travail Transcendence et Combinatoire

May 2021

Joint work with Joel Dahne

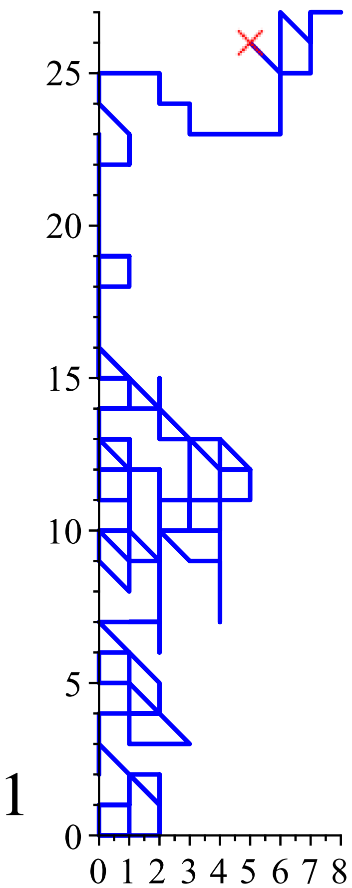
SIAM J. Sci. Comp. 2020

doi: 10.1137/20M1326520

arXiv: abs/2003.08095

I. Long Introduction:
Walks in Cones

Lattice Walks: a Mine of Linear Recurrences Waiting for Tools



Walks from 0 to $P \in \mathbb{Z}^d$ staying in $K \subset \mathbb{R}^d$ using n steps in $\mathcal{S} \subset \mathbb{Z}^d$

Ex.: $d = 2,\ \mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \nwarrow\},\ K = \mathbb{N}^2$

$$u_{i,j,n} = u_{i-1,j,n-1} + u_{i,j-1,n-1} + u_{i+1,j,n-1} + u_{i,j+1,n-1} + u_{i+1,j+1,n-1}$$

$$u_{i,j,n} = 0 \quad \text{for} \quad (i,j) \notin K$$

Generating functions:

excursions

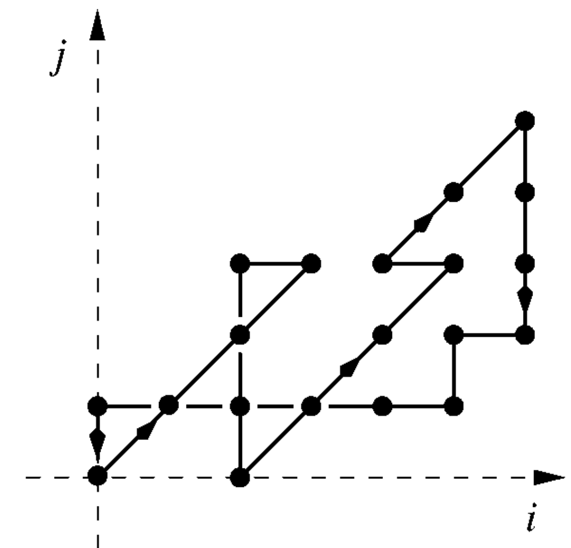
total number

$$U_K(x,y;z) := \sum_{i,j,n} u_{i,j,n} x^i y^j z^n, \quad U_K(0,0;z) = \sum_{n \geq 0} e_n z^n, \quad U_K(1,1;z) = \sum_{n \geq 0} u_n z^n.$$

Applications: queuing theory, statistical physics, combinatorics,…

Questions: $\mathcal{S}, K \rightarrow$ asymptotics? nature of these series?

Example: Kreweras Walks



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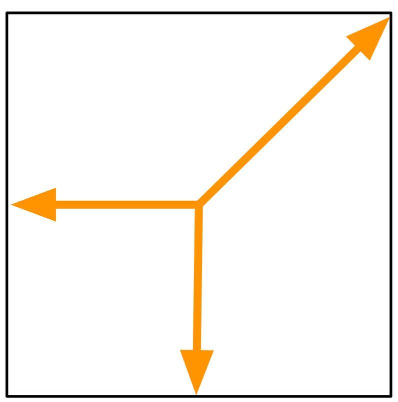


$$\mathcal{S} := \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$K = \mathbb{N}^2$$

Excursions:

$$e_{3n} = \frac{4^n\,(3n)!}{(n+1)!(2n+1)!} \sim C\,\frac{3^{3n}}{n^{5/2}}$$

Total number: $u_n \sim C' \dfrac{3^n}{n^{3/2}}$

$U_{\mathbb{N}^2}(x, y; z)$ is algebraic

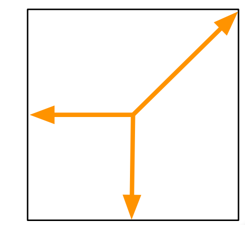
[Kreweras65, Niederhausen83, Gessel86, BousquetMelou05, Bernardi07, BernardiBousquet-MélouRaschel21]

Main Character: Generating Polynomial

$$\chi_{\mathcal{S}}(x_1, \ldots, x_d) := \sum_{s \in \mathcal{S}} x_1^{s_1} \cdots x_d^{s_d}$$

Ex. Kreweras:

$$\chi := \frac{1}{x} + \frac{1}{y} + xy$$



For $k \in \mathbb{N}$, $\chi_{\mathcal{S}}^k = \sum_{m \in \mathbb{Z}^d} c_{k,m} x_1^{m_1} \cdots x_d^{m_d}$

$c_{k,m}$: num. walks from 0 to m in $\mathbb{Z}^d$ in k steps

Summing over k, $\frac{1}{1 - z\chi_{\mathcal{S}}} = \sum_{k,m} c_{k,m} x^m z^k = U_{\mathbb{Z}^d}(x; z)$

For an arbitrary cone K,

Kernel equation

$$(1 - z\chi_S)U_K(x; z) = 1 + \text{correcting terms encoding } \partial K$$

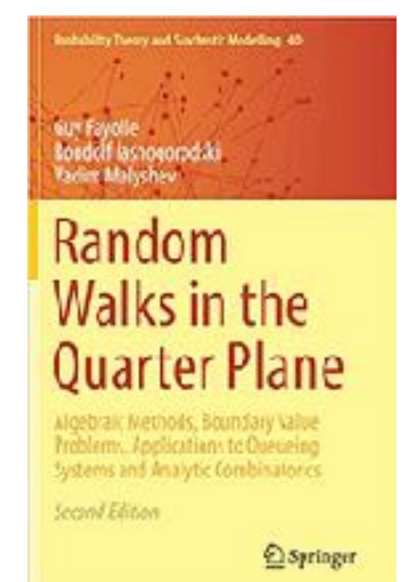
A Mysterious Secondary Character: Group of the Walk

For *small-step* walks ($\max_i |s_i| = 1$, for all $s \in \mathcal{S}$)

For all $i \in \{1, \dots, d\}$, $\chi_{\mathcal{S}} = A_i^- x_i^{-1} + A_i^0 + A_i^+ x_i$,

$$A_i^-, A_i^0, A_i^+ \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_{i+1}^{\pm 1}, \dots, x_d^{\pm 1}]$$

$$\psi_i : \left(x_j \mapsto x_j \text{ for } j \neq i, x_i \mapsto \frac{A_i^-}{A_i^+} \frac{1}{x_i} \right) \text{ fixes } \chi_{\mathcal{S}}$$

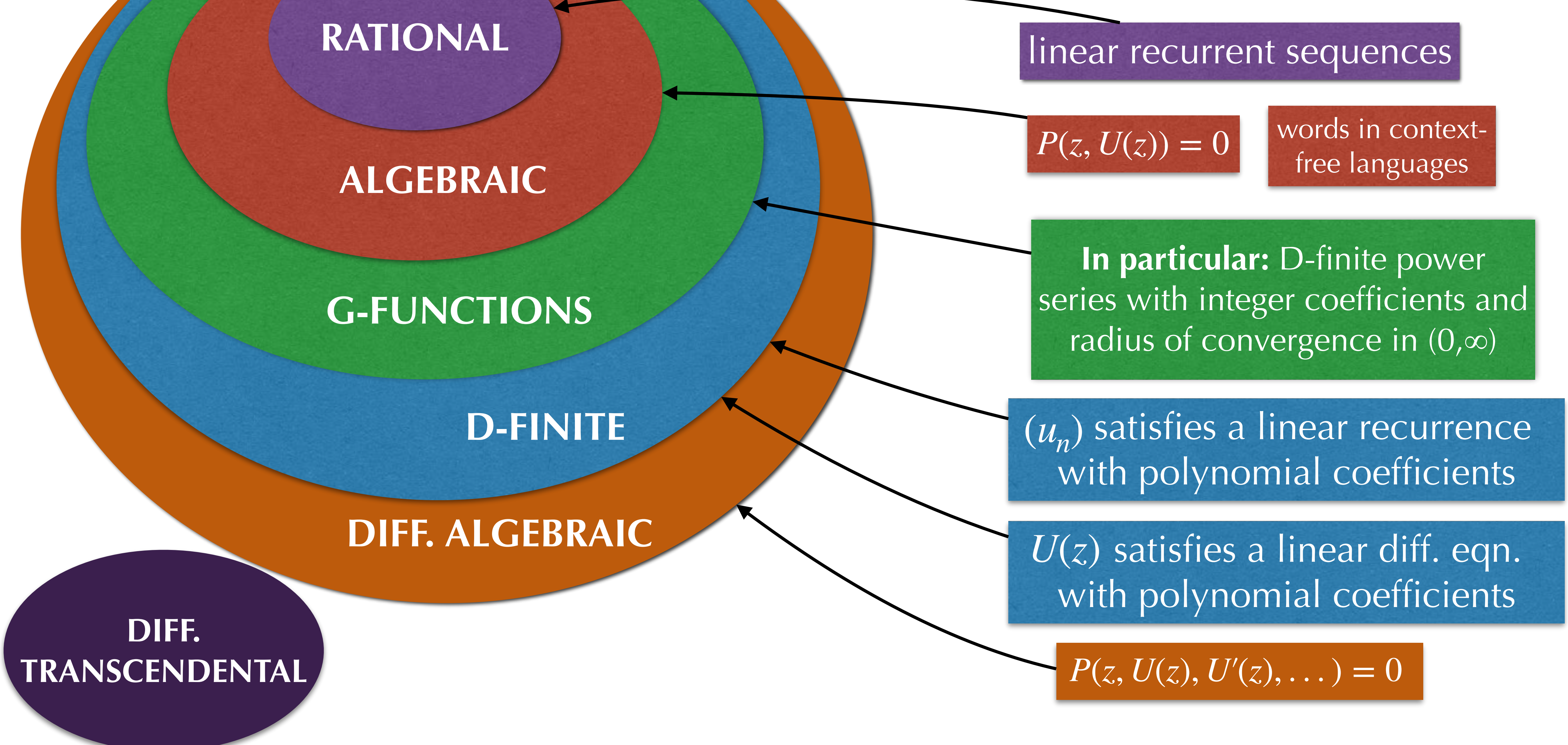


Group: $\mathcal{G}_{\mathcal{S}} := \langle \psi_1, \dots, \psi_d \rangle$ generated by the ψ_i.

[Malyshev, Fayolle, Iasnogordski, Mishna, Bousquet-Mélou]

Classes of Univariate Power Series

$$U(z) := \sum_{n=0}^{\infty} u_n z^n$$



Knowing where $U(z)$ fits helps deduce properties of (u_n) .

Generating Functions and Asymptotics

$$\text{For } U(z) := \sum_{n=0}^{\infty} u_n z^n \in \mathbb{Q}[[z]]$$

$$\text{if } u_n \sim C\rho^n n^{\alpha}, \quad n \to \infty$$

$$U \text{ rational} \implies \rho \text{ algebraic}, \alpha \in \mathbb{N}, \quad \text{Fibonacci}$$

$$\left.\begin{array}{r} U \text{ algebraic} \\ U \text{ G-function} \end{array}\right\} \implies \rho \text{ algebraic}, \alpha \in \mathbb{Q}, \quad \text{Catalan}$$

$$U \text{ D-finite} \implies \rho \text{ algebraic}, \alpha \in \overline{\mathbb{Q}}.$$

Conversely, asymptotics help classify.

Walks in $\mathbb{N}^2$: Recent Progress

79 models
with small steps

$\mathcal{G}_{\mathcal{S}}$ finite

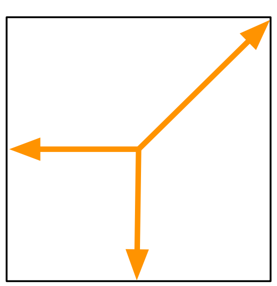
$\mathcal{G}_{\mathcal{S}}$ infinite

Starting point for this work

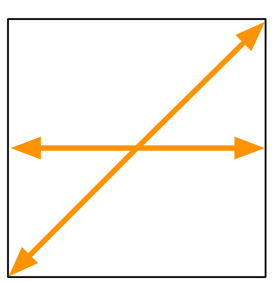
Guess & Prove (4)
algebraic
(1965,1985,2010)

Creative telescoping (19)
D-finite
(2010)

Probabilities & number theory (56)
non-D-finite
(2014)



Kreweras



Gessel

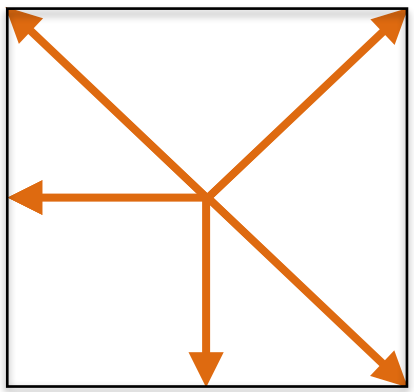
Tutte invariants
(2021)

Tutte invariants (9)
Diff. algebraic
(2021)

Differential Galois theory (47)
Diff. transcendental
(2018)

[Bernardi,Bostan,Bousquet-Mélou,Gessel,Gouyou-Beauchamps,Hardouin,Kauers, Kreweras,Melczer,Mishna,Raschel,Rechnitzer,Roques,Salvy,Singer]

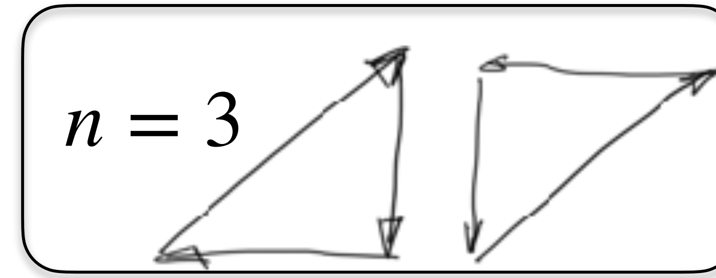
Example in $\mathbb{N}^2$

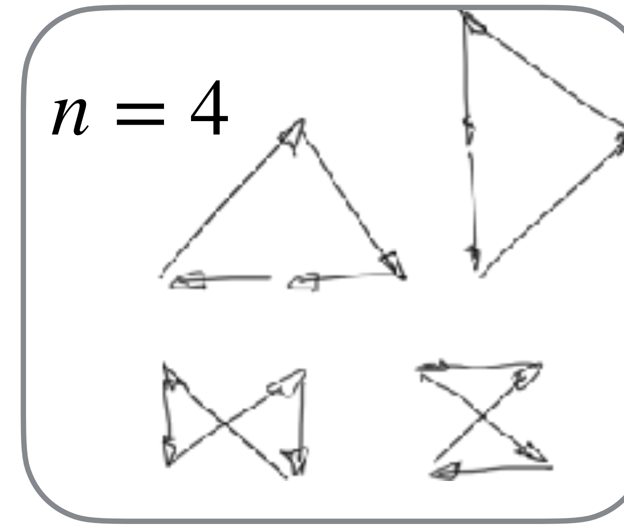


$\mathcal{S}$

Excursions:

$$(e_n) = 1,0,0,2,4,8,28,108,\ldots$$





Asymptotics: $e_n \sim C\, 5^n n^\alpha$ with $\alpha = -1 - \dfrac{\pi}{\arccos(1/4)}$

Next 2 slides

$\alpha \notin \mathbb{Q}$

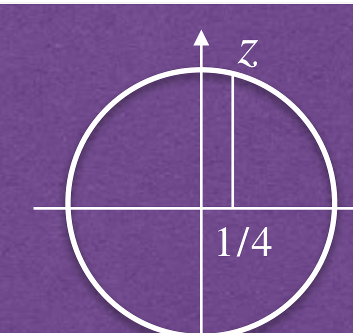
$\Rightarrow U(0,0,z)$ not D-finite

$\Rightarrow U(x,y,z)$ not D-finite

Proof: $\alpha \in \mathbb{Q}$

$\Rightarrow z$ root of unity, with

$$\frac{z + 1/z}{2} = \frac{1}{4}$$



$\Rightarrow 2z^2 - z + 2$ divisible by a cyclotomic pol.

Contradiction.

[BostanRaschelSalvy2014]

Asymptotics from Probabilities (1/2)

Hyp.

$\mathcal{S} := \{s_1, \dots, s_N\} \subset \mathbb{Z}^d$

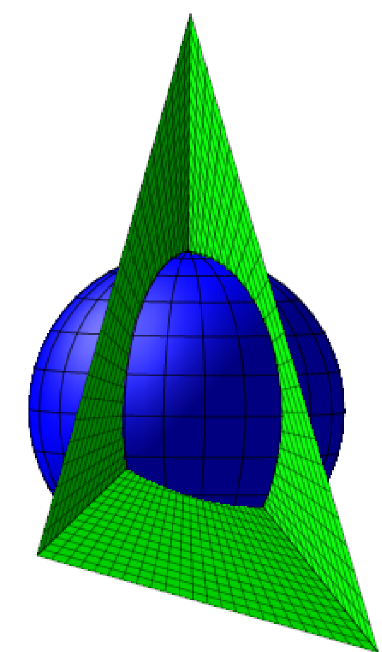
+ non-degeneracy conditions

K a cone with apex at 0

+ regularity conditions on ∂K

No drift: $\sum_i s_i = 0$

Normalized: $\left(\sum_{k=1}^{d} s_i^{(k)} s_j^{(k)} \right)_{i,j} = \mathbf{Id}$



Conclusion

$$\#\text{excursions} \sim C \,|\mathcal{S}|^n n^{-\alpha_{\mathcal{S}}},$$

$\alpha_{\mathcal{S}} := \sqrt{\lambda_1 + (d/2-1)^2} + 1,$

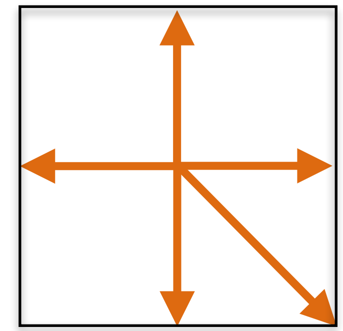
λ_1: fundamental eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1} \cap K$.

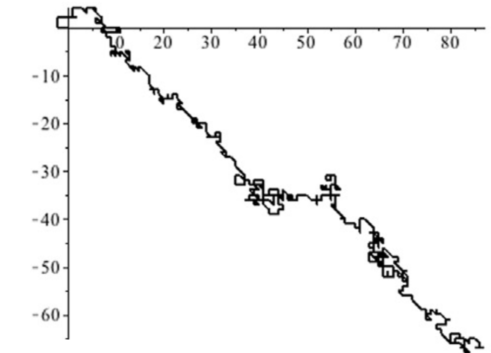
def soon

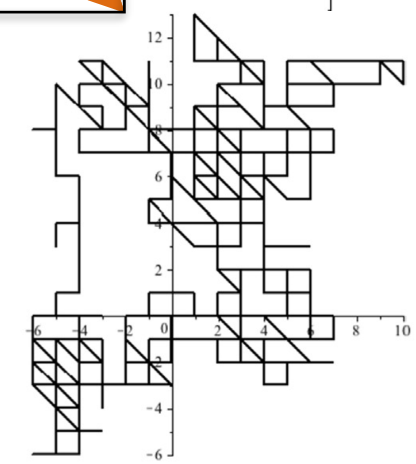
Principle: reduce to Brownian motion in K

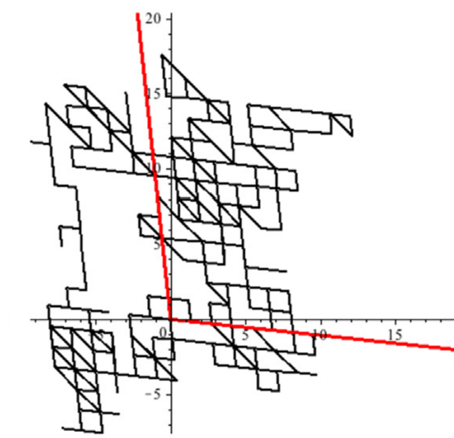
[DenisovWachtel2015]

Asymptotics from Probabilities (2/2)









Reduce cases with drifts and covariance by

1. Adding weights $1/w_i, w_i$ to the steps in direction i, for $i = 1, \ldots, d$

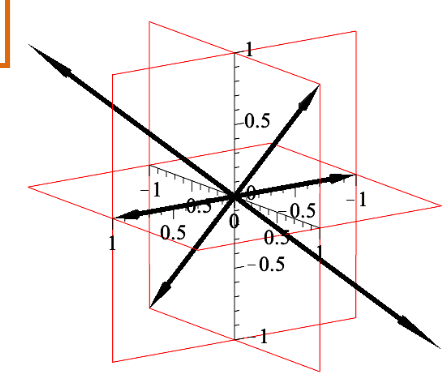
2. A linear change M of coordinates

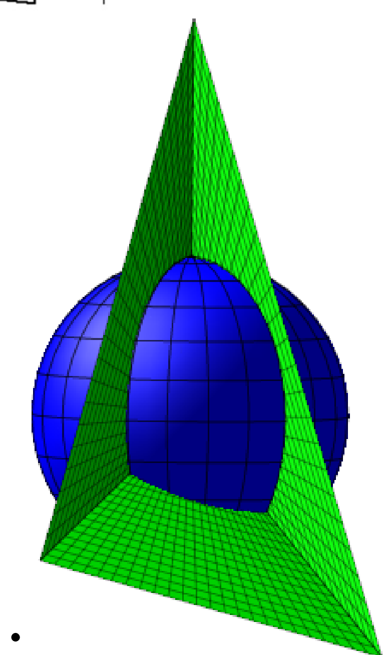
Conclusion

$$\#\text{excursions} \sim C\,\rho^n n^{-\alpha_{\mathcal{S}}},$$

$$\rho := |\mathcal{S}| \min_{(x_1,\ldots,x_d)\in\mathbb{R}_+^d} \chi(x_1, \ldots, x_d)$$

$$\alpha := \sqrt{\lambda_1 + (d/2-1)^2} + 1,$$





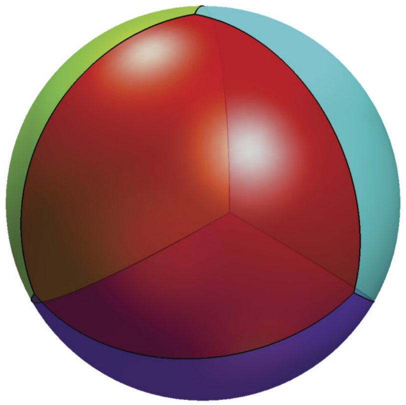
Partial combinatorial explanation for ρ

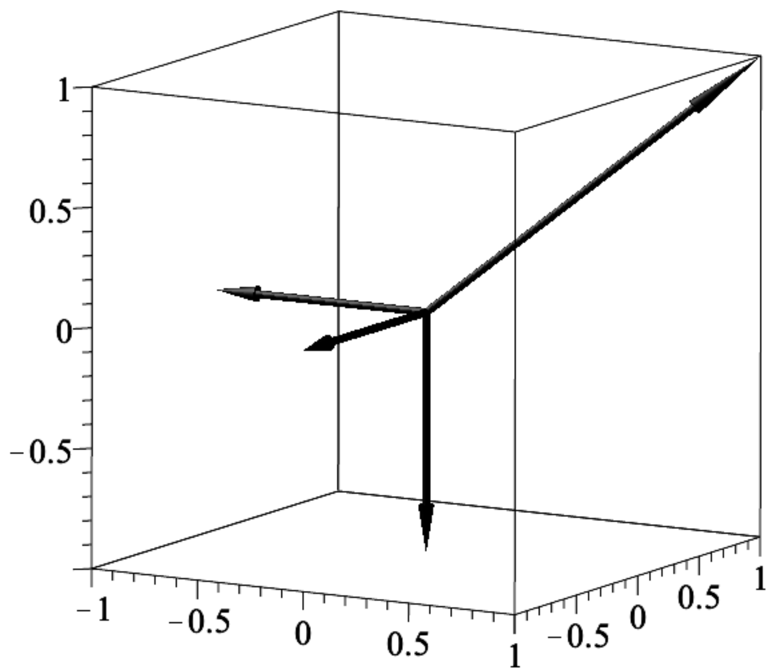
λ_1: fundamental eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1} \cap MK$.

[DenisovWachtel15,JohnsonMishnaYeats18]

Example: Kreweras 3D

The group of the walk is finite





$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Excursions: $e_n \sim C\, 4^n n^{-\alpha_\kappa}$

Previous estimates lead to:

$\alpha_\kappa \in [3.323, 3.326]$ (Costabel, 2008)

$\alpha_\kappa \simeq 3.32572$ (Ratzkin, Treibergs, 2009)

$\alpha_\kappa \simeq 3.3261$ (Balakrishna, 2013)

$\alpha_\kappa \simeq 3.325757004174456$ (Guttmann, 2015)

$\alpha_\kappa \simeq 3.3257569$ (Bacher et al., 2016)

$\alpha_\kappa \simeq 3.325757004175$ (Bogosel et al., 2020)

Initial goal of this work

New: If $\alpha_\kappa = p/q \in \mathbb{Q}$, then $q > 10^{51}$.

D-finiteness more and more doubtful

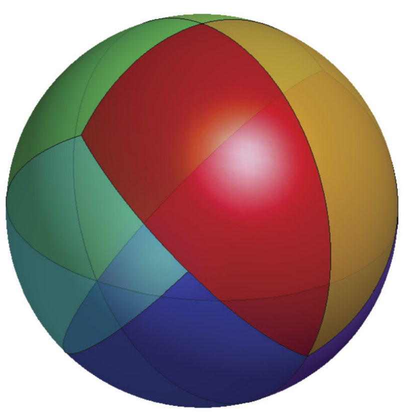
Walk-mining in $\mathbb{N}^3$

(Bogosel, Perrollaz, Raschel, Trotignon 2020)

17 spherical triangles associated to finite groups

+ computation of the corresponding angles (all in $\{\pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4\}$)

+ exact or estimated value of the exponent



$(\pi/3, \pi/2, 2\pi/3)$

They all correspond to tilings of the sphere

Results

Angles	BPRT	new
$(3\pi/4,\pi/3,\pi/2)$	12.400051	$12.40005165284337790 5\ldots \pm 10^{-47}$
$(2\pi/3,\pi/3,\pi/2)$	13.744355	$13.7443552132132318 35\ldots \pm 10^{-84}$
$(2\pi/3,\pi/4,\pi/2)$	20.571973	$20.5719735379847305 57\ldots \pm 10^{-30}$
$(2\pi/3,\pi/3,\pi/3)$	21.309407	$21.3094076301904452 60\ldots \pm 10^{-206}$
$(3\pi/4,\pi/4,\pi/3)$	24.456913	$24.4569137962991116 94\ldots \pm 10^{-73}$
$(2\pi/3,\pi/4,\pi/4)$	49.109945	$49.1099452632846099 20\ldots \pm 10^{-153}$
$(2\pi/3,3\pi/4,3\pi/4)$	4.261734	$4.26173475529398708 57\ldots \pm 10^{-22}$
$(2\pi/3,2\pi/3,2\pi/3)$	5.159145	$5.15914564246654171 12\ldots \pm 10^{-104}$
$(\pi/2,2\pi/3,3\pi/4)$	6.241748	$6.24174833072633423 68\ldots \pm 10^{-20}$
$(\pi/2,2\pi/3,2\pi/3)$	6.777108	$6.77710805459830095 73\ldots \pm 10^{-35}$

finite elements & convergence acceleration

[BogoselPerrollazRaschelTrotignon20]

Next: How do we do it?

and why are the precisions so different?

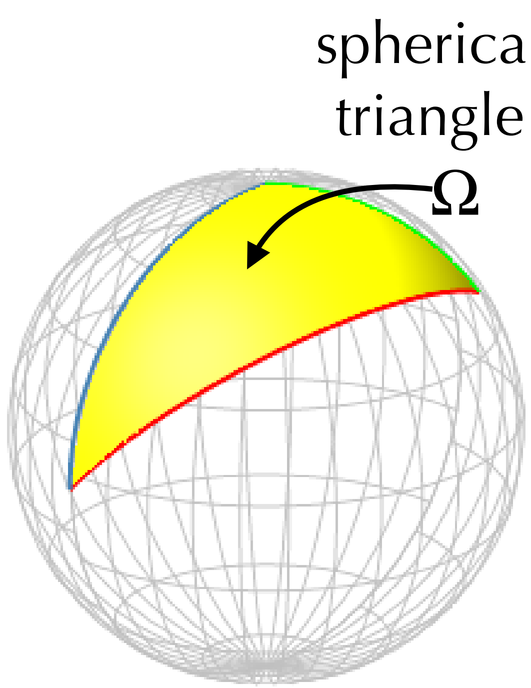
II. Laplacian on Spherical Triangles

Fundamental Eigenvalue of the Laplace-Beltrami Operator on the Unit Sphere

Laplace operator in spherical coordinates in $\mathbb{R}^d$

$$\Delta f = r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right) + r^{-2}\Delta_{\mathbb{S}^{d-1}} f$$

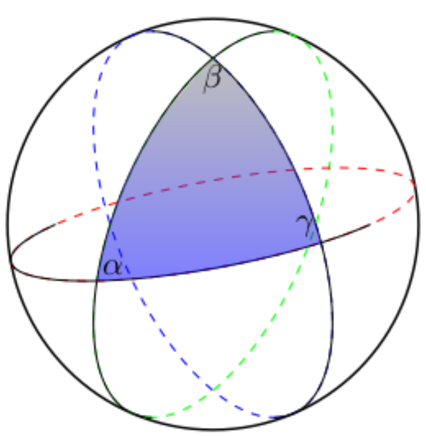
Laplace-Beltrami on the sphere



Eigenvalue problem for $\Omega \subset \mathbb{S}^{d-1}$:

$$\Delta_{\mathbb{S}^{d-1}} f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial\Omega} = 0.$$

Dirichlet condition



Goal: $(\alpha, \beta, \gamma) \mapsto \lambda_1$ with high precision

(dimension d=3)

Basic Properties of the Laplacian over a Bounded Domain $\Omega \subset \mathbb{R}^d$

Hold also for $\Delta_{\mathbb{S}^{d-1}}$

$$(u, v) := \frac{1}{\mathrm{Vol}\,\Omega} \int_\Omega uv \, d\sigma$$

. self-adjoint: $(\Delta u, v) = (u, \Delta v)$ on $\{f \in C^2(\Omega),\ f|_{\partial\Omega} = 0\}$

. discrete spectrum with no accumulation point

fundamental eigenvalue

$$0 < \lambda_1 < \lambda_2 \leq \cdots, \quad \lambda_n \to \infty$$

. corresponding eigenfunctions (u_n) Hilbert basis of $L_2(\Omega)$

. maximum principle: $\Delta u \geq 0$ in $\Omega \Rightarrow \sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x)$

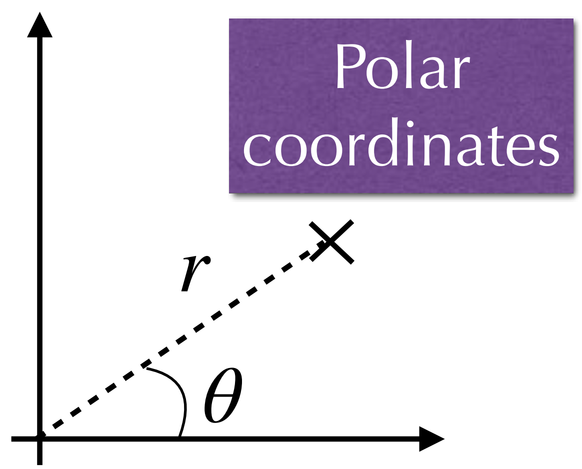
. monotonicity: $\Omega \subset \Omega' \Rightarrow \lambda_i(\Omega) \geq \lambda_i(\Omega')$, for all i

. Faber-Krahn inequality: $\lambda_1(\Omega) \geq \lambda_1(\Omega^\star)$,

$$\Omega^\star = \begin{cases} \text{ball with the same volume for } \Delta, \\ \text{spherical cap with the same area for } \Delta_{\mathbb{S}^{d-1}}. \end{cases}$$

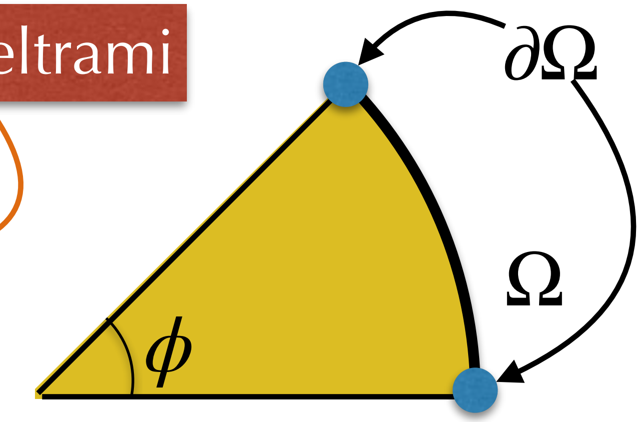
Matlab logo

Planar Case

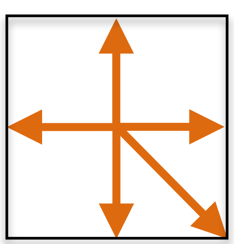


Laplace-Beltrami

$$\Delta f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$



Eigenfunctions: $\sin(\sqrt{\lambda}\theta + c)$



Boundary: $\begin{cases} \theta = 0 \to c = 0 \\ \theta = \phi \to \lambda = \lambda_k := \left(\frac{k\pi}{\phi}\right)^2, k \in \mathbb{N}^\star. \end{cases}$

Ex. $\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \searrow\}$, $\phi = \arccos(u)$, $\mu_u(u) = 0$, $\mu_u = 8t^3 - 8t^2 + 6t - 1$

exponent $\alpha = 1 + \frac{\pi}{\arccos(u)} \notin \mathbb{Q} \Rightarrow$ U not D-finite.

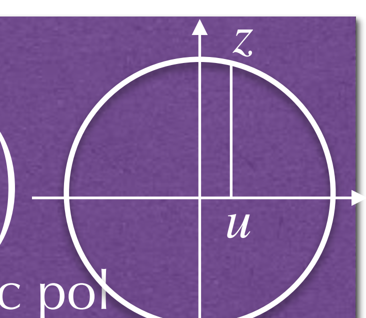
Automatic proof of 51 of the 56 non-Dfinite cases

Proof: $\alpha \in \mathbb{Q}$

$\Rightarrow$ numer $\mu_u\left(\frac{z+1/z}{2}\right)$

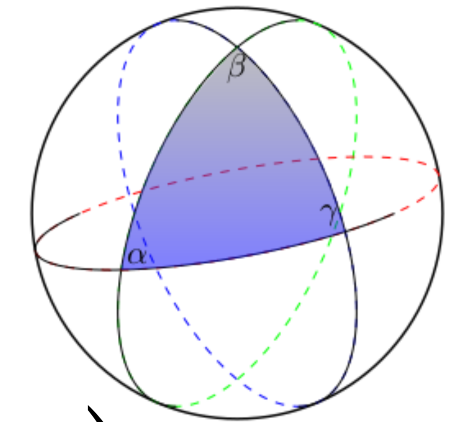
divisible by a cyclotomic pol

Contradiction.



[BostanRaschelSalvy14]

Spherical Triangles



Eigenvalues known when $(\alpha, \beta, \gamma) = \left(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}\right)$

Only possible (p, q, r) that give triangles:

. $(2,3,3) \longrightarrow \lambda = k(k+1),\ k \in 6 + 3\mathbb{N} + 4\mathbb{N}$

. $(2,3,4) \longrightarrow \lambda = k(k+1),\ k \in 9 + 6\mathbb{N}$

. $(2,3,5) \longrightarrow \lambda = k(k+1),\ k \in 15 + 6\mathbb{N} + 10\mathbb{N}$

. $(2,2,r) \longrightarrow \lambda = k(k+1),\ k \in r + 1 + 2\mathbb{N} + r\mathbb{N}$

This solves 7 of the 17 triangles with finite groups for 3D walks

$$\alpha := \sqrt{\lambda_1 + (d/2 - 1)^2} + 1 \in \mathbb{Q}$$

No other value known —> turn to numerical computation

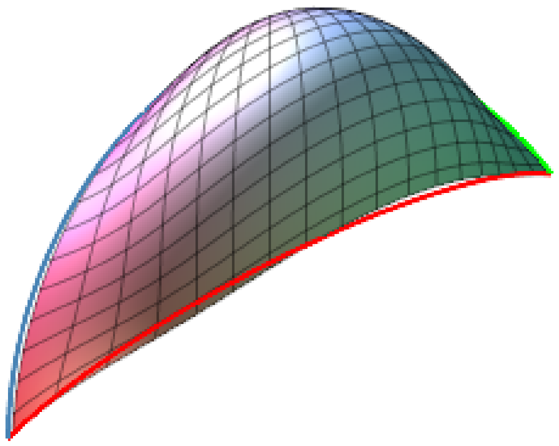
[Berard1983,BogoselPerrollazRaschelTrotignon18]

Bounds from an Approximate Eigenvalue

Eigenvalue problem for Ω :

$\Delta f + \lambda f = 0$ in Ω, $f|_{\partial\Omega} = 0.$

replace by f small



Thm. If $\Delta f^\star + \lambda^\star f^\star = 0$ in Ω, then there exists λ s.t.

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

Method:

1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Cost of certification: not large

[FoxHenriciMoler67,MolerPayne68]

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

Proof

Use

(u_n) orthonormal basis, with $\Delta u_n = \lambda_n u_n$, $u_n|_{\partial_\Omega} = 0$

w solution of $\Delta w = 0$, $w|_{\partial\Omega} = f^\star|_{\partial\Omega}$

If small, $\|w\|$ small

Coefficients of w: $(w, u_n) = (w - f^\star, u_n) + (f^\star, u_n) = \frac{1}{\lambda_n}(w - f^\star, \Delta u_n) + (f^\star, u_n)$,

Δ self-adjoint

$$= \frac{1}{\lambda_n}(\Delta w - \Delta f^\star, u_n) + (f^\star, u_n) = \left(1 - \frac{\lambda^\star}{\lambda_n}\right)(f^\star, u_n).$$

Take λ where min is reached

$$|(w, u_n)| \geq \left|1 - \frac{\lambda^\star}{\lambda}\right| |(f^\star, u_n)|,$$

discrete spectrum with no accumulation point

Square and sum

$$\left(\sup_{x\in\partial\Omega} f^\star\right)^2 \geq \|w\|^2 \geq \left|1 - \frac{\lambda^\star}{\lambda}\right|^2 \|f^\star\|^2.$$

max. ppl

Orthonormality and completeness

[MolerPayne68]

1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 1. Find a good approximate pair $(f^\star, \lambda^\star)$

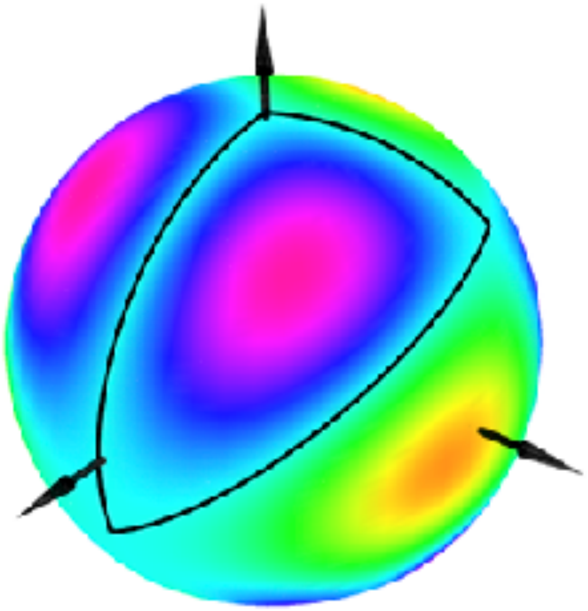
$$\frac{|\lambda - \lambda^\star|}{\lambda} \le \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

High precision needed, and no guarantee

Method of Particular Solutions

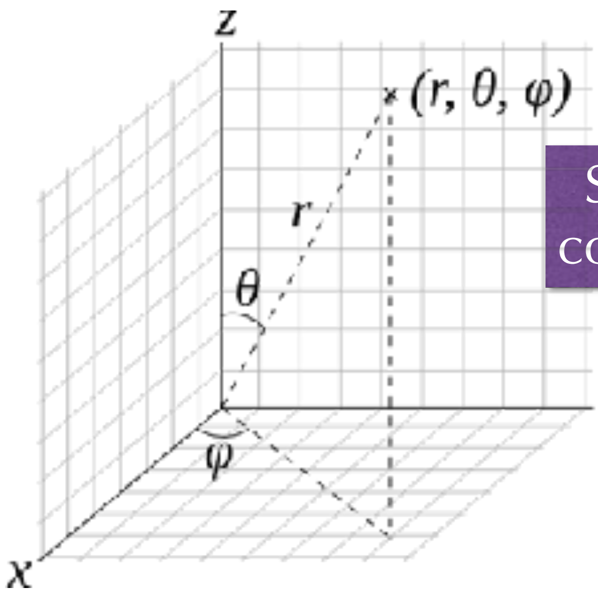
Target: $\Delta f^\star + \lambda^\star f^\star = 0$ in Ω, $\sup_{x\in\partial\Omega} |f^\star|$ small.

1. Fix λ
2. Find a set $(u_\lambda^{(k)})_{k=1}^N$ of solutions of $\Delta f + \lambda f = 0$ in Ω
3. Find a linear combination $\sum_{k=1}^N c_k u_\lambda^{(k)}$ that is
   . small on $\partial\Omega$
   . *not too small on* Ω
4. Repeat to minimize $\sup_{x\in\partial\Omega}$ over λ

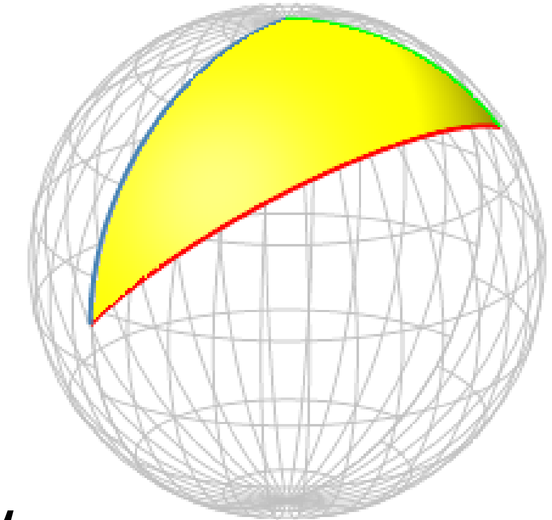


[Bergman47,Vekua48,FoxHenriciMoler67,BetckeTrefethen05,Betcke08]

2. Set of Eigenfunctions



Spherical coordinates



$$\Delta_{\mathbb{S}^2} u(\theta, \phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Separation of variables: $u = f(\phi)g(\theta)$ gives

$$\frac{g''(\phi)}{g(\phi)} + \sin^2\theta\frac{f''(\theta)}{f(\theta)} + \sin\theta\cos\theta\frac{f'(\theta)}{f(\theta)} + \lambda\sin^2\theta = 0$$

$$g(\phi) = \sin(\mu\phi + c), \quad f(\theta) = \mathsf{P}_\nu^\mu(\cos\theta) \quad (\mu \leq 0)$$

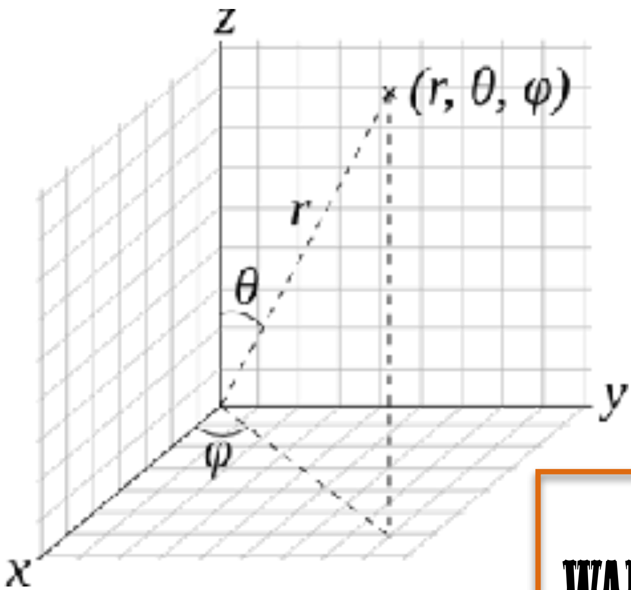
with $\lambda = \nu(\nu+1)$.

Ferrers function of the 1st kind (D-finite; generalize the Legendre functions)

First 2 boundaries:
$$\begin{cases} \phi = 0 & \to c = 0 \\ \phi = \phi_{\max} & \to \mu = \mu_k := -\frac{k\pi}{\phi_{\max}},\ k \in \mathbb{N}\,. \end{cases}$$

$$u_\lambda^{(k)}(\theta, \phi) = \sin(\mu_k\phi)\mathsf{P}_\nu^{\mu_k}(\cos\theta)$$

3. Small on $\partial\Omega$, Not too Small on Ω

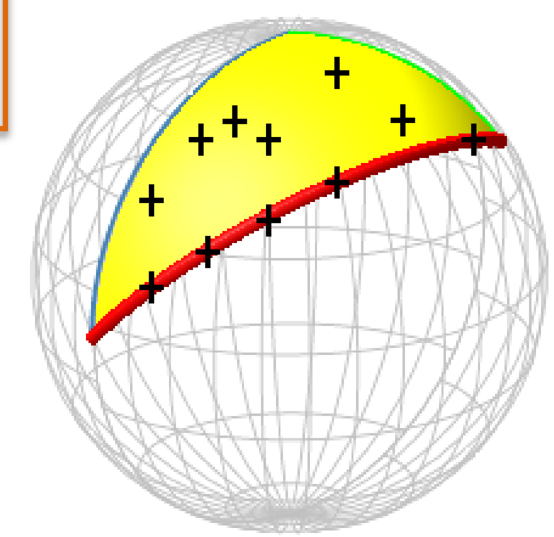


$$u_\lambda(\phi) := \sum_{k=1}^{N} c_k \sin(\mu_k \phi) \mathsf{P}_\nu^{\mu_k}(\cos\theta(\phi))$$

with $\lambda = \nu(\nu+1)$

satisfies
$\Delta u_\lambda + \lambda u_\lambda = 0$

WANTED: c_k s.t. $u_\lambda|_{[0,\phi_{\max}]} \approx 0$, $\|u_\lambda\|_\Omega \approx 1$.



Choose $x_1, \ldots, x_{m_b}$ on $\partial\Omega$, $y_1, \ldots, y_{m_i}$ inside Ω;

Form a matrix $A := \begin{pmatrix} u_\lambda^{(k)}(x_i) \\ u_\lambda^{(k)}(y_i) \end{pmatrix}$

Orthonormal basis of Im A

Compute its QR factorization $A = \begin{pmatrix} Q_{\partial\Omega} \\ Q_\Omega \end{pmatrix} R$

$\sigma := \min_{\|v\|=1} \|Q_{\partial\Omega} v\|$ found together with v by SVD (least squares)

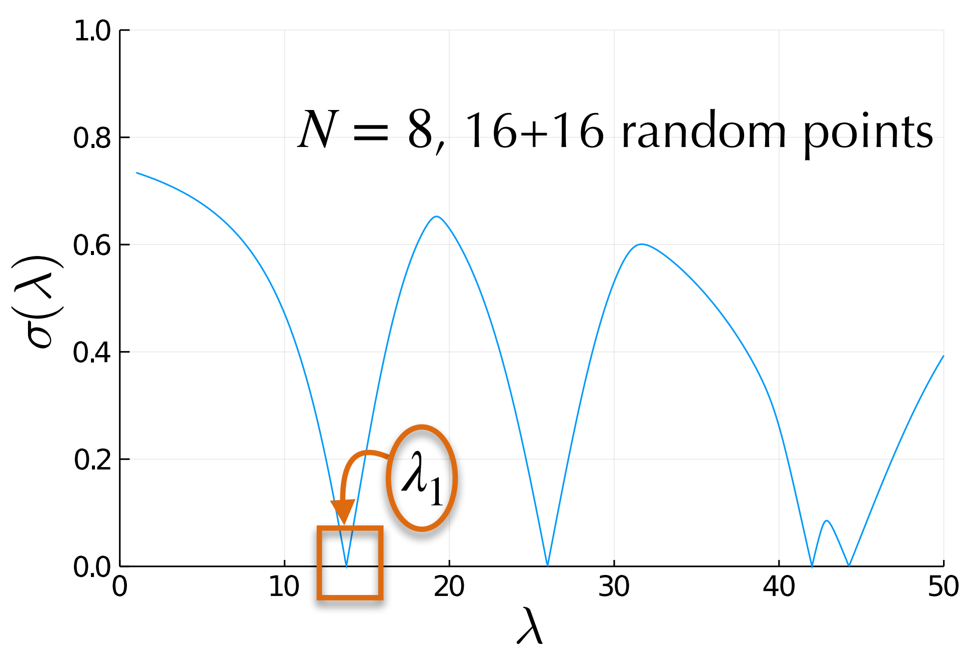
Then
$\|Q_\Omega v\|^2 = 1 - \sigma^2$
not too small

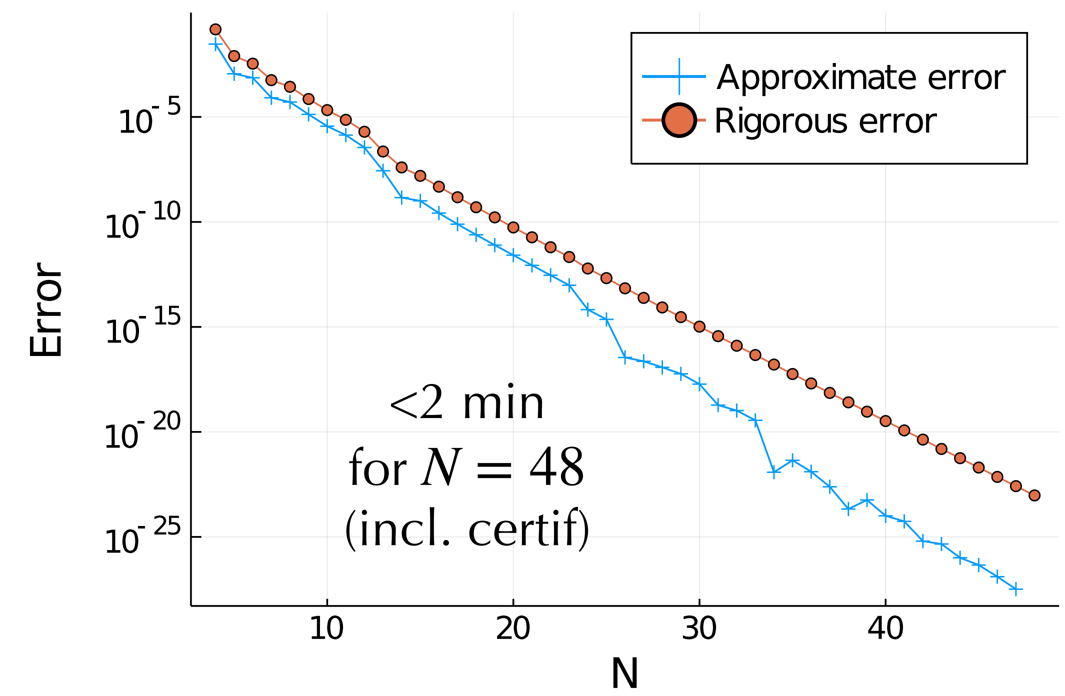
Recover c by solving $Rc = v$.

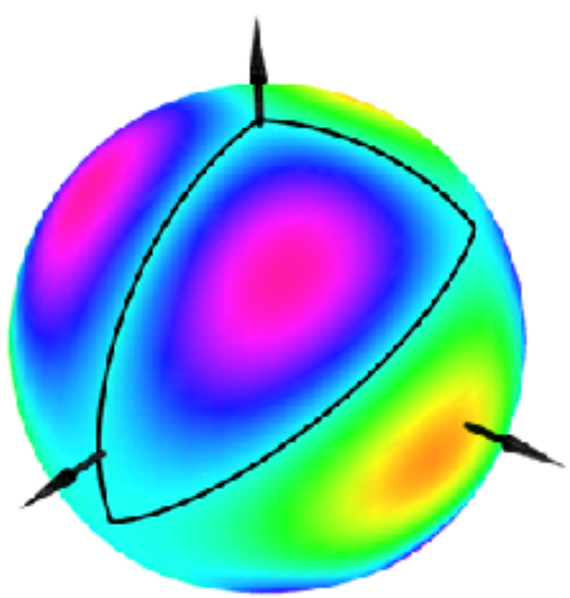
[BetckeTrefethen05]

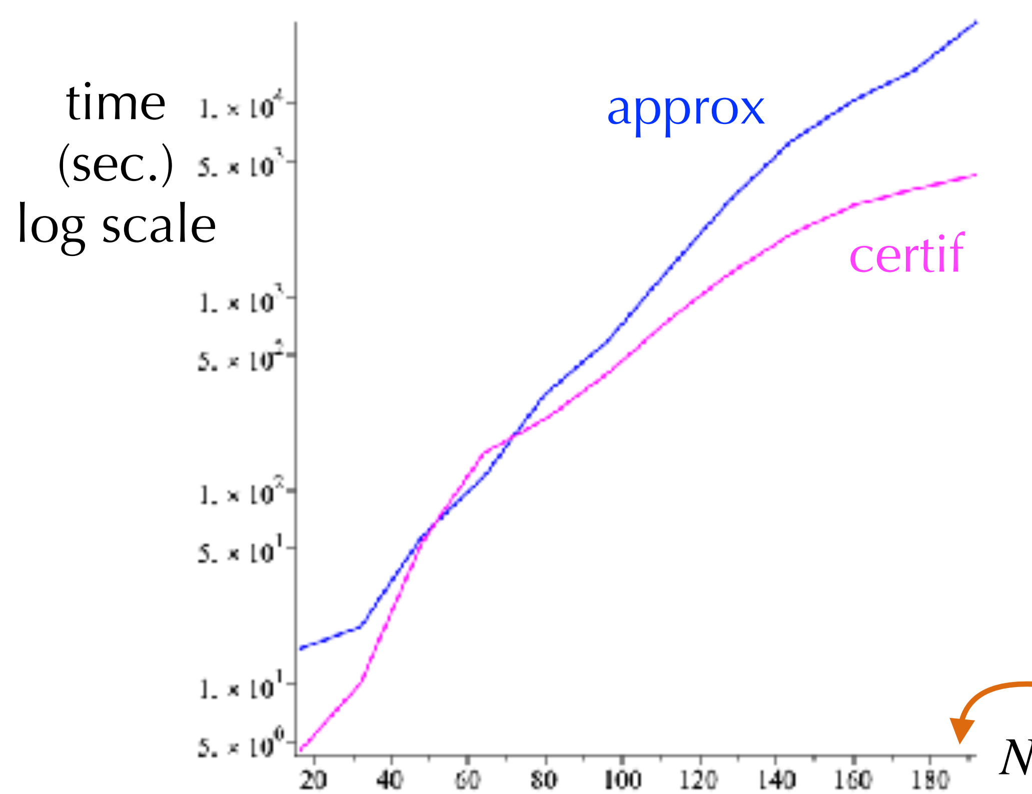
4. Optimize over λ

Ex. Regular Triangle: $(2\pi/3, \pi/3, \pi/2)$



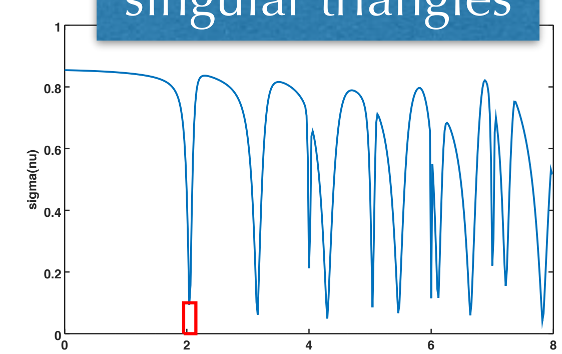






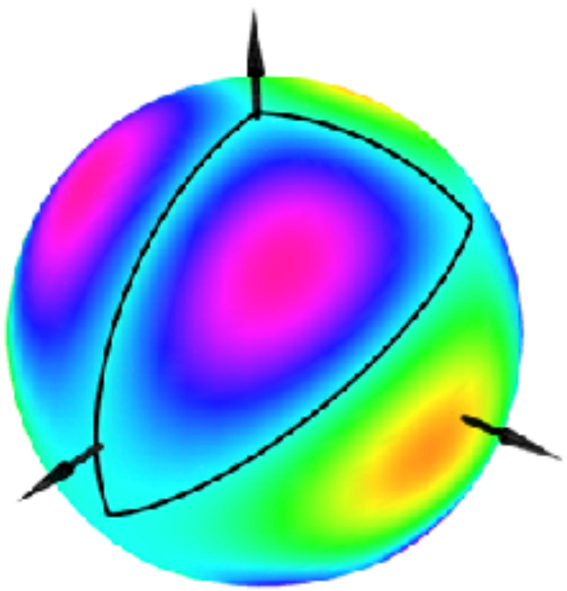
≈ 80 digits

Not so nice for singular triangles



$(\pi/2, 2\pi/3, 3\pi/4)$

Regular vs Singular Triangles



Def. a corner is regular if its angle is π/k, $k \in \mathbb{N}^\star$, singular otherwise.

At a regular corner, eigenfunctions can be continued analytically (by reflection).

Def. Regular triangle: ≤1 singular corners.

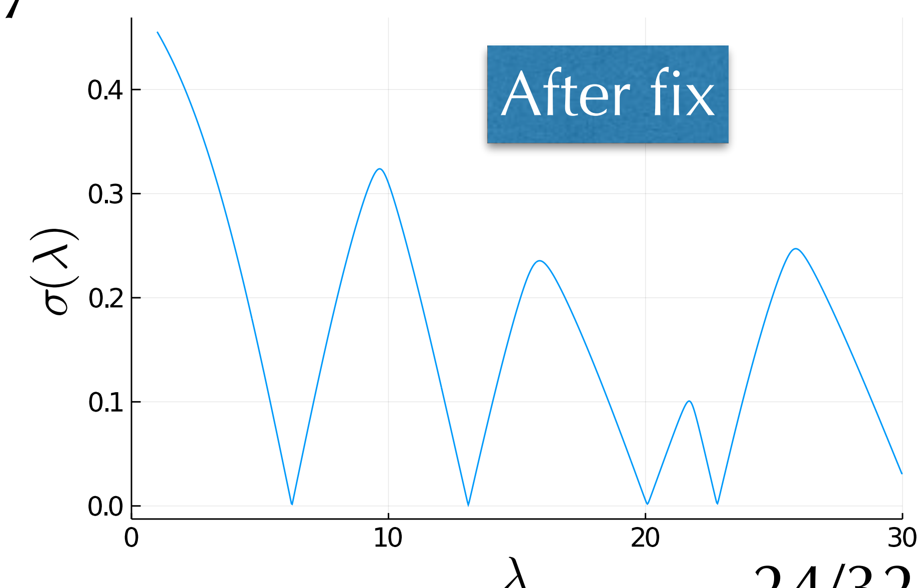
3d-Kreweras is singular

Expansions from a (singular) corner converge well when the other corners are regular, poorly otherwise.

Fix: use a sum of 4 expansions

$$f^\star(\theta,\phi) = u_1(\theta_1,\phi_1) + u_2(\theta_2,\phi_2) + u_3(\theta_3,\phi_3) + u_{\text{int}}(\theta_{\text{int}},\phi_{\text{int}})$$

One from each corner, one from an interior point



1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 2. Rigorous Bounds

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|_2}.$$

Basic Tool: Interval Arithmetic

Replace all floating-point operations by set operations

$$[1.2,1.3] + [2.0,2.1] = [3.2,3.4]$$

$$[1.2,1.3] \times [2.0,2.1] = [2.40,2.73]$$

provides certified enclosures

Implementation requires care with rounding modes

We use https://arblib.org/

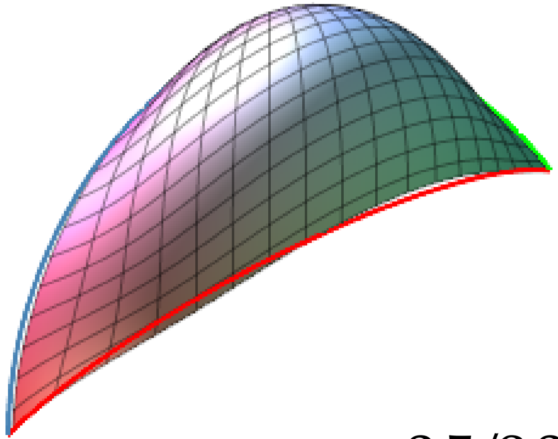
Weakness: wrapping effect

$$f := e^{-t} - (1 - t + t^2/2! + \cdots - t^9/9!)$$

$$f([1.0,1.1]) = [-0.161,0.161] \text{ while } f : [1.0,1.1] \mapsto [2.5\,10^{-7},6.5\,10^{-7}]$$

Situation very similar to our

$$f^\star = \sum c_k u_\lambda^{(k)} \text{ on } \partial\Omega$$



Upper Bound on the Boundary

$$\frac{|\lambda - \lambda^\star|}{\lambda} \le \frac{\sup_{x\in\partial\Omega}|f^\star(x)|}{\|f^\star\|_2}$$

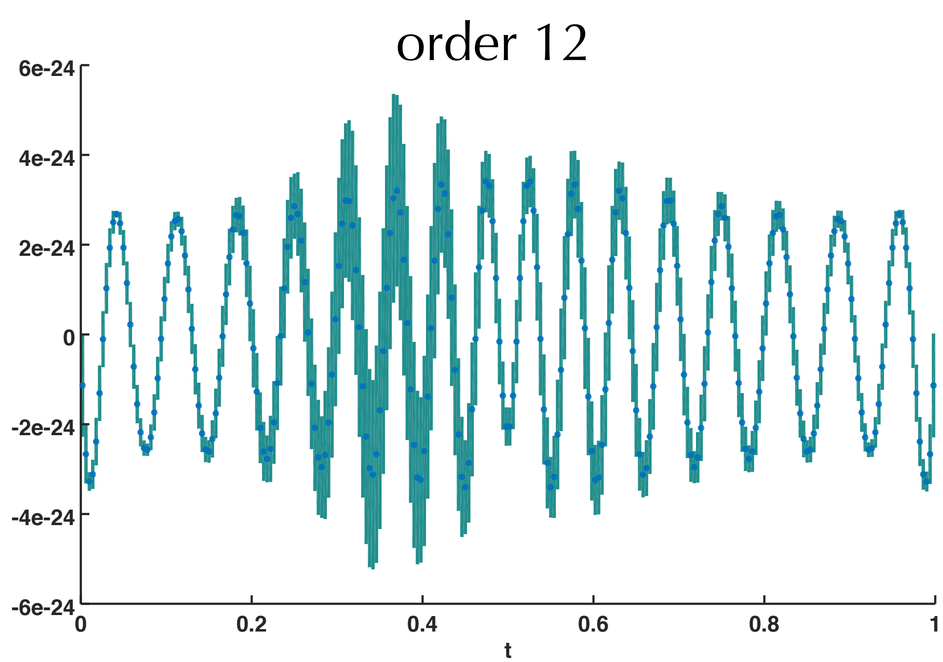
Taylor model:

$$\max_{t\in I} f^\star(\gamma(t)) \le \max_{t\in I} P_{\ell-1}(t) + \frac{(|I|/2)^\ell}{\ell\,!} \max_{t\in I} \left| \frac{d^\ell}{dt^\ell} f^\star(\gamma(t)) \right|$$

Taylor expansion at the midpoint
Coefficients via linear rec.

small

interval evaluation



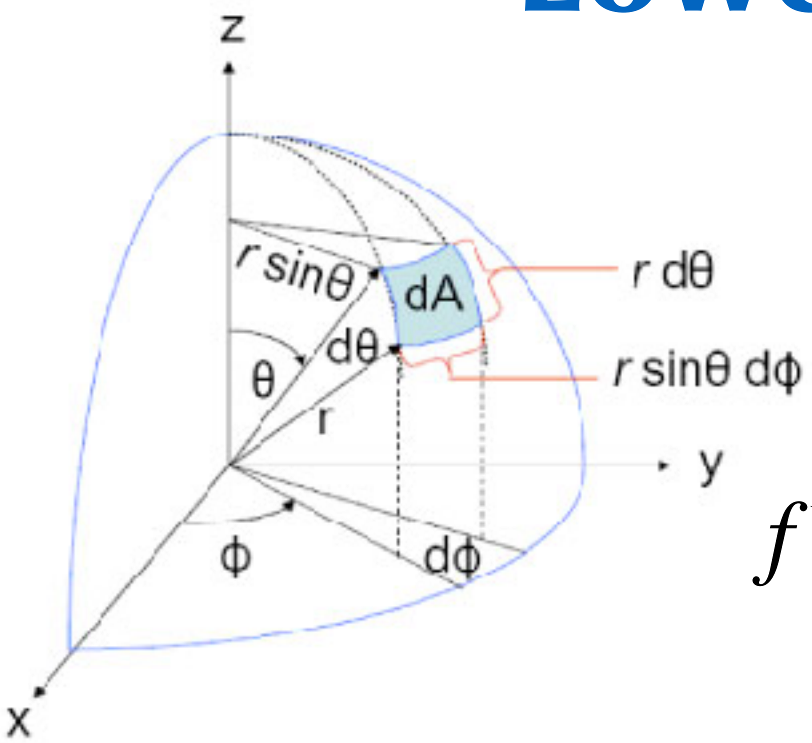
A lower bound on $\min f^\star$ also follows

Interval-evaluate
at enclosures of roots of the
derivative

The expensive part of the certification

[MakinoBerz03]

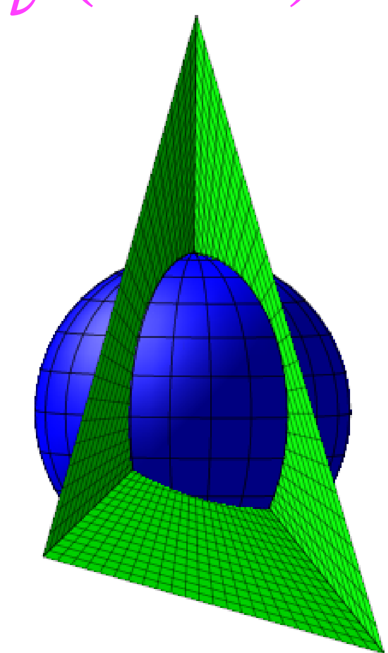
Lower Bound on the Norm



Regular Triangles

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x \in \partial\Omega} |f^\star(x)|}{\|f^\star\|_2}$$

$$f^\star = \sum c_k u_k^{(\lambda)} \qquad u_\lambda^{(k)}(\theta, \phi) = \sin(\mu_k \phi) \mathsf{P}_\nu^{\mu_k}(\cos\theta)$$



$$\int_0^{\phi_{\max}} \int_0^{\theta(\phi)} f^\star(\theta,\phi)^2 \, \sin\theta \, d\theta \, d\phi \geq \int_0^{\phi_{\max}} \int_0^{\theta_{\min}} f^\star(\theta,\phi)^2 \, \sin\theta \, d\theta \, d\phi$$

Orthogonality of $\sin(\mu_k \phi)$

$$\geq \frac{\phi_{\max}}{2} \sum c_k^2 \underbrace{\int_0^{\theta_{\min}} (\mathsf{P}_\nu^{\mu_k}(\cos\theta))^2 \sin\theta \, d\theta}$$

Rigorous interval evaluation (Arb)

Lower Bound on the Norm

Singular Triangles

$$\frac{|\lambda - \lambda^\star|}{\lambda} \leq \frac{\sup_{x\in\partial\Omega}|f^\star(x)|}{\|f^\star\|_2}$$

1. When $f^\star \geq 0$ on $\Omega' \subset \Omega$, $\min_{\Omega'} f^\star \geq \min_{\partial\Omega'} f^\star$

Maximum principle

2. $f^\star|_{\partial\Omega'} > 0$ and Ω' sufficiently small $\Rightarrow f^\star > 0$ on Ω'

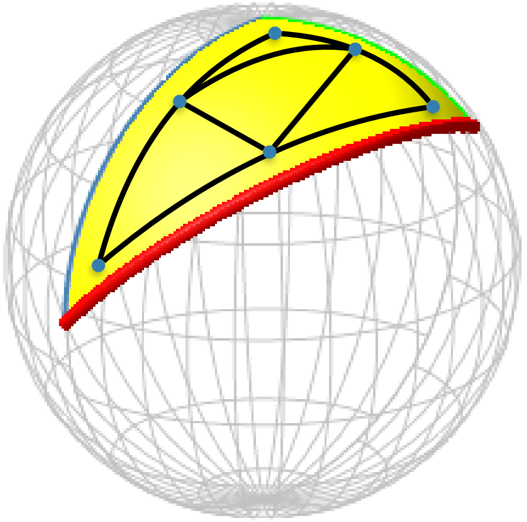
Consequence of Faber-Krahn

$$\Delta f^\star|_{\partial\Omega''} = 0 \text{ for } \Omega'' \subset \Omega' \Rightarrow \mathrm{Vol}\,\Omega'' \geq \mathrm{Vol}\,\Omega^\star_{\lambda^\star}$$

can be computed from zeros of Legendre functions

$\Omega^\star_{\lambda^\star}$: spherical cap with fundamental eigenvalue $\lambda^\star$

Conclusion: subdivide and minimize with Taylor models



4 triangles sufficient in our computations

[Gómez-SerranoOrriols21]

Results

Angles	eigenvalue
$(3\pi/4,\pi/3,\pi/2)$	$12.40005165284337790 5\ldots \pm 10^{-47}$
$(2\pi/3,\pi/3,\pi/2)$	$13.74435521321323183 5\ldots \pm 10^{-84}$
$(2\pi/3,\pi/4,\pi/2)$	$20.57197353798473055 7\ldots \pm 10^{-30}$
$(2\pi/3,\pi/3,\pi/3)$	$21.30940763019044526 0\ldots \pm 10^{-206}$
$(3\pi/4,\pi/4,\pi/3)$	$24.45691379629911169 4\ldots \pm 10^{-73}$
$(2\pi/3,\pi/4,\pi/4)$	$49.10994526328460992 0\ldots \pm 10^{-153}$
$(2\pi/3,3\pi/4,3\pi/4)$	$4.26173475529398708 57\ldots \pm 10^{-22}$
$(2\pi/3,2\pi/3,2\pi/3)$	$5.15914564246654171 12\ldots \pm 10^{-104}$
$(\pi/2,2\pi/3,3\pi/4)$	$6.24174833072633423 68\ldots \pm 10^{-20}$
$(\pi/2,2\pi/3,2\pi/3)$	$6.77710805459830095 73\ldots \pm 10^{-35}$

regular triangles

more work for this one

singular triangles

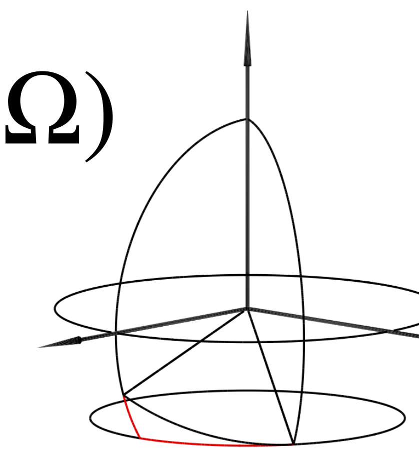
1. Find a good approximate pair $(f^\star, \lambda^\star)$
2. Compute the bound in a certified way
3. Certify the index

Step 3. Certify the index

Certification of the index

monotonicity

$$\lambda < \lambda_2(\Omega') \leq \lambda_2(\Omega) \text{ with } \Omega' \supset \Omega \Rightarrow \lambda = \lambda_1(\Omega)$$



For the domain $0 \leq \phi \leq \phi_{\max}, 0 \leq \theta \leq \theta_{\max}$, the eigenvalues are $\nu(\nu+1)$ with ν root of

P_ν^μ satisfies $((1-x^2)w')' + q_{\mu,\nu}(x)w = 0$, $q_{\mu,\nu}(x) = \nu(\nu+1) - \dfrac{\mu^2}{1-x^2}$

$$\mathsf{P}_\nu^{\mu_k}(\cos\theta_{\max}) = 0$$

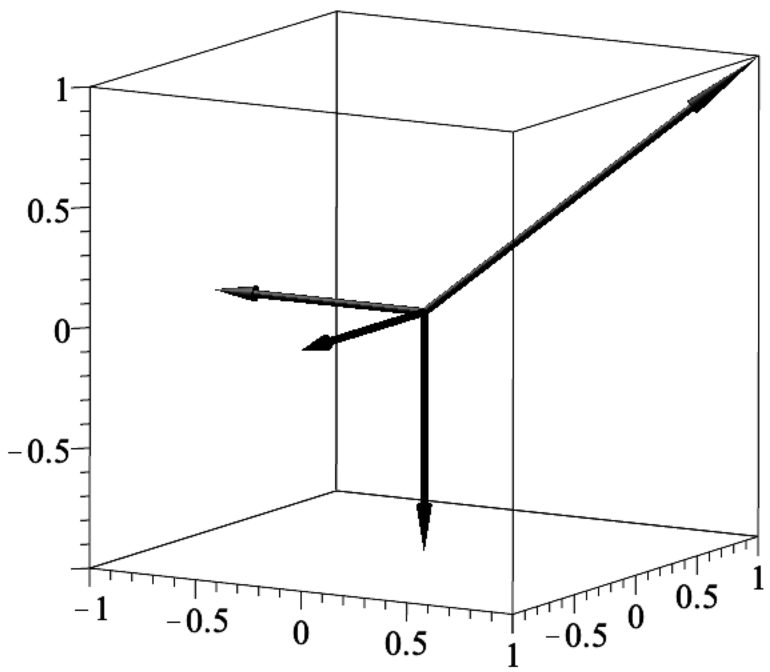
For each $k \in \mathbb{N}^\star$, an infinity of roots $\nu_{k,j}$, $j \in \mathbb{N}^\star$

It is sufficient to check λ against $\nu_{1,2}$ and $\nu_{2,1}$

Proof via Sturm's comparison theorem

Sometimes this fails, but another orientation of the triangle works

Bound on Denominator



$$\#\text{excursions} \sim C\,4^n n^{-\alpha_\kappa}$$

$$\lambda = 5.159145642466541\ldots \pm 10^{-104}$$

certified enclosure

$$\alpha_\kappa = 1 + \sqrt{\lambda + 1/4} = 3.32575700417\ldots \pm 10^{-104}$$

Interval arithmetic

Continued fraction: $\alpha_\kappa = 3 + \cfrac{1}{3 + \cfrac{1}{14 + \cfrac{1}{3 + \cfrac{1}{100 + \cfrac{1}{12 + \cfrac{1}{\ddots}}}}}}$

Stop when a partial quotient cannot be certified

Last convergent: P/Q with $Q = 95716\ldots26933 > 10^{51}$

If $\alpha_\kappa = p/q \in \mathbb{Q}$, then $q > 10^{51}$.

Summary & Conclusion

Linear recurrences with constant coefficients remain mysterious;

lattice walks provide a simple source of examples;

more and more tools are available;

numerical computation can yield rigorous results, useful in experimental mathematics.

Thank you.