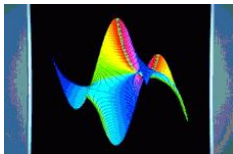


Gfun — 15 Years Later

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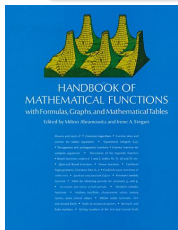
I Introduction

Framework: D-finite Series & Sequences

Definition

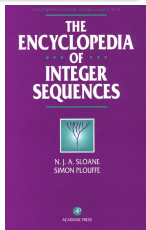
A series $f(x) \in \mathbb{K}[[x]]$ is **D-finite** over \mathbb{K} when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$. (LDE)

A sequence u_n is **D-finite** over \mathbb{K} when its shifts (u_n, u_{n+1}, \dots) generate a finite-dimensional vector space over $\mathbb{K}(n)$. (LRE)



About **25%** of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Gfun [SaZi94]: Maple package to **guess**, **manipulate** and **prove** D-finite identities.

References: Stanley vol. 2, $A = B$ [PeWiZe96]

II Guessing Identities

Mehler's Identity on Hermite Polynomials (1866)

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = ?$$

Answer: compute the first 10 polynomials and guess!



```
> L:= [seq(orthopoly[H](n,x)*orthopoly[H](n,y),n=0..9)];
```

```
L
:= [1, 4xy, (-2 + 4x^2) (-2 + 4y^2), (8x^3 - 12x) (8y^3 - 12y), (12 + 16x^4 - 48x^2) (12 + 16y^4 - 48y^2), (32x^5 - 160x^3
+ 120x) (32y^5 - 160y^3 + 120y), (-120 + 64x^6 - 480x^4 + 720x^2) (-120 + 64y^6 - 480y^4 + 720y^2), (128x^7 - 1344x^5
+ 3360x^3 - 1680x) (128y^7 - 1344y^5 + 3360y^3 - 1680y), (1680 + 256x^8 - 3584x^6 + 13440x^4 - 13440x^2) (1680
+ 256y^8 - 3584y^6 + 13440y^4 - 13440y^2), (512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x) (512y^9 - 9216y^7
+ 48384y^5 - 80640y^3 + 30240y)]
```

```
> deq:=gfun[listtodiffeq](L,F(u),['egf']);
```

```
deq := [ [ (1 - 8u^2 + 16u^4) (d/dx F(u)) + (-4xy + 8ux^2 - 4u + 8uy^2 - 16u^2xy + 16u^3) F(u), F(0) = 1 ], egf ]
```

```
> dsolve(deq[1],F(u)) assuming 0<u,u<1/2;
```

$$F(u) = \frac{e^{\frac{-4xyu + x^2 + y^2}{(2u+1)(2u-1)}} \sqrt{\frac{1}{(-2u+1)(2u+1)}}}{e^{-x^2-y^2}}$$

```
>
```

Guess = Good Approximant

Definition (Padé-Hermite Approximant)

The vector of polynomials (P_1, \dots, P_k) with $\deg P_i \leq d_i$ is a **Padé-Hermite approximant** of type (d_1, \dots, d_k) for a vector of power series (f_1, \dots, f_k) when

$$P_1 f_1 + \dots + P_k f_k = O(x^{d_1 + \dots + d_k + k - 1}).$$

Special cases: (given one series y)

- $k = 2$, $f_1 = -1$, $f_2 = y$: Padé-approximant;
- $f_i = y^{i-1}$, $i = 1, \dots, k$: algebraic approximants;
- $f_i = y^{(i-1)}$, $i = 1, \dots, k$: differential approximants.

Guess = Good Approximant

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The vector of polynomials (P_1, \dots, P_k) with $\deg P_i \leq d_i$ is a **Padé-Hermite approximant** of type (d_1, \dots, d_k) for a vector of power series (f_1, \dots, f_k) when

$$P_1 f_1 + \dots + P_k f_k = O(x^{d_1 + \dots + d_k + k - 1}).$$

Algorithms and complexity ($D = d_1 + \dots + d_k$):

- Linear algebra: $O(D^\omega)$ ops ($\omega \leq 3$ complexity of matrix product);
- minimal basis of approximants in $O(k^\omega D^{1+\epsilon})$ ops [BeLa94];
- genset in $O(k^\omega (D/k)^{1+\epsilon})$ ops [Storjohann06].

III Computing Identities

Proof of Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

- 1 Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$):
recurrence of order 2
- 2 Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!$, $k \in \mathbb{N}$
generated over $\mathbb{Q}(x, n)$ by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order **at most 4**;

- 3 Translation into a differential equation



I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\} :$

> $R_2 := \text{subs}(H=H_2, x=y, R_1);$

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

II. Product

> $R_3 := \text{gfun} :- \text{poltorec}(H(n) \cdot H_2(n) \cdot v(n), [R_1, R_2, \{v(n+1) \cdot (n+1) = v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n));$

$$R_3 := \left\{ c(0)=1, c(1)=4xy, c(2)=8x^2y^2 + 2 - 4y^2 - 4x^2, c(3) = \frac{32}{3}x^3y^3 + 24xy - 16xy^3 - 16x^3y, (16n \right.$$

$$\left. + 16)c(n) - 16xyc(n+1) + (-8n - 20 + 8y^2 + 8x^2)c(n+2) - 4xc(n+3)y + (n+4)c(n+4) \right\}$$

III. Differential Equation

> $\text{gfun} :- \text{rectodiffeq}(R_3, c(n), f(u));$

$$\left\{ (16u^3 - 16u^2yx - 4u + 8uy^2 + 8ux^2 - 4xy)f(u) + (16u^4 - 8u^2 + 1) \left(\frac{d}{du} f(u) \right), f(0)=1 \right\}$$

> $\text{dsolve}(\%, f(u));$

$$f(u) = \frac{\text{Ie} \left(\frac{-4xyu + y^2 + x^2}{(2u-1)(2u+1)} \right)}{e^{(-y^2-x^2)} \sqrt{2u+1} \sqrt{2u-1}}$$

Closure Properties

Theorem (XIXth century)

- *D*-finite series and sequences over \mathbb{K} form \mathbb{K} -algebras;
- *y* algebraic, *f* is *D*-finite $\implies y, f \circ y, \exp \int y$ *D*-finite.

Proof.

Linear algebra in finite dimension. □

Corollary

D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform ($ogf \leftrightarrow egf$).

All implemented in Gfun.

IV Multivariate Functions & Sequences

Definition (D-finiteness)

A series $F(x, y) \in \mathbb{K}[[x, y]]$ is **D-finite** when its derivatives generate a finite-dimensional space over $\mathbb{K}(x, y)$.

More generally, 0-dimensional ideals of operators in suitable Ore algebras [ChSa98].

New Operation: **Creative telescoping** (Zeilberger)

If F satisfies an equation $A(x, \partial_x)F = \partial_y B(x, y, \partial_x, \partial_y)F$, then

$$A(x, \partial_x) \int_{\Omega} F(x, y) dy = 0, \quad \text{for "good" } \Omega.$$

B is called the *certificate* of this identity.

Applications: definite summation and integration, computation of generating functions, extraction of coefficients, . . .

Mgfun: Multivariate Gfun [Chyzak98]

Example: an identity on Bessel Functions [GIMo94]

$$\int_0^{\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = \frac{1}{2\pi a^2} \ln \frac{1}{1-a^4}.$$



```
> libname:="/Users/salvy/lib/maple/Algolib", libname:
> f:=x*BesselJ(1,a*x)*BesselI(1,a*x)*BesselY(0,x)*BesselK(0,x);
      f:=x BesselJ(1, a x) Bessel(1, a x) BesselY(0, x) BesselK(0, x)
> sys:=Mgfun:-dfinite_expr_to_sys(f,y(x)::diff,a::diff));
```

sys

$$\begin{aligned}
 := & \left\{ (-4 a^4 x^4 + 4 x^4 + 3) y(x, a) + 12 x^2 a \left(\frac{\partial^3}{\partial x^2 \partial a} y(x, a) \right) - 4 a^3 x \left(\frac{\partial^4}{\partial x \partial a^3} y(x, a) \right) - 3 x \left(\frac{\partial}{\partial x} y(x, a) \right) \right. \\
 & + 26 a \left(\frac{\partial}{\partial a} y(x, a) \right) - 26 a x \left(\frac{\partial^2}{\partial x \partial a} y(x, a) \right) + 40 a^2 \left(\frac{\partial^2}{\partial a^2} y(x, a) \right) + 6 a^2 x^2 \left(\frac{\partial^4}{\partial x^2 \partial a^2} y(x, a) \right) \\
 & + x^2 \left(\frac{\partial^2}{\partial x^2} y(x, a) \right) - 24 a^2 x \left(\frac{\partial^3}{\partial x \partial a^2} y(x, a) \right) + 8 a^3 \left(\frac{\partial^3}{\partial a^3} y(x, a) \right) - 4 x^3 a \left(\frac{\partial^4}{\partial x^3 \partial a} y(x, a) \right) + x^4 \left(\frac{\partial^4}{\partial x^4} y(x, a) \right), \\
 & \left. 4 x^4 a^3 y(x, a) + a^3 \left(\frac{\partial^4}{\partial a^4} y(x, a) \right) + 3 \left(\frac{\partial}{\partial a} y(x, a) \right) - 3 a \left(\frac{\partial^2}{\partial a^2} y(x, a) \right) + 4 a^2 \left(\frac{\partial^3}{\partial a^3} y(x, a) \right) \right\}
 \end{aligned}$$

```
> deq:=Mgfun:-int_of_sys(sys,x=0..infinity,_takayama_algo);
```

$$\begin{aligned}
 deq := & \left\{ 32 a^3 y(a) + (16 a^6 - 4 a^2) \left(\frac{d^3}{da^3} y(a) \right) + (-a^3 + a^7) \left(\frac{d^4}{da^4} y(a) \right) + (73 a^5 + 3 a) \left(\frac{d^2}{da^2} y(a) \right) \right. \\
 & \left. + (103 a^4 - 3) \left(\frac{d}{da} y(a) \right) \right\}
 \end{aligned}$$

```
> normal(eval(deq,y(a)=1/2/Pi/a^2*ln(1/(1-a^4))));
      {0}
```

```
> sol:=subs(dsolve(deq,y(a)),y(a)) assuming a>0,a<1;
```

$$\begin{aligned}
 sol := & \frac{C1}{a^2} + \frac{C2 \ln((-1+a)(a+1)(a^2+1))}{a^2} + \frac{C3 (\ln(a+1) + \ln(-1+a) - \ln(a^2+1))}{a^2} \\
 & + \frac{1}{a^2} \left(-C4 (2 \operatorname{dilog}(a+1) + 2 \ln(a) \ln(a+1) - 2 \operatorname{dilog}(a) + 2 \ln(a) \ln(-1+a)) + 2 \ln(a) \ln(-1+a) \right) \\
 & + 2 \operatorname{dilog}(-1+a) + 2 \operatorname{dilog}(-1+a) - \ln((-1+a)(a+1)(a^2+1))
 \end{aligned}$$

V Guess as Proof

Guess + Bound = Proof = Algorithm

```
> series(sin(x)2 + cos(x)2, x, 4);
```

$$1 + O(x^4)$$

Why is this a proof?

Guess + Bound = Proof = Algorithm

> `series(sin(x)2 + cos(x)2, x, 4);`

$$1 + O(x^4)$$

Why is this a proof?

- 1 sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
- 2 their squares (and their sum) satisfy a 3rd order LDE;
- 3 the constant 1 satisfies a 1st order LDE: $y' = 0$;
- 4 $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
- 5 it is not singular at 0, Cauchy's theorem concludes.

Same for Creative Telescoping

Zeilberger *et alii* [WiZe92, Yen93, Yen97, MoZe05, ApZe06]:

Bounds for classes of

- hypergeometric sums;
- hyperexponential integrals;
- q -analogues, . . .

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Prop. Linear Recurrences for Algebraic Series (BoChLeSaSc07)

If $P(x, y)$ has degree at most d then the coefficients of its series solutions obey a recurrence of order at most $(d + 1)^2$.

(First observed experimentally, then proved rigorously).

Proof by bounds for creative telescoping on

$$\frac{1}{2\pi i} \oint \frac{yP_y}{P} dy.$$

VI Conclusion

In the Next Versions of Gfun

- Minimal recurrences and differential equations by removing apparent singularities [ChDuLeMaMiSa07];
- Better/faster guessing;
- Initial conditions at singular points (see also <http://algo.inria.fr/esf>);
- Fast evaluation of D-finite sequences.

Do not hesitate to ask for more, or provide code!