Polynomial and Rational Solutions of Linear Differential or Difference Equations

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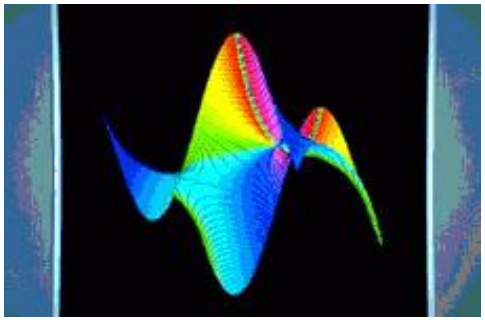


Joint work with Alin Bostan and Thomas Cluzeau

Liouville (1833) did most of it!

$$a_0(x)y^{(d)}(x) + \cdots + a_d(x)y(x) = 0.$$

- Polynomial solutions:

  Si l'on se bornait à demander les intégrales entières, le problème n'offrirait aucune difficulté.

- Algorithm for rational solutions, later improved by Abramov *et alii*.

Our result: better complexity (still exponential).

Timings

$$(2x+1)^3 y'' + (2x+1)(8x+3)y' + 2N((4x+2)N - 4x - 1)y = 0$$

N	LFS	BinSplit	BsGs
2^{10}	1.58	0.10	0.02
2^{12}	62.59	0.44	0.03
2^{14}	4597.2	2.40	0.07
2^{16}	$>$ 4Gb	14.67	0.19
2^{18}		89.05	0.44
2^{20}		528.73	1.07
2^{22}		3060.1	2.54
2^{24}		$> 1h$	6.09

LFS: Liouville's method (Maple) $\tilde{O}(N^2)$

BinSplit: Our deterministic algo (Maple) $\tilde{O}(N)$

BsGs: Our probabilistic algo (Magma) $\tilde{O}(\sqrt{N})$

Output different

Applications

- Indefinite hypergeometric summation [Gosper78]

$$\sum_{k=0}^{n} \frac{(3k)!}{k!(k+1)!(k+2)!27^k} = \frac{(81n^2+261n+200)(3n+2)!}{40(n+2)!(n+1)!n!27^n} - \frac{9}{2}$$

- Definite summation and integration [Zeilberger90, Chyzak00]

$$\sum_{n=0}^{\infty} J_{2n+1/2}(x) = \int_0^x \frac{\cos t}{\sqrt{2\pi t}}\, dt$$

- Liouvillian solutions of LDEs [Marotte1898, Kovacic86, Singer81,...]
- Hypergeometric solutions of LREs [Petkovšek90]
- Desingularization of LDEs and LREs [ChDuLeMaMiSa05]

They all

- need rational or polynomial solutions of LDEs or LREs;
- waste time when none exists.

Basic Algorithm

Liouville: bound on degree and undeterminate coefficients

$$\begin{pmatrix} * & * & * & & & & \\ * & * & * & * & & & \\ & * & * & * & * & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & * & * & * & * \\ & & & & * & * & * \end{pmatrix}$$

Abramov *et alii*: alternate view via recurrence

$$(n+3)(n+2)u_{n+3} + \cdots + 8(n-N)(n-N+1)u_n = 0$$

and basis of power series solutions of the LDE

$$1 + 0x + \cdots + *x^N + *x^{N+1} + *x^{N+2} + *x^{N+3} + O(x^{N+4})$$
$$0 + 1x + \cdots + *x^N + *x^{N+1} + *x^{N+2} + *x^{N+3} + O(x^{N+4})$$

Same Complexity

Singular Points & Indicial Polynomial

$$\mathcal{L}y(x) = a_0(x)y^{(d)}(x) + \cdots + a_d(x)y(x) = 0.$$

- Cauchy: singular points only at roots of a_0.
- Indicial polynomial $P_\alpha(\lambda)$:

$$\mathcal{L}(x-\alpha)^\lambda(1 + c_1(x-\alpha) + \cdots) = P_\alpha(\lambda)(x-\alpha)^{\lambda+c}(1 + \cdots).$$

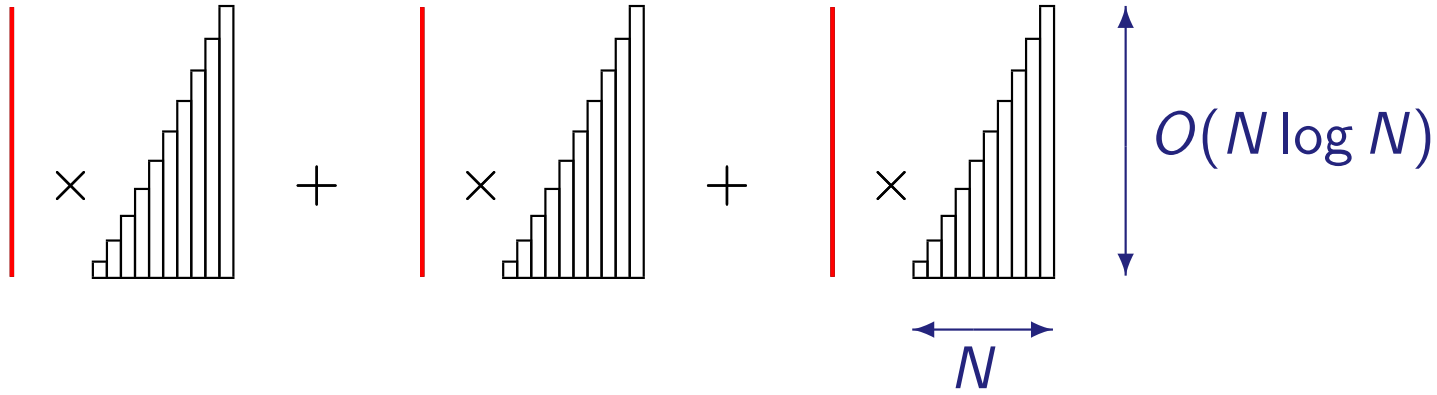
$$\underbrace{(n+3)(n+2)}_{P_0(n+3)}\, u_{n+3} + \cdots + 8\underbrace{(n-N)(n-N+1)}_{P_\infty(n)}\, u_n = 0$$

$\longrightarrow$ Exponential bound on degree and orders of poles

$\longrightarrow$ Rational solutions [Liouville1833]: patch up local polar behaves

$$y(x) := \tilde{y}(x) \prod_{\substack{a_0(\alpha)=0,\\ P_\alpha(N_\alpha)=0,\\ N_\alpha\in\mathbb{Z}^-}} (x-\alpha)^{N_\alpha}.$$

Shape of Polynomial Solutions



- Size of Solutions: $O(N^2 \log N)$ bits
- Compact representationCompact representation: $O(N \log N)$ bits

Theorem (BoCISa05)

Our algorithm BinSplitPolySols *computes compact representation and degree in* $\tilde{O}(N)$ *bit operations. Knowing the degree* D*, the expanded polynomial is computed in* $\tilde{O}(D^2)$ *bit operations.*

Quasi-optimal!

Fast Multiplication

- Balanced input: $\text{size}\, T \times \text{size}\, T$
  - Naïve: $I(T) = O(T^2)$
  - Karatsuba (1963): $I(T) = O(T^{1.59})$
  - Fast Fourier Transform: $I(T) = O(T \log T \log\log T)$ [Schönhage-Strassen71]
- Same for polynomials ($M(T)$).
- Many applications via Newton iteration, including division.

Algorithms	Ours	Before
Degrees (probabilistic)	$O(M(\sqrt{N})I(\log N))$	—
Compact and degree	$O(I(N \log N) \log N)$	$O(N^2 I(\log N))$
Expanded, degree D	$O(DI(D \log D))$	

Factorial

Problem (Fast computation of $N! = 1 \times \cdots \times N$)

Naïve way: complexity $O(N^2 I(\log N))$

Binary Splitting:

$$N! = \underbrace{(1 \times \cdots \times \lfloor N/2 \rfloor)}_{\text{size } \frac{1}{2} N \log N} \times \underbrace{(\lceil N/2 \rceil \times \cdots \times N)}_{\text{size } \frac{1}{2} N \log N}$$

and recurse.
Complexity $O(I(N \log N) \log N)$.
Numerous applications [Hakmem72, Brent75, Chudnovsky^{2}88]
Even faster way [Borwein85,Schönhage94] $O(I(N \log N))$.

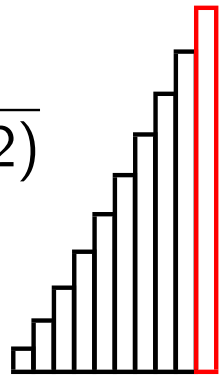
Binary Splitting for Polynomial Solutions

$$* + *x + \cdots + *x^N + *x^{N+1} + *x^{N+2} + *x^{N+3} + O(x^{N+4})$$

$$(n+3)(n+2)u_{n+3} + \cdots + 8(n-N)(n-N+1)u_n = 0$$

First order recurrence on vectors $U_n = {}^t(u_{n+3}, u_{n+2}, u_{n+1})$:

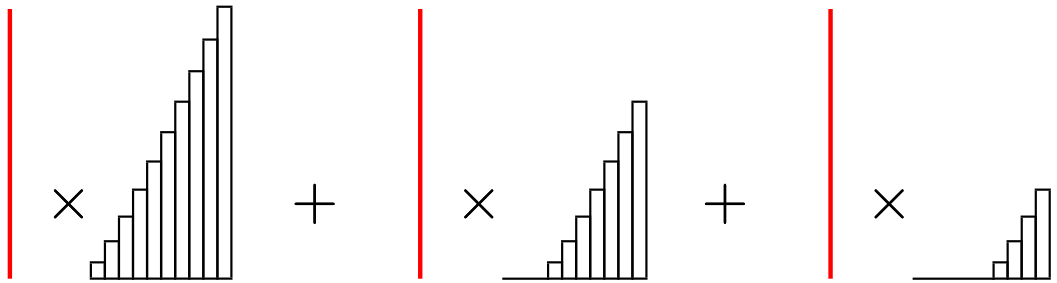
$$U_N = \underbrace{\begin{pmatrix} * & * & * \\ (N+3)(N+2) & 0 & 0 \\ 0 & (N+3)(N+2) & 0 \end{pmatrix}}_{A(N)} \frac{U_{N-1}}{(N+3)(N+2)}$$



$$= \underbrace{A(N)\cdots A(-1)}_{\textbf{matrix factorial}} \frac{U_{-2}}{(N+2)(N+1)!^2}.$$

Complexity: like for $N!$.

Operations on the Compact Representation



Compact Representation: recurrence & generalized initial conditions.

- Division by a power of x: easy
- Evaluation: linear recurrence for $v_k = \sum_{n \le k} u_n x^n$
  $\longrightarrow v_N$ by binary splitting (for x algebraic).
  Applications: $\exp(1), \gamma, \pi, \ldots$ [Hakmem,Brent,ChCh]
- Evaluation of a derivative: *idem*

$\longrightarrow$ Compact representation at another point

$\longrightarrow$ Rational solutions.

From Recurrence to Recurrence

$$a_0(n)u(n+d) + \cdots + a_d(n)u(n) = 0.$$

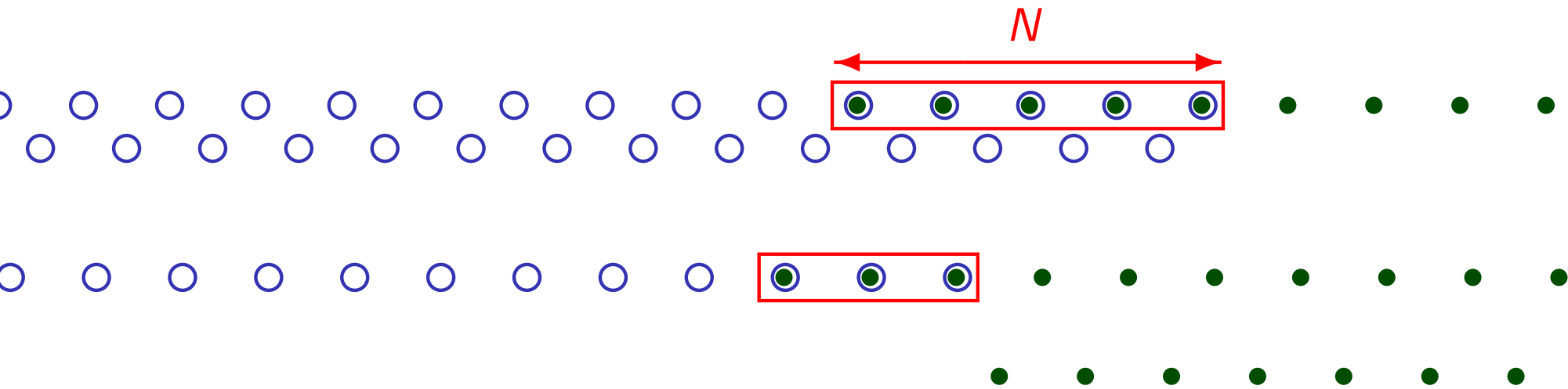
In general, coefficients of u do not satisfy a linear recurrence.
They do on a binomial basis [Abramov-Bronstein-Petkovšek 95]:

$$u(n) = \sum_{k=0}^{N} c_k \binom{n-a}{k}.$$

$\longrightarrow$ same algorithm for the compact representation

Poles of Rational Solutions

$$a_0(n)a_0(n)u(n+d) + \cdots + a_d(n)a_d(n)u(n) = 0.$$



Rational solutions

Abramov 89: bound the poles and look for polynomials.

Probabilistic Algorithm

Theorem (BoCISa05)

Given $c \geq 0$, our algorithm BsGsPolySols *computes the degrees of polynomial solutions with $\tilde{O}(\sqrt{N})$ bit operations. The result is correct with probability $\geq 1 - 1/(2\log^c N)$.*

Idea:

1. Compute the matrix factorial
   – with only $O(\sqrt{N})$ operations
   – modulo a prime of bit size only $O(\log N)$
2. Bound the probability that the rank drops.

Baby-Steps/Giant-Steps

Problem ($N!$ mod p in less than N operations)

Naïve: N arithmetic operations

Strassen 76: $\tilde{O}(\sqrt{N})$.

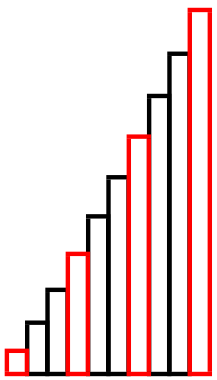
1. Compute $Q(x) = (x+1)\cdots(x+k)$ — $O(M(k)\log k)$;
2. Evaluate $Q(0), Q(k), \ldots, Q(N-k)$ — $O(NM(k)\log(k)/k^2)$;
3. Compute their product — $O(N/k)$.

Recurrences via matrix factorials [Chudnovsky^{2}88]:

$$O(m^\omega M(\sqrt{dN}) + m^2 M(\sqrt{dN})\log(dN))$$

We improve this to

$$O(m^\omega\sqrt{dN} + m^2 M(\sqrt{dN}))$$



Multipoint Evaluation

Problem (Evaluate P at $x_1, \ldots, x_d$, $\deg P = d$.)

Naïve: $O(d^2)$ operations.

Borodin & Moenck 74: Recursive computation

$$Q_i^j := (x - x_i) \cdots (x - x_j)$$

$$P(x_i) = \begin{cases} (P \bmod Q_1^{\lfloor d/2 \rfloor})(x_i), & i \leq d/2 \\ (P \bmod Q_{\lceil d/2 \rceil}^{d})(x_i), & i > d/2. \end{cases}$$

$$\text{Complexity: } C(d) = \underbrace{2C(d/2)}_{\text{recursion}} + \underbrace{2M(d/4)}_{Q} + \underbrace{\frac{7}{2}M(d)}_{\text{divisions}}$$

$$= O(M(d) \log d).$$

Better constant in [Bostan-Lecerf-Schost 03] (Tellegen).

Polynomial Extrapolation

Problem $((P(0),\dots,P(d)) \mapsto (P(a),P(a+1),\dots,P(a+d)),$ $\deg P = d)$

Interpolation / evaluation: $O(M(d)\log d)$.

Bostan-Gaudry-Schost 03:

$$P(a+k) = \sum_{i=0}^{d} P(i)\frac{\prod_{j\neq i}(a+k-j)}{\prod_{j\neq i}(j-i)}, \quad \text{(Lagrange)}$$

$$= \prod_j (a+k-j)\ \underbrace{\sum_{i=0}^{d} \frac{P(i)}{\prod_{j\neq i}(j-i)}\frac{1}{a+k-i}}_{[x^k]\sum_{i=0}^{d}\frac{P(i)}{\prod_{j\neq i}(j-i)}x^i \sum_{i=0}^{2d}\frac{1}{a+i-d}x^i}.$$

Complexity: $O(M(d))$.

Faster $N!$ mod p

Bostan-Gaudry-Schost 03:

- Recall Strassen:
  1. Compute $Q(x) = (x+1)\cdots(x+k)$ $\quad O(M(k)\log k)$;
  2. Evaluate $Q(0), Q(k), \ldots, Q(N-k)$ $\quad O(NM(k)\log(k)/k^2)$;
- Recursion:
  from $(x+1)\cdots(x+k/2)$ evaluated at $(0,\ldots,(N-k)/2)$,
  3 extrapolations.
- Complexity ($k=\sqrt{N}$): $O(M(k))$
- Generalizes to matrix factorials in arbitrary degree.

Probability Estimate

Kernel of a matrix factorial

Lemma (Zippel)

A, B positive integers. If p prime is chosen uniformly at random in $\mathcal{S} = \{B, \ldots, 2B\}$ then $\Pr(p|A) \leq 2\log(A)/B$.

Proof:

- Number of distinct divisors of A in $\mathcal{S}$: $< \log(A)/\log(B)$
- Size of $\mathcal{S}$: $\geq B/2\log(B)$.

Matrix factorials:

Size of the entries $O(N \log N)$, size of minors *idem*.

Precise estimates for B available

(and required for an implementation).

Conclusion

- Compact representation & fast operations
  $\longrightarrow$ quasi-optimal algorithms for rational functions.
- Also, a criterion in polynomial complexity for a large class.
- Extensions
  - Non-homogeneous case: $\mathcal{L}y = r$
  - Parameterized non-homogeneous: $\mathcal{L}y = \lambda_1 r_1 + \cdots + \lambda_k r_k$

  $\longrightarrow$ Faster Zeilberger, Chyzak.
  - Possible extension to linear differential systems.