Automatic Proofs of Identities

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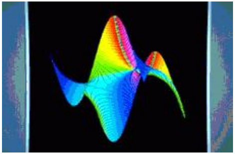
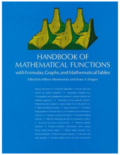
Algorithms Project, Inria

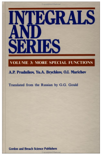


Séminaire Philippe Flajolet, May 26, 2011

I Introduction

Dynamic Dictionary of Mathematical Functions



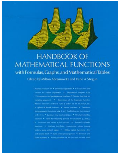


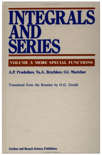
1. In the beginning, there were handbooks of identities.

   Among the most cited documents in the scientific literature.

   Thousands of useful mathematical formulas,
   computed, compiled and edited by hand.

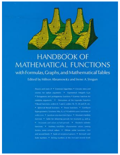
Dynamic Dictionary of Mathematical Functions

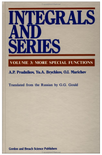




1. In the beginning, there were handbooks of identities.
2. Then, came computer algebra. Computation with exact mathematical objects.
   Several million users.
   30 years of algorithmic progress in effective mathematics.

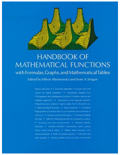
Dynamic Dictionary of Mathematical Functions

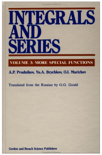




1. In the beginning, there were handbooks of identities.
2. Then, came computer algebra.
3. Last, came the Web. New kinds of interaction with documents.

Dynamic Dictionary of Mathematical Functions





1. In the beginning, there were handbooks of identities.
2. Then, came computer algebra.
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Aim of the project

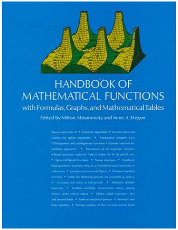
DDMF = Mathematical Handbooks + Computer Algebra + Web

1. Develop and use computer algebra algorithms to generate the formulas;
2. Provide web-like interaction with the document and the computation.

http://ddmf.msr-inria.inria.fr/

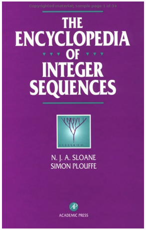
Equations Are a Good Data Structure

- Classical:
  polynomials represent their roots better than radicals.
  Algorithms: Euclidean division and algorithm, Gröbner bases.
- Recent:
  same for linear differential or recurrence equations.
  Algorithms: non-commutative analogues.



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Examples of Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \quad \text{[Strehl92]}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{[GlMo94]}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \quad \text{[Doetsch30]}$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{[Andrews74]}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{[Paule85].}$$

More Identities

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle, \quad \text{[Frobenius1910}$$

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \quad \text{[Gessel03]}$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha),$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

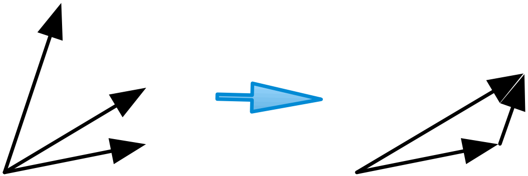
- 1st slide: Zeilberger's algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: recent generalization of previous ones (with Chyzak & Kauers).

Ideas

Confinement in finite dimension + Creative telescoping.

II Confinement in Finite Dimension

Confinement Provokes Identities



Obvious

$k+1$ vectors in dimension $k \to$ an identity.

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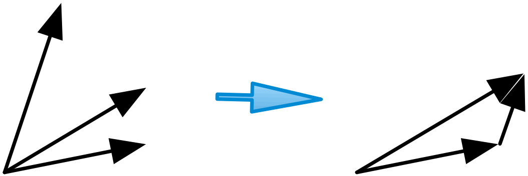

Obvious

$k+1$ vectors in dimension $k \to$ an identity.

Idea: confine a function and all its derivatives.

First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$$O(x^4)$$

Why is this a proof?

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Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

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What about $\sin' = \cos$?

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Second algorithmic proof (same idea): $F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$

```
> for n to 5 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
```

Third Proof: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

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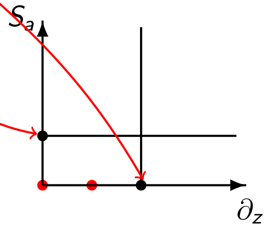

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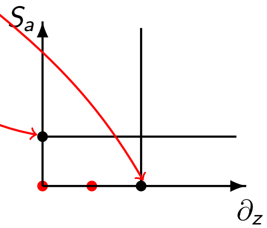


Gauss 1812: contiguity relation.

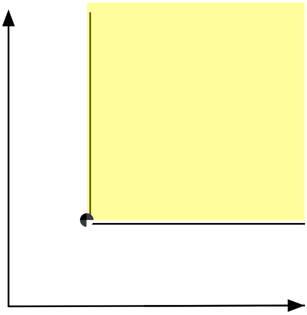
dim=2 $\Rightarrow$ S_a^2F, S_aF, F linearly dependent:

(Coordinates in $\mathbb{Q}(a,b,c,z)$.)

$$(a+1)(z-1)S_a^2F+((b-a-1)z+2-c+2a)S_aF+(c-a-1)F=0.$$

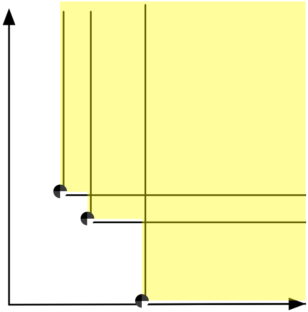
Gröbner Basis: Euclidean Division in Several Variables

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.



Gröbner Basis: Euclidean Division in Several Variables

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.
2. Gröbner basis of a (left) ideal $\mathcal{I}$: corners of stairs.



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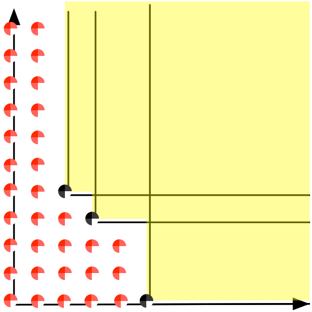

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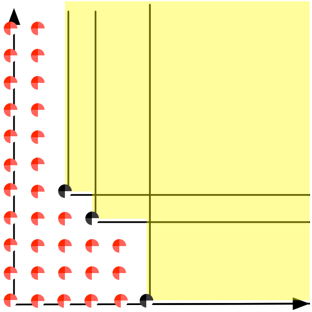
4. Reduction of P: Rewrite P mod $\mathcal{I}$ on this basis.



$\rightarrow$ An access to (finite dimensional) vector spaces

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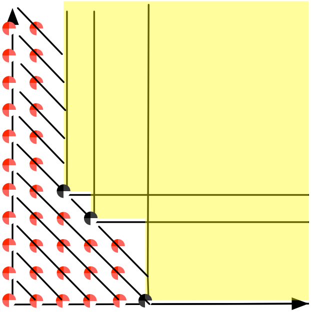

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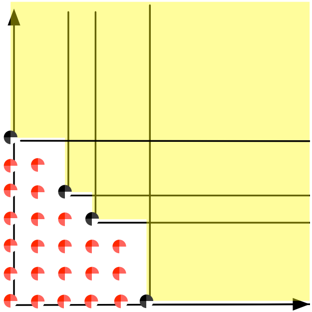
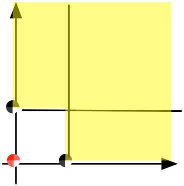
6. D-finiteness: $\dim = 0$.



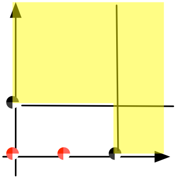
$\rightarrow$ An access to (finite dimensional) vector spaces

Examples

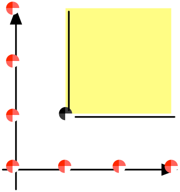
Binomial coeffs $\binom{n}{k}$ wrt S_n, S_k
Hypergeometric sequences



Bessel $J_\nu(x)$ wrt S_ν, ∂_x
Orthogonal pols wrt S_n, ∂_x

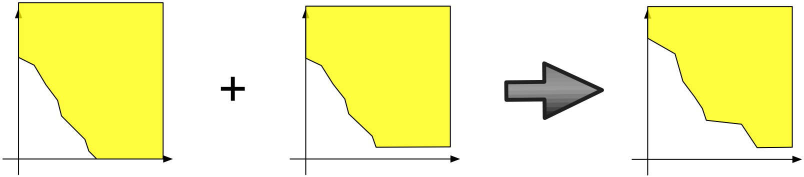


Stirling nbs wrt S_n, S_k



Abel type wrt S_m, S_r, S_k, S_s
$\mathrm{hgm}(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$
dim $= 2$ in space of dim 4.

Closure Properties



Proposition

$$\dim \operatorname{ann}(f+g) \le \max(\dim \operatorname{ann} f, \dim \operatorname{ann} g),$$
$$\dim \operatorname{ann}(fg) \le \dim \operatorname{ann} f + \dim \operatorname{ann} g,$$
$$\dim \operatorname{ann} \partial f \le \dim \operatorname{ann} f.$$

Algorithms by linear algebra.

Fourth Algorithmic Proof: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x, n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\rightarrow$ recurrence of order at most 4;
3. Translate into differential equation.

III Creative Telescoping

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping (Zeilberger 90)

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum “telescopes”, leading to $A(n, S_n) \cdot F_n = 0.$

Creative Telescoping (Zeilberger 90)

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

IF one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = 0.$

Creative Telescoping (Zeilberger 90)

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IF one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = 0.$

> *Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*
>
> Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

Diff. under $\int$ + Integration by Parts $\to$ Algorithm?

Ex.: $\displaystyle\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,\ J_0(0) = 1).$

Diff. under $\int$ + Integration by Parts $\to$ Algorithm?

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$$I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt, \quad I'(z) = \int_0^1 -t\frac{\sin zt}{\sqrt{1-t^2}}\,dt,$$

$$I''(z) = \int_0^1 -t^2\frac{\cos zt}{\sqrt{1-t^2}}\,dt = -I(z) + \int_0^1 \sqrt{1-t^2}\cos zt\,dt,$$

$$I''(z) + I(z) = \underbrace{\left[\sqrt{1-t^2}\frac{\sin zt}{z}\right]_0^1}_{0} + \int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{\sin zt}{z}\,dt = -\frac{I'(z)}{z}.$$

Diff. under $\int$ + Integration by Parts → Algorithm?

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$$I''(z)=\int_0^1 -t^2\frac{\cos zt}{\sqrt{1-t^2}}\,dt = -I(z)+\int_0^1 \sqrt{1-t^2}\cos zt\,dt,$$

$$I''(z)+I(z)=\underbrace{\left[\sqrt{1-t^2}\frac{\sin zt}{z}\right]_0^1}_{0}+\int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{\sin zt}{z}\,dt = -\frac{I'(z)}{z}.$$

$$\operatorname{ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,\partial_z)}_{\text{no } t,\partial_t} - \partial_t \underbrace{\frac{t^2-1}{t}\partial_z}_{\text{anything}}$$

Diff. under $\int$ + Integration by Parts $\rightarrow$ Algorithm?

Ex.: $$\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,\ J_0(0) = 1).$$

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Creative Telescoping

Input: generators of (a subideal of) ann f;
Output: A, B such that $A - \partial_t B \in \operatorname{ann} f$, A free of t, ∂_t.
Algorithm: sometimes. (Why would they exist?)

Telescoping of $\mathcal{I}$ wrt t:

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle \partial_z, \partial_t \rangle) \cap \mathbb{Q}(z)\langle \partial_z \rangle.$$

Diff. under $\int$ + Integration by Parts → Algorithm?

Ex.: $$\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z), \quad (\underbrace{zJ_0''+J_0'+zJ_0}_{A(z,\partial_z)\cdot J_0}=0,\ J_0(0)=1).$$

$$\operatorname{ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,\partial_z)}_{\text{no } t,\partial_t} - \partial_t \underbrace{\frac{t^2-1}{t}\partial_z}_{\text{anything}}$$

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Note: holonomy is a sufficient condition for

$$0 \neq (\mathcal{I} + \partial_t \mathbb{Q}(z)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle.$$

Example: Pascal's Triangle Again

$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (\underbrace{S_n - 2}_{\text{no } k,\, S_k} + (S_k-1)(S_n-1))\cdot\binom{n}{k}.$$

Sum over $k \Rightarrow (S_n - 2)\sum_k \binom{n}{k} = 0.$

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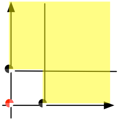
Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1,\quad S_n \to \frac{n+1}{n+1-k}1,\quad S_k \to \frac{n-k}{k+1}1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1,\quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1,$$

$$S_nS_k \to \frac{n+1}{k+1}1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41)$.

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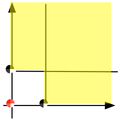

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This has to happen for some degree: $\deg D_s = O(s)$.

Polynomial Growth

Definition (Polynomial Growth p)

There exists a sequence of polynomials P_s, s.t. for all $(a_1,\dots,a_k)$ with $a_1+\dots+a_k\le s$, $P_s\partial_1^{a_1}\cdots\partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

Theorem (ChyzakKauersSalvy2009)

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$$\dim T_t(\mathcal{I}) \leq \max(\dim \mathcal{I} + p - 1, 0).$$

Proof. Same as above. Set $q := \dim \mathcal{I} + p$.

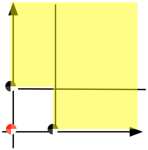
- In degree s, dim $O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \ldots, \partial_{i_q}$: $O(s^{q+1})$;

$\Rightarrow$ any q variables linearly dependent $\Rightarrow$ dim $\leq q - 1$.

This proof gives an algorithm. Also, bounds available.

Examples (all with $p = 1$)

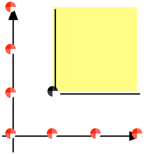
- Proper hypergeometric [Wilf & Zeilberger 1992]:



$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{v}(u_i n + v_i k + w_i)!},$$

Q polynomial, $\xi \in \mathbb{C}$, a_i, b_i, u_i, v_i integers.

- Differential D-finite (definite integration);
- Stirling: ok for $n \geq 3$, e.g., Frobenius:



$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle .$$

  - Abel type: dim $= 2 \to$ ok for $n \geq 4$, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

IV Conclusion

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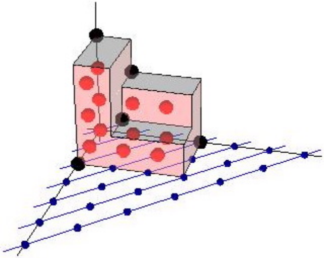
- Summary:
  - Linear differential/recurrence equations as a data structure;
  - Confinement in vector spaces + creative telescoping → identities.
- Also:
  - *q*-analogues;
  - Fast algorithms: Zeilberger 1990 (hypergeom); Chyzak 2000 (D-finite) Us 2009 (non-D-finite).
  - Bounds → identities;
  - Fast algorithms for special classes;
  - Efficient numerical evaluation.



- Open questions:
  - Replace polynomial growth by something intrinsic;
  - Exploit symmetries;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.

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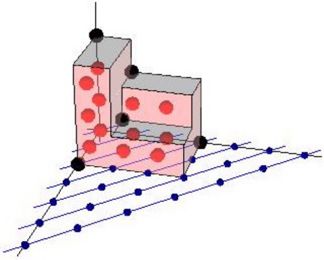

THE END

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