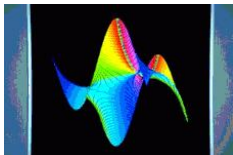


# D-finiteness: Algorithms and Applications

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July 25, 2005

# I D-finiteness in One Variable

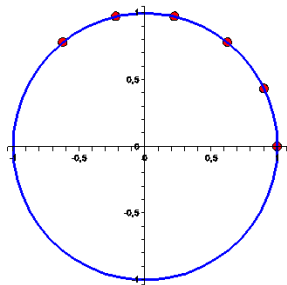
## Warm-up: Algebraic Numbers and Finite Dimension

$$x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.$$

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$\mathbb{Q}(\exp(i\pi/7))$  has dim 6 over  $\mathbb{Q}$

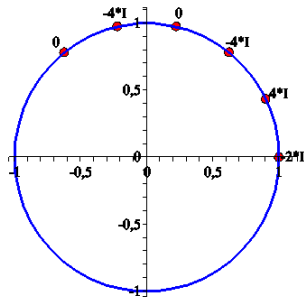


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Coordinates of  $x$



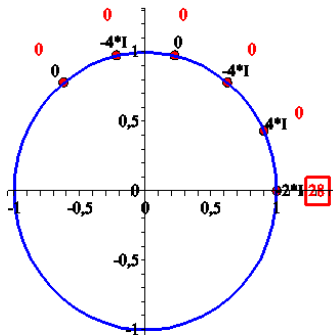
Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

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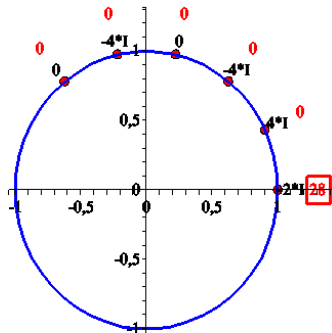
$\mathbb{Q}(\exp(i\pi/7))$  has dim 6 over  $\mathbb{Q}$

Coordinates of  $x^2$

### Definition

A number  $x \in \mathbb{C}$  is *algebraic* when its powers generate a finite-dimensional vector space over  $\mathbb{Q}$ .

**Tools:** Euclidean division, (extended) Euclidean algorithm, linear algebra.

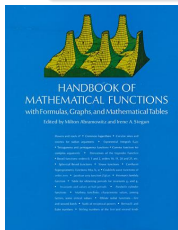


# D-finite Series & Sequences

## Definition

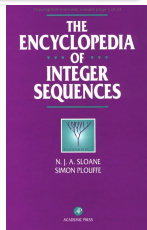
A series  $f(x) \in \mathbb{K}[[x]]$  is **D-finite** over  $\mathbb{K}$  when its derivatives generate a finite-dimensional vector space over  $\mathbb{K}(x)$ . (LDE)

A sequence  $u_n$  is **D-finite** over  $\mathbb{K}$  when its shifts  $(u_n, u_{n+1}, \dots)$  generate a finite-dimensional vector space over  $\mathbb{K}(n)$ . (LRE)



About **25%** of Sloane's encyclopedia,  
**60%** of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Tools: **right** Euclidean division; **right** (extended) Euclidean algorithm; linear algebra; equivalence via generating series.

Implemented in **gfun** [SaZi94].



# Example: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

- 1 Definition of Hermite polynomials (D-finite over  $\mathbb{Q}(x)$ ):  
recurrence of order 2
- 2 Product by linear algebra:  $H_{n+k}(x)H_{n+k}(y)/(n+k)!$ ,  $k \in \mathbb{N}$   
generated over  $\mathbb{Q}(x, n)$  by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order **at most 4**;

- 3 Translation into a differential equation



## I. Definition

$$> R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\} :$$

$$> R_2 := \text{subs}(H=H_2, x=y, R_1);$$

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

## II. Product

$$> R_3 := \text{gfum} :- \text{poltorec}(H(n) \cdot H_2(n) \cdot v(n), [R_1, R_2, \{v(n+1) \cdot (n+1) = v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n));$$

$$R_3 := \left\{ c(0)=1, c(1)=4xy, c(2)=8x^2y^2 + 2 - 4y^2 - 4x^2, c(3) = \frac{32}{3}x^3y^3 + 24xy - 16xy^3 - 16x^3y, (16n \right.$$

$$\left. + 16)c(n) - 16xyc(n+1) + (-8n - 20 + 8y^2 + 8x^2)c(n+2) - 4xc(n+3)y + (n+4)c(n+4) \right\}$$

## III. Differential Equation

$$> \text{gfum} :- \text{rectodiffeq}(R_3, c(n), f(u));$$

$$\left\{ (16u^3 - 16u^2yx - 4u + 8uy^2 + 8ux^2 - 4xy)f(u) + (16u^4 - 8u^2 + 1) \left( \frac{d}{du} f(u) \right), f(0)=1 \right\}$$

$$> \text{dsolve}(\%, f(u));$$

$$f(u) = \frac{\text{Ie} \left( \frac{-4xyu + y^2 + x^2}{(2u-1)(2u+1)} \right)}{e^{(-y^2-x^2)} \sqrt{2u+1} \sqrt{2u-1}}$$

# Euclidean Division & Finite Dimension

## Theorem (XIXth century)

*D*-finite series and sequences over  $\mathbb{K}$  form  $\mathbb{K}$ -algebras.

## Proof.

Linear algebra □

## Corollary

*D*-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform ( $ogf \leftrightarrow egf$ ).

## Euclidean Division &amp; Finite Dimension

Theorem (Tannery 1874)

*D*-finite series composed with algebraic power series are *D*-finite.

Proof.

$$\begin{aligned}
 P(x, y) = 0 \text{ and } AP + BP_y = 1 &\Rightarrow y' = -\frac{P_x}{P_y} = -BP_x \text{ mod } P \\
 &\Rightarrow y^{(k)} \in \underbrace{\bigoplus_{i < \deg_y P} \mathbb{K}(x)y^i}_{\text{finite dim}}.
 \end{aligned}$$

$(f \circ y)^{(p)}$  linear combination of  $(f^{(j)} \circ y)y^k$ .



# Example: Airy Ai at Infinity

$$\begin{aligned} \text{Ai}(z) &= \frac{\sqrt{z}e^{-\xi}}{2\pi} \int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]} dv, \quad \xi = \frac{2}{3}z^{3/2}, u = \sqrt{1 + \frac{v^2}{3}} \\ &\sim \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{\Gamma(3n + \frac{1}{2})}{54^n n! \Gamma(n + \frac{1}{2})}. \end{aligned}$$

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## Computation:

- ① **algebraic** change of variables  $t^2 = (u-1)(4u^2+4u+1)$ ;

$$\rightarrow \int_{-\infty}^{\infty} e^{-\xi t^2} f(t) dt, \quad f(t) = \frac{dv}{dt},$$

- ② recurrence satisfied by the coefficients of  $f$  (**generating series**);  
 ③ termwise integration (**Hadamard product**).



## I. Algebraic change of variables

$$eq_u := u^2 - \left(1 + \frac{v^2}{3}\right);$$

$$eq_t := t^2 - (u - 1) \cdot (4 \cdot u^2 + 4 \cdot u + 1);$$

$$res := \text{resultant}(eq_u, eq_t, u);$$

$$t^4 + 2t^2 - 3v^2 - \frac{8}{3}v^4 - \frac{16}{27}v^6$$

$$gfun := \text{algeqtodiffeq}\left(res, v(t), \left\{v(0)=0, D(v)(0)=\text{sqrt}\left(\frac{2}{3}\right)\right\}\right);$$

$$\left\{-4v(t) + 9t \left(\frac{d}{dt}v(t)\right) + (9t^2 + 18) \left(\frac{d^2}{dt^2}v(t)\right), v(0)=0, (D(v))(0) = \frac{1}{3}\sqrt{6}\right\}$$

## II. Recurrence satisfied by the coefficients of f

$$gfun := \text{poltoDiffeq}(\text{diff}(v(t), t), [\%], [v(t)], f(t));$$

$$\left\{5f(t) + 27t \left(\frac{d}{dt}f(t)\right) + (9t^2 + 18) \left(\frac{d^2}{dt^2}f(t)\right), f(0) = \frac{1}{3}\sqrt{6}, (D(f))(0) = 0\right\}$$

$$R_f := gfun := \text{diffeqtorec}(\%, f(t), c(n));$$

$$\left\{(5 + 18n + 9n^2)c(n) + (18n^2 + 54n + 36)c(n+2), c(0) = \frac{1}{3}\sqrt{6}, c(1) = 0\right\}$$

### III. Hadamard product

assume ( $\xi > 0$ );  $s := \text{Int}(\exp(-\xi * t^2) * t^n, t = -\infty .. \infty)$ ;  $s = \text{student}[\text{intparts}](s, \exp(-\xi * t^2))$ ;

$$\int_{-\infty}^{\infty} e^{(-\xi - t^2)} t^n dt = - \int_{-\infty}^{\infty} \frac{2 \xi \sim t e^{(-\xi - t^2)} t^{(n+1)}}{n+1} dt$$

$$R_i := \left\{ c(n) = \frac{2 \cdot \xi}{(n+1)} \cdot c(n+2), c(0) = \text{value}(\text{eval}(s, n=0)), c(1) = \text{value}(\text{eval}(s, n=1)) \right\};$$

$$\left\{ c(n) = \frac{2 \xi \sim c(n+2)}{n+1}, c(0) = \frac{\sqrt{\pi}}{\sqrt{\xi \sim}}, c(1) = 0 \right\}$$

> **FinalRec:=gfun:-`rec\*rec`(R[i],R[f],c(n));**

$$\text{FinalRec} := \left\{ (5 + 18n + 9n^2) c(n) + (36 \xi \sim n + 72 \xi \sim) c(n+2), c(1) = 0, c(0) = \frac{1}{3} \frac{\sqrt{\pi} \sqrt{6}}{\sqrt{\xi \sim}} \right\}$$

Sol := rsolve(FinalRec, c(n));

$$\left\{ \begin{array}{ll} \frac{1}{3} \frac{(-1)^{\left(\frac{1}{2}n\right)} {}_2\left(-1 - \frac{1}{2}n\right) \Gamma\left(\frac{1}{2}n + \frac{5}{6}\right) \Gamma\left(\frac{1}{2}n + \frac{1}{6}\right) \xi \sim^{\left(-\frac{1}{2}n\right)} \sqrt{6}}{\sqrt{\pi} \Gamma\left(\frac{1}{2}n + 1\right) \sqrt{\xi \sim}} & n::\text{even} \\ 0 & n::\text{odd} \end{array} \right.$$



# Analytic Properties

$$\mathcal{L}y = a_0(x)y^{(k)} + \cdots + a_k(x)y = 0$$

- ① Singular points: roots  $\rho$  of  $a_0$ ;
- ② Indicial polynomial:  $\mathcal{L}(x - \rho)^\sigma \sim P(\sigma)(x - \rho)^{\sigma+m}$  [Fuchs1868]
- ③ Basis of formal solutions [Fabry1885]
  - $\deg P = k$ : *regular* singular point.

$$\Psi_i(z) = (z - \rho)^{\sigma_i} \sum_{j=0}^{d_i} \log^j(z - \rho) \underbrace{\Phi_{i,j}(z - \rho)}_{\text{convergent p. s.}}, \quad P(\sigma_i) = 0.$$

- $\deg P < k$ : *irregular* singular point

$$y_i(t) = \exp\left(\underbrace{P_i(1/t)}_{\text{polynomial}}\right) \underbrace{\Psi_i(t)}_{\text{as above}}, \quad \underbrace{t^{\mu_i}}_{\mu_i \in \mathbb{N}^*} = (z - \rho).$$

Algorithms for everything [Tournier87,vanHoeij97]

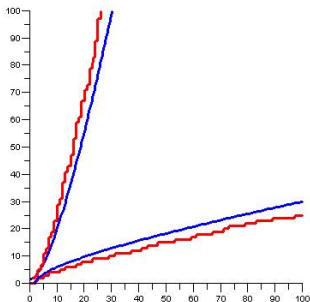
Incorporated in ESF: <http://algo.inria.fr/esf> [MeSa03]

# Not Everything is D-finite

## Analytic behaviour

- $\tan z$ , Lambert  $W$  are not D-finite;
- Bell numbers are not D-finite ( $\sum B_n z^n / n! = \exp(e^z - 1)$ );
- sequences  $\log n$ ,  $n^\alpha$  ( $\alpha \notin \mathbb{N}$ ),  $p_n$  not D-finite [FIGeSa05];

$$\pi(x) \sim \text{Li}(x) + R(x) \Rightarrow p_n - nH_n \sim n \log \log n \Rightarrow \text{g.f.} \sim \frac{\log \log(1-z)}{(1-z)^2}$$



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## Algebraic behaviour (mostly Galois theory)

- $f$  and  $1/f$  D-finite iff  $f'/f$  algebraic [HaSi85];
- $f_n$  and  $1/f_n$  D-finite iff  $f_n$  interlacing of hypergeometric sequences (= rec. of order 1) [vdPSi97];
- $f$  and  $\exp \int f$  D-finite iff  $f$  algebraic;
- $g$  algebraic of genus  $\geq 1$ .  $f$  and  $g \circ f$  D-finite iff  $f$  is algebraic.

[Singer86]

# Fast Algorithms & Applications

- 1 Power series expansion:  $O(n)$  arithmetic ops (no product)
- 2  $n$ th coefficient:  $\tilde{O}(\sqrt{n})$  arithmetic ops (baby steps/giant steps)
- 3  $n$ th coefficient over  $\mathbb{Q}$ :  $\tilde{O}(n)$  **binary** ops (binary splitting)
- 4 evaluation at an algebraic point.

## Examples:

- hypergeometric formula for  $1/\pi$  [ChCh87]
- Sigsam Challenges'97, Problem 4.
- Rational solutions of LDEs and LREs [BoCISa05].

[Hakmem, Brent76, ChCh86, vdH00, BoGaSc04]

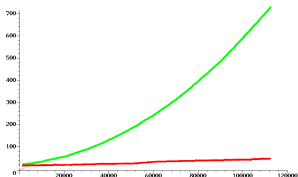
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$$\frac{1}{\pi} = \frac{1}{53360\sqrt{640320}} \sum_{n \geq 0} \frac{(-1)^n (6n)! (13591409 + 545140134n)}{n!^3 (3n)! (8 \cdot 100100025 \cdot 327843840)^n};$$



Used in Maple & Mathematica. Recurrence of order 1.

- Sigsam Challenges'97. Problem 4.

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## Examples:

- hypergeometric formula for  $1/\pi$  [ChCh87]
- Sigsam Challenges'97, Problem 4. Coefficient of  $x^{3000}$  in

$$(x+1)^{2000}(x^2+x+1)^{1000}(x^4+x^3+x^2+x+1)^{500}$$

(first compute a rec. of order 7, then use binary splitting).

- Rational solutions of LDEs and LREs [BoCISa05].

[Hakmem, Brent76, ChCh86, vdH00, BoGaSc04]

## II D-finiteness in Several Variables

# Ore Polynomials & Ore Algebras

- **Skew polynomial ring:**  $\mathbb{A}[\partial; \sigma, \delta]$ ,  $\mathbb{A}$  integral domain and commutation  $\partial P = \sigma(P)\partial + \delta(P)$ ,  $P \in \mathbb{A}$   
 (ex.  $\partial_x P(x) = P(x)\partial_x + P'(x)$ ,  $S_n P(n) = P(n+1)S_n$ ).  
 Technical conditions on  $\sigma, \delta$  to make product associative.
- **Ore algebra:**  $\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ ,  $\sigma, \delta$  s. t.  $\partial_i \partial_j = \partial_j \partial_i$ .  
**Aim** [ChSa98]: manipulate (solutions of) systems of mixed linear ( $q$ -)differential or ( $q$ -)difference operators.



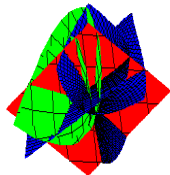
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- **Ore algebra:**  $\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ ,  $\sigma, \delta$  s. t.  $\partial_i \partial_j = \partial_j \partial_i$ .
- **Main property:** the leading term of a product is (up to a cst) the product of leading terms.
- **Consequences:**
  - ① Univariate: Right Euclidean division and extended Euclidean algorithm [Ore 33];
  - ② Multivariate: Buchberger's algorithm for Gröbner bases works in Ore algebras [Kredel93].

# 0-dimensionality & D-finiteness



Polynomial algebra

0-dimensional ideal



quotient is a **finite dimensional** vector space

Ore algebra

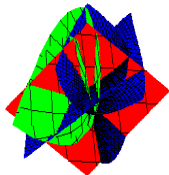
**D-finite** left ideal



**Exs:** Orthogonal polynomials, hypergeometric series, their  $q$ -analogues,...

system+ini. cond.=data structure

# 0-dimensionality & D-finiteness



Polynomial algebra

0-dimensional ideal



quotient is a **finite dimensional** vector space



polynomial expressions  
are algebraic

Ore algebra

**D-finite** left ideal



polynomials and  $\partial$ 's  
are D-finite

Tools: linear algebra, Gröbner bases. Implemented in **Mgfun** [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their  $q$ -analogues,...

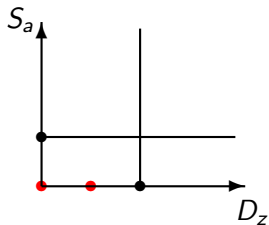
system+ini. cond.=data structure

# Example: Contiguity of Hypergeometric Series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n (b)_n}{(c)_n n!}}_{u_{a,n}} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.$$



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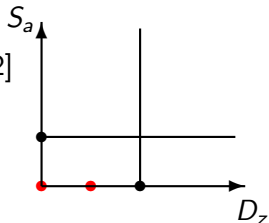
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$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.$$

$\dim=2 \Rightarrow S_a^2 F, S_a F, F$  linearly dependent [Gauss1812]

Also:

- $S_a^{-1}$  in terms of  $\text{Id}, D_z$ ;
- relation between any three polynomials in  $S_a, S_b, S_c$ ;
- generalizes to any  ${}_pF_q$  and multivariate case [Takayama89].



## Creative Telescoping [Zeilberger 90]

$$F_n = \sum_k u_{n,k} = ?$$

**IF** one knows  $A(n, S_n)$  and  $B(n, k, S_n, S_k)$  such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum “telescopes”, leading to  $A(n, S_n) \cdot F_n = 0$ .

## Creative Telescoping [Zeilberger 90]

$$I(x) = \int_{\Omega} u(x, y) dy = ?$$

**IF** one knows  $A(x, \partial_x)$  and  $B(x, y, \partial_x, \partial_y)$  such that

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then the integral “telescopes”, leading to  $A(x, \partial_x) \cdot I(x) = 0$ .

*Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*

Richard P. Feynman 1985

Creative telescoping = “differentiation” under integral + “integration” by parts



## Creative Telescoping [Zeilberger 90]

$$I(x) = \int_{\Omega} u(x, y) dy = ?$$

**IF** one knows  $A(x, \partial_x)$  and  $B(x, y, \partial_x, \partial_y)$  such that

$$(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,$$

then the integral “telescopes”, leading to  $A(x, \partial_x) \cdot I(x) = 0$ .

Creative telescoping = “differentiation” under integral + “integration” by parts

- General case: Find annihilators of

$$I(x_1, \dots, x_{n-1}) = \partial_n^{-1} \Big|_{\Omega} f(x_1, \dots, x_n)$$


knowing generators of  $\text{Ann}_f$  in

$$\mathbb{O}_n = \mathbb{K}(x_1, \dots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n];$$

- Crucial step: compute  $(\mathbb{O}_n \text{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$ .

# Example: $\zeta(3)$ is Irrational [Apéry78]

$$a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n = a_n \sum_{k=1}^n \frac{1}{k^3} + \sum_{k=1}^n \sum_{m=1}^k \frac{(-1)^{m+1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

- ①  $b_n/a_n \rightarrow \zeta(3)$ ,  $n \rightarrow \infty$ ;  $d_n^3 b_n \in \mathbb{Z}$ , where  $d_n = \text{lcm}(1, \dots, n)$ ;
- ② By **creative telescoping**, both  $a_n$  and  $b_n$  satisfy 

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1;$$

- ③  $0 < \zeta(3) - \frac{b_n}{a_n} = \sum_{k \geq n+1} \frac{b_k}{a_k} - \frac{b_{k-1}}{a_{k-1}}: b_k a_{k-1} - b_{k-1} a_k = \frac{6}{k^3}$ ;

- ④  $\lambda a_n + \mu b_n \approx \alpha_{\pm}^n$ , with  $\alpha_{\pm}^2 = 34\alpha_{\pm} - 1$ ;

- ⑤ Conclusion:  $0 < \underbrace{a_n d_n^3}_{\in \mathbb{N}} \zeta(3) - \underbrace{d_n^3 b_n}_{\in \mathbb{N}} \approx \alpha_-^n e^{3n} \rightarrow 0$ .

Algolib can be downloaded from <http://algo.inria.fr/libraries>.

> `libname := "/Users/salvy/lib/maple/Algolib", libname :`

> `a := binomial(n, k)^2 · binomial(n + k, k)^2;`

`a := binomial(n, k)^2 binomial(n + k, k)^2`

> `Mgfun[creative_telescoping](a, n :: shift, k :: shift);`

$$\left[ \begin{aligned} & (-n^3 - 3n^2 - 3n - 1) f(n, k) + (34n^3 + 153n^2 + 231n + 117) f(n + 1, k) + (-n^3 - 6n^2 - 12n - 8) f(n \\ & + 2, k), - \frac{4k^4 (4n^2 + 12n + 8 + 3k - 2k^2) (2n + 3) f(n, k)}{4 + 12n - 12k - 4nk^3 + 13n^2 + 13k^2 + k^4 - 26nk + n^4 + 6n^3 - 6k^3 + 6n^2k^2 - 18n^2k + 18nk^2 - 4n^3k} \end{aligned} \right]$$

Neither Cohen nor I had been able to prove [this] in the intervening two months. [Van der Poorten]

## Applications of Creative Telescoping

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad [\text{Strehl92}]$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2} \quad [\text{GIMo94}]$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!} \quad [\text{Doetsch30}]$$

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \quad [\text{Andrews74}]$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \quad [\text{Paule85}]$$

## (Partial) Algorithms for Creative Telescoping

Aim:  $\mathcal{I} = (\mathbb{O}_n \text{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$

- By Gröbner bases, eliminate  $x_n$  and set  $\partial_n$  to 0 [ChSa98]  
 $\rightarrow (\mathbb{O}_n \text{Ann}_f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I}$
- Differential case: algorithms from  $\mathcal{D}$ -module theory [SaStTa00, Tsai00], Gröbner bases with negative weights.
- Shift case,  $n = 2$ ,  $\dim 1$  (= hypergeometric): [Zeilberger91]  
 For increasing  $k$ , search for  $a_i$  and  $B$

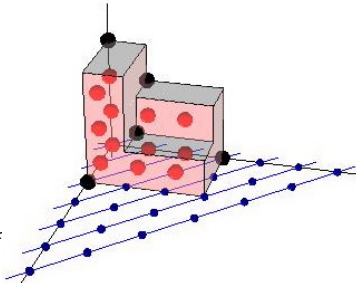
$$\mathbb{O}_{n-1} \ni \sum_{i=0}^k a_i \partial_{n-1}^i f = \partial_n B f$$

Termination [Abramov03].

- Arbitrary  $n$  and  $\mathbb{O}_n$ : [Chyzak00]

$$\mathbb{O}_{n-1} \ni \sum_{\lambda} a_{\lambda} \partial^{\lambda} = \partial_n B \text{ mod } \text{Ann}_f$$

$B$  is given by rational solutions of a linear system in  $\sigma_n, \partial_n$ .



## III D-finiteness in Infinitely Many Variables

# $k$ -uniform Young Tableaux

4	4					
3	3	5				
2	2	3	4			
1	1	1	2	5	5	

**Question:** Asymptotic number of semi-standard Young tableaux filled with  $k$  1's,  $k$  2's, ...,  $k$   $n$ 's?

Result [ChMiSa05]

$$\frac{1}{\sqrt{2}} \left( \frac{e^{k-2}}{2\pi} \right)^{k/4} n!^{k/2-1} \left( \frac{k^{k/2}}{k!} \right)^n \frac{\exp \sqrt{kn}}{n^{k/4}}, \quad n \rightarrow \infty.$$

**Method:**

Combinatorics  $\rightarrow$  Symmetric functions  $\rightarrow$  LDE  $\rightarrow$  Asymptotics.

# D-finite Symmetric Series

Algebra of symmetric functions:  $\Lambda := \mathbb{K}[[p_1, p_2, \dots]]$

power  $p_k$   $p_3 = x_1^3 + x_2^3 + x_3^3 + \dots$

homogeneous  $h_k$   $h_3 = x_1^3 + x_2^3 + \dots + x_1^2 x_2 + \dots + x_1 x_2 x_3 + \dots$

monomial  $m_\lambda$   $m_{(3,2,1)} = x_1^3 x_2^2 x_3 + x_2^3 x_1^2 x_3 + \dots$

## Definition

$F \in \Lambda[[t]]$  **D-finite** if for any  $n$ ,  $F(p_1, \dots, p_n, 0, \dots; t)$  D-finite.

## Theorem (Gessel 90)

- Closed under  $+$ ,  $\times$ ,  $\partial/\partial p_i$ , algebraic substitution.



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## Definition

$F \in \Lambda[[t]]$  **D-finite** if for any  $n$ ,  $F(p_1, \dots, p_n, 0, \dots; t)$  D-finite.

## Theorem (Gessel 90)

- Closed under  $+$ ,  $\times$ ,  $\partial/\partial p_i$ , algebraic substitution.
- [Technical conds] closed under plethysm and **scalar product**.

Scalar product:  $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ , where  $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$

Adjoint:  $\langle \phi F, G \rangle = \langle F, \phi^\perp G \rangle$ , with  $p_k^\perp = k \frac{\partial}{\partial p_k}$ ,  $\left( \frac{\partial}{\partial p_k} \right)^\perp = \frac{p_k}{k}$ .

## Algorithm

$$\left\langle \exp \left( \sum_{n=1}^{\infty} \frac{p_n^2}{2n} + \sum_{n \text{ odd}} \frac{p_n}{n} \right), \sum_{n \geq 0} h_k^n t^n \right\rangle$$

All semi-standard  
Young tableauxExtract coeffs  
corresponding  
to  $k$ -uniform

$$= \left\langle \exp \left( \sum_{n=1}^k \frac{p_n^2}{2n} + \sum_{n \text{ odd}} \frac{p_n}{n} \right), \sum_{n \geq 0} h_k^n t^n \right\rangle =: \langle F, G \rangle$$

$$\text{Wanted: } \left( \underbrace{\text{Ann}_F^\perp}_{\text{right ideal}} + \underbrace{\text{Ann}_G}_{\text{left ideal}} \right) \cap \mathbb{K}[t, \partial_t].$$

## Algorithm [ChMiSa05]

For increasing  $D$ , from GB's of  $\text{Ann}_F$  and  $\text{Ann}_G$ , compute bases of  $\text{Ann}_F^\perp$  and  $\text{Ann}_G$  up to degree  $D$  as vector spaces.  
Stop when Gaussian elimination yields an element of  $\mathbb{K}[t, \partial_t]$ .

**Termination:** granted by holonomy.

## IV Conclusion

# Future Work

## Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93,GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants.

## Understand non-minimality

- Remove **apparent** singularities by Ore closure, a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (extend [Paule94]).

## Easy-to-use Implementations

- Improve `gfun` and `Mgfun`. Make the ESF interactive.