Special Function Identities: D-Finiteness and Beyond

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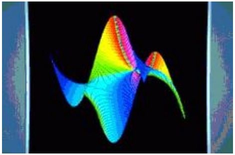


Journées de Combinatoire de Bordeaux, Feb. 6, 2009
Joint work with Frédéric Chyzak and Manuel Kauers

I Introduction

Special Function Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \quad \text{[Strehl92]}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{[GlMo94]}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \quad \text{[Doetsch30]}$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{[Andrews74]}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{[Paule85].}$$

More Special Function Identities

$$\sum_{k=0}^{n}\binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n}(-1)^{m-k}k!\binom{n-k}{m-k}\left\{{n+1 \atop k+1}\right\} = \left\langle {n \atop m} \right\rangle, \quad \text{[Frobenius1910}$$

$$\sum_{k=0}^{m}\binom{m}{k}B_{n+k} = (-1)^{m+n}\sum_{k=0}^{n}\binom{n}{k}B_{m+k}, \quad \text{[Gessel03]}$$

$$\int_0^\infty x^{k-1}\zeta(n,\alpha+\beta x)\,dx = \beta^{-k}B(k,n-k)\zeta(n-k,\alpha),$$

$$\int_0^\infty x^{\alpha-1}\operatorname{Li}_n(-xy)\,dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{k-1}\exp(xy)\Gamma(n,xy)\,dx = \frac{\pi y^{-k}}{\sin((n+k)\pi)}\frac{\Gamma(k)}{\Gamma(1-n)}$$

Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

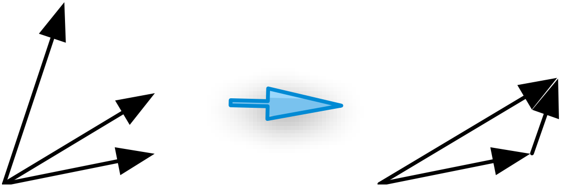
- 1st slide: Zeilberger's algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: new generalization of Zeilberger, Majewicz,. . . .

Ideas

Confinement in finite dimension + Creative telescoping.

II Confinement in Finite Dimension

Confinement Provokes Identities



Obvious

$k+1$ vectors in dimension $k \longrightarrow$ an identity.

Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n (b)_n}{(c)_n n!}}_{u_{a,n}} z^n,$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow \; z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1,b;c;z) = \frac{z}{a}F' + F.$$

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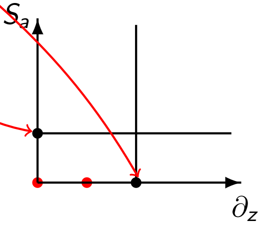

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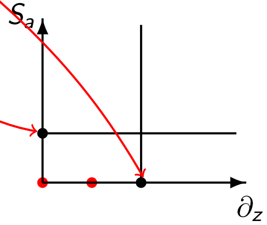
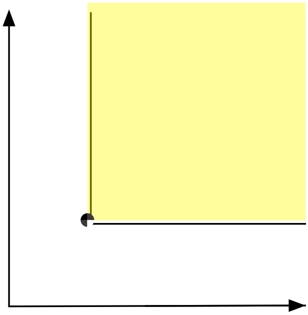
Gauss 1812: contiguity relation.

dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent.



Gröbner Basis: Euclidean Division in Several Variables

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, no $\infty \searrow$ chain.



Gröbner Basis: Euclidean Division in Several Variables

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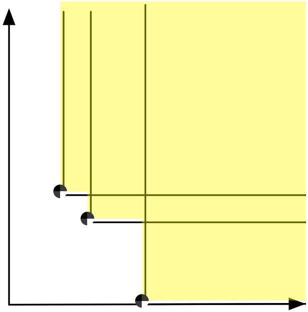
2. Gröbner basis of a (left) ideal $\mathcal{I}$: corners of stairs.



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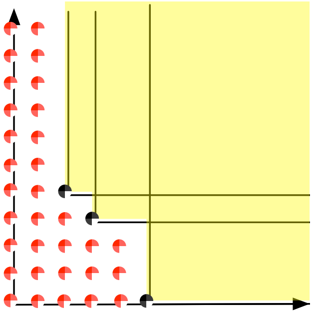

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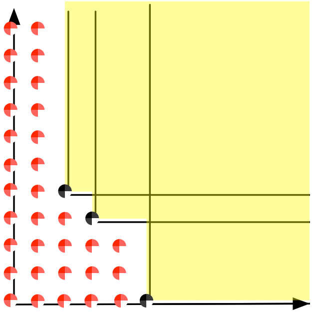
4. Reduction of P: Rewrite P mod $\mathcal{I}$ on this basis.



$\rightarrow$ An access to (finite dimensional) vector spaces

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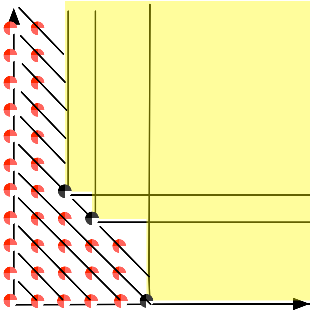

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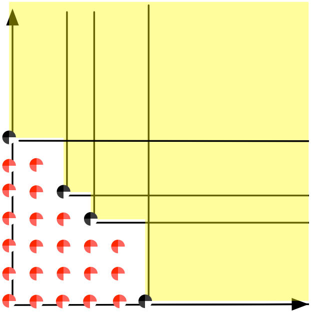
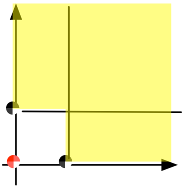
5. Dimension of $\mathcal{I}$: "size" of the quotient ∞ly far.
6. D-finiteness: dim $= 0$.



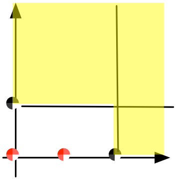
$\rightarrow$ An access to (finite dimensional) vector spaces

Examples

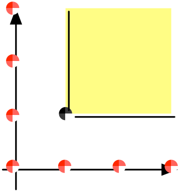
Binomial coeffs $\binom{n}{k}$ wrt S_n, S_k
Hypergeometric sequences



Bessel $J_\nu(x)$ wrt S_ν, ∂_x
Orthogonal pols wrt S_n, ∂_x



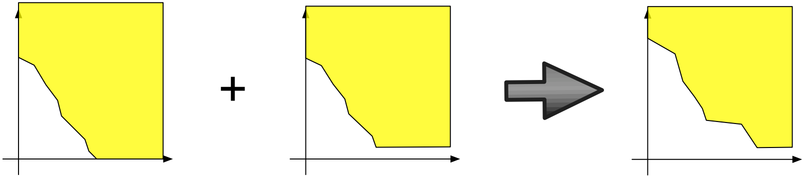
Stirling nbs wrt S_n, S_k



Abel type wrt S_m, S_r, S_k, S_s
$\mathrm{hgm}(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$

dim = 2 in space of dim 4.

Closure Properties



Proposition

$$\dim \operatorname{ann}(f+g) \leq \max(\dim \operatorname{ann} f, \dim \operatorname{ann} g),$$
$$\dim \operatorname{ann}(fg) \leq \dim \operatorname{ann} f + \dim \operatorname{ann} g,$$
$$\dim \operatorname{ann} \partial f \leq \dim \operatorname{ann} f.$$

Algorithms by linear algebra.

III Creative Telescoping

Diff. under $\int$ + Integration by Parts → Algorithm?

Ex.: $$\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{2}{\pi}J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,\ J_0(0) = 1).$$

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Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Diff. under $\int$ + Integration by Parts $\to$ Algorithm?

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$$I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt, \quad I'(z) = \int_0^1 -t\frac{\sin zt}{\sqrt{1-t^2}}\,dt,$$

$$I''(z) = \int_0^1 -t^2\frac{\cos zt}{\sqrt{1-t^2}}\,dt = -I(z) + \int_0^1 \sqrt{1-t^2}\cos zt\,dt,$$

$$I''(z) + I(z) = \underbrace{\left[\sqrt{1-t^2}\frac{\sin zt}{z}\right]_0^1}_{0} + \int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{\sin zt}{z}\,dt = -\frac{I'(z)}{z}.$$

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$$\operatorname{ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,\partial_z)}_{\text{no } t,\partial_t} - \partial_t \underbrace{\frac{t^2-1}{t}\partial_z}_{\text{anything}}$$

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Creative Telescoping

Input: generators of (a subideal of) ann f;
Output: A, B such that $A - \partial_t B \in \operatorname{ann} f$, A free of t, ∂_t.
Algorithm: sometimes. (Why would they exist?)

Telescoping of $\mathcal{I}$ wrt t:

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t\mathbb{Q}(z,t)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle.$$

Example: Pascal's Triangle

$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (\underbrace{S_n - 2}_{\text{no } k,\, S_k} + (S_k - 1)(S_n - 1))\cdot\binom{n}{k}.$$

Sum over $k \Rightarrow (S_n - 2)\sum_k \binom{n}{k} = 0.$

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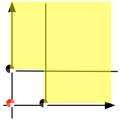
Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1, \quad S_n \to \frac{n+1}{n+1-k}1, \quad S_k \to \frac{n-k}{k+1}1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1, \quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1,$$

$$S_nS_k \to \frac{n+1}{k+1}1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41).$$

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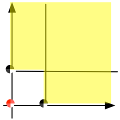

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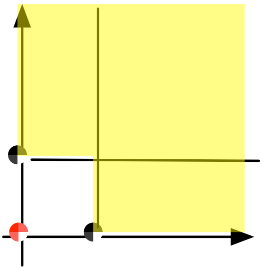
This has to happen for some degree: $\deg D_s = O(s)$.

Proper Hypergeometric Sequences



Definition (Proper hypergeometric)

$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{v}(u_i n + v_i k + w_i)!},$$

Q polynomial, $\xi \in \mathbb{C}$, a_i, b_i, u_i, v_i integers.

Theorem (Wilf-Zeilberger 1992)

Creative telescoping works for proper hypergeometric sequences.

Proof.

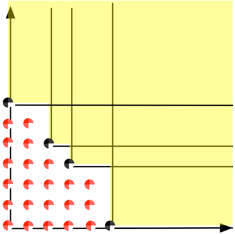
The degree of the lcm of the denominators of $\{S_n^a S_k^b \mid a+b \leq s\}$ grows linearly with s. □

Differential Case: D-finite ⇒ Creative Telescoping Works

- D-finite ⇒ $b_1, \dots, b_m$ with $m < \infty$ below stairs;
- Take $L := \operatorname{lcm}_{i,j}(\operatorname{denom}(\partial_i b_j \bmod \mathcal{I}))$;
- By induction, if $a + b \leq s$, then

$$\partial_x^a \partial_y^b - \sum \frac{c_{a,b,i}}{L^s} b_i \in \mathcal{I},$$

with $c_{a,b,i}$ polynomials of degree $O(s)$;
- ⇒ Identity exists by confinement.



Related to holonomy in D-module theory.

IV Beyond D-Finiteness

Polynomial Growth

Definition (Polynomial Growth p)

There exists a sequence of polynomials P_s, s.t. for all $(a_1, \dots, a_k)$ with $a_1 + \cdots + a_k \leq s$, $P_s \partial_1^{a_1} \cdots \partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

Theorem (Telescoping one variable)

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Proof. Same as above. Set $q := \dim \mathcal{I} + p$.

- In degree s, dim $O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \dots, \partial_{i_q}$: $O(s^{q+1})$;

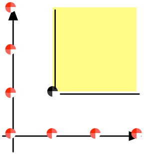
$\Rightarrow$ any q variables linearly dependent $\Rightarrow$ dim $\leq q - 1$.

This proof gives an algorithm. Also, bounds available.

Examples

- Proper hypergeometric: dim $=0, p=1$;
- Differential D-finite: dim $=0, p=1$;
- Stirling: dim $=1, p=1$, $\to$ ok for $n\geq 3$, e.g., Frobenius:

$$\sum_{k=0}^{n}(-1)^{m-k}k!\binom{n-k}{m-k}\left\{\begin{matrix}n+1\\k+1\end{matrix}\right\}=\left\langle\begin{matrix}n\\m\end{matrix}\right\rangle.$$



- Abel type: dim $=2, p=1$ ok for $n\geq 4$, e.g., Abel:

$$\sum_{k=0}^{n}\binom{n}{k}i(k+i)^{k-1}(n-k+j)^{n-k}=(n+i+j)^n.$$

- ...

V Conclusion

Fast Algorithms

1. Zeilberger 1990: fast algorithm for hypergeometric sequences.
2. Chyzak 2000: extension to D-finite.
3. New: extension to non-D-finite.