Minimization of Differential Equations and Algebraic Values of E-Functions

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Minimal Linear Differential Equation

Input:

a linear differential eqn $\mathscr{L}(y(z)) = 0$, coefficients in $\mathbb{Q}[z]$;
initial conditions specifying a unique solution $S \in \mathbb{Q}[[z]]$.

Output:

linear differential equation of minimal order with coefficients in $\mathbb{Q}[z]$ having S as a solution.

$$\text{deq} := \{z^2(z-3)y''(z) + (4z^7 + z^2 + 3z - 9)y'(z) + 4z^5(5z+3)y(z) = 0,\ y(0) = 1\}$$

```
> minimizediffeq(deq,y(z));
```

$$\{(z-3)y'(z) + 4z^5y(z),\ y(0) = 1\}$$

Two variants: homogeneous/non-homogeneous

Minimal $\neq$ Irreducible

$$S = \ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$$

satisfies

$$\left((1-z)\frac{d}{dz} - 1\right)\frac{d}{dz}\,y(z) = 0$$

but no 1st order homogeneous LDE

Minimal $\neq$ `Smallest'

$$(x^2 - 6x + 1)\, y'''(x) + 3(x-3)\, y''(x) = 0$$

has for solution

$$S = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2} = x + 2x^2 + \cdots$$

that satisfies the minimal equation

$$(x+1)(x^2 - 6x + 1)y''(x) + 4(x-1)y'(x) - 4y(x) = 0$$

with smaller order but coefficients of higher degree

Apparent singularity

Motivation: Values of E-functions (1/2)

$Y = {}^t(f_1, \ldots, f_n)$ vector of E-functions

$Y' = AY$ with $A \in \overline{\mathbb{Q}}(z)^{n\times n}$

$T(z)$ the common denominator of the entries of A

If $(f_1, \ldots, f_n)$ are $\overline{\mathbb{Q}}(z)$-linearly independent, then

1. for any $\xi \in \overline{\mathbb{Q}}$ s.t. $\xi T(\xi) \neq 0$,
   $(f_1(\xi), \ldots, f_n(\xi))$ are $\overline{\mathbb{Q}}$-linearly independent;
2. there exists $Z = {}^t(e_1, \ldots, e_n)$ vector of E-functions s.t.
   . $Z' = BZ$ with $B \in \overline{\mathbb{Q}}[z, 1/z]$;
   . $Y = MZ$ with $M \in \overline{\mathbb{Q}}[z]$.

The computation of M is effective

The only linear relations between $(f_1(\xi), \ldots, f_n(\xi))$ with $\xi \neq 0$ come from the left kernel of M

[Beukers 2006] using [André 2000]

Motivation: Values of E-functions (2/2)

Input: E-function f

Output: either `f is polynomial′ or the finite set

$$\{(\alpha, f(\alpha)) \mid \alpha \in \overline{\mathbb{Q}}^* \text{ and } f(\alpha) \in \overline{\mathbb{Q}}\}\,.$$

1. Compute the minimal non-homogeneous LDE
$$v_0 f^{(s)} + \cdots + v_s f = g$$
2. If $s = 0$ then return `f is polynomial′ else $R := \{\,\}$
3. Compute Beukers′ matrix M for $(1, f, f', \ldots, f^{(s-1)})$
4. For all roots α of v_0 s.t. the left kernel of M contains a vector $(\beta, -1, 0, \ldots, 0)$, set $R := R \cup \{f(\alpha) = \beta\}$
5. Return R

Demo

[Adamczewski-Rivoal 2018]

Variant: Canonical Decomposition

Exceptional values: $\mathrm{Exc}(f) = \{\alpha \in \overline{\mathbb{Q}} \backslash \{0\} \mid f(\alpha) \in \overline{\mathbb{Q}}\}$

Prop. A transcendental E-function $f \in \mathbb{Q}[[z]]$ can be written

$$f = p + qg$$

with $p, q \in \mathbb{Q}[z]$, $\deg p < \deg q$, q monic, $q(0) \neq 0$, g E-function, $\mathrm{Exc}(g) = \varnothing$. The decomposition is unique.

$$\mathrm{Exc}(f) = \{\alpha \mid q(\alpha) = 0\}.$$

Exceptional values are `trivial'

I. Minimization Toolbox

Input

$\mathscr{L}(y(z)) = 0 \qquad \mathscr{L} \in \mathbb{Q}[z]\langle \partial_z \rangle$, not a right multiple of z

Initial conditions specifying a unique solution $S \in \mathbb{Q}[[z]]$?

There exists $g \leq 0$, $p_i \in \mathbb{Q}[z]$, s.t. for all monomials z^s,

indicial polynomial at 0 $\operatorname{ind}^0_{\mathscr{L}}(s)$

$$\mathscr{L}(z^s) = z^{s+g}(p_0(s) + p_1(s)z + \cdots),$$

Ordinary case: $g = -r$, $p_0(s) = s(s-1)\cdots(s-r+1)$

$$\mathscr{L}\big(\underbrace{\sum s_i z^i}_{S(z)}\big) = \underbrace{\sum d_i z^{g+i}}_{0}, \qquad \underbrace{d_i}_{0} = s_i p_0(i) + s_{i-1}p_1(i-1) + \cdots, \qquad i \geq 0.$$

If $\operatorname{ind}^0_{\mathscr{L}}(i) \neq 0$, s_i determined from the previous ones.

Initial conditions: $y^{(i)}(0)$ for $i \in \mathscr{E}_{\mathscr{L}} := \{k \in \mathbb{N} \mid \operatorname{ind}^0_{\mathscr{L}}(k) = 0\}$.

Check that a Factor Vanishes at S

$$\mathscr{L} = \mathscr{N}\mathscr{M}$$

$$\mathscr{L}(z^s) = \mathscr{N}(\mathscr{M}(z^s)) \Rightarrow \mathrm{ind}^0_{\mathscr{M}} \mid \mathrm{ind}^0_{\mathscr{L}} \Rightarrow \mathscr{Z}_{\mathscr{M}} \subset \mathscr{Z}_{\mathscr{L}}$$

Lemma. If T is a polynomial s.t. $T^{(i)}(0) = S^{(i)}(0)$ for $i \in \mathscr{Z}_{\mathscr{L}}$ and $\mathscr{M}(T) = O(z^{\max \mathscr{Z}_{\mathscr{L}}+1})$, then $\mathscr{M}(S) = 0$.

Proof. T extends to a unique power series solution of $\mathscr{M}$, hence of $\mathscr{L}$. Same initial conditions as S, thus $T = S$.

Guess Factor

From $\mathscr{L}$, initial conditions, a target order m,
compute $S + O(z^{N+m})$ for large N
and a Hermite-Padé approximant
$(a_0, \ldots, a_m) \in \mathbb{Q}[z]^{m+1}$ s.t. $a_0 S + a_1 S' + \cdots + a_r S^{(m)} = O(z^N)$

(Structured) linear algebra on the coefficients of the a_i

Given bounds on the possible degrees of the a_i,
this gives a proof when no factor of order m exists,
whence also minimality proofs.

[Beckermann-Labahn 1994; Zhou-Labahn 2012; Jeannerod-Neiger-Schost-Villard 2016; Jeannerod-Neiger-Villard 2020]

Right Factor From a Guess

From $\mathscr{L}(S) = 0,\ \mathscr{A}(S) = O(z^N)$,

compute the greatest common right divisor of $\mathscr{L}, \mathscr{A}$

$$\mathscr{M} := \operatorname{gcrd}(\mathscr{L}, \mathscr{A})$$

by a non-commutative variant of Euclid's algorithm,
or linear algebra on a Sylvester matrix

[Ore 1933; Grigoriev 1990; Bostan 2003; van der Hoeven 2016]

Minimization Algorithm (Given Bounds)

Input: $\mathcal{L} = a_r(z)\partial_z^r + \cdots + a_0(z)$ in $\mathbb{Q}[z]\langle\partial_z\rangle$;
ini: S_0 a truncated power series at precision $\geq \max \mathcal{Z}_{\mathcal{L}}$
specifying a unique solution $S \in \mathbb{Q}[[z]]$ of $\mathcal{L}(S) = 0$.

Output: a right factor of $\mathcal{L}$ in $\mathbb{Q}[z]\langle\partial_z\rangle$ of minimal order that vanishes at S

$\mathcal{M} := \mathcal{L}$; $T := S_0$; $m := r$; $p := \max \mathcal{Z}_{\mathcal{L}} + r$
while $m > 1$ **do**
  $m := m - 1$
  if $N :=$ BoundDegreeCoeffs($\mathcal{L}$, m)$\neq$FAIL **then**
    $p := \max(p, m(N+1))$
    while true **do**
      $T :=$ SeriesSolution($\mathcal{L}, T, p + m$);
      $\mathcal{H} :=$ ApproximantBasis($T, T', \ldots, T^{(m)}; N, \ldots, N; p$)
      if $\mathcal{H} = \emptyset$ **then** break *// No right factor of order m*
      else *// $\mathcal{H}$ contains at least a candidate factor h*
        $\mathcal{M} :=$ GreatestCommonRightDivisor($\mathcal{L}, h$);
        if $\mathcal{M}(T) = O(z^{\max \mathcal{Z}_{\mathcal{L}}+1})$ **then** $m := \operatorname{ord} \mathcal{M}$; break
        else $p := 2p$
return $\mathcal{M}$

Demo

Similar to [van Hoeij 1997]

Non-homogeneous Minimization

Input: $\mathscr{L}$ of minimal order s.t. $\mathscr{L}(S) = 0$

Output: $\mathscr{M}, b$ s.t. $\mathscr{M}(S) = b,\ b \in \mathbb{Q}[z]$,
$\mathscr{M}$ of minimal order.

Differentiate & combine: $(b\partial - b')\mathscr{M}(S) = 0$

$\Rightarrow b = 0$ or $\operatorname{ord} \mathscr{M} = \operatorname{ord} \mathscr{L} - 1$

If $b \neq 0$, up to dividing $\mathscr{M}$ by b: $\partial\mathscr{M}(S) = 0$

Minimality of $\mathscr{L}$: $\partial\mathscr{M} = R(z)\mathscr{L}$ for some $R \in \mathbb{Q}(z)$

Integrating factor

Adjoint: $\mathscr{L}^\star(R) = 0$

Find rational solution (if any) and reconstruct $\mathscr{M}$.

\`Trivial' Relations for ${}_2F_1$

Gauss differential equation for $y(z) = {}_2F_1[a,b;c;z]$

$$z(z-1)y'' + ((a+b+1)z-c)y' + aby = 0$$

Its adjoint has for solution

$$R(z) := {}_2F_1[1-a,1-b;2-c;z]$$

Thus y satisfies the non-homogeneous 1st order equation

$$z(z-1)R(z)y' + (z(1-z)R' + ((a+b-1)z+1-c)R)y = 1-c$$

Simple roots of R yield special values, e.g.,

$${}_2F_1\left(\begin{matrix}-7/2 \; 97/16 \\ 49/16\end{matrix}\,\middle|\,\frac{5}{11}\right) = -\frac{2248\sqrt{66}}{297297}.$$

Can also be found by contiguity relations

II. Computation of Bounds

Tools from factorization of operators

[van Hoeij 1997]

Singular Points

$$\mathscr{L} = \partial^r + \frac{a_{r-1}}{a_r}\partial^{r-1} + \cdots + \frac{a_0}{a_r} = \mathscr{N}\mathscr{M}$$

Singular points of $\mathscr{L}$: roots of a_r

Indicial polynomial $\mathrm{ind}^{\alpha}_{\mathscr{L}}$: leading coeff of $\mathscr{L}((z-\alpha)^s)$ $\deg \le r$

Regular singular point: $\deg \mathrm{ind}^{\alpha}_{\mathscr{L}} = r$ $\mathrm{val}_{\alpha}(a_i/a_r) \ge i - r$

Exponents at α: roots of $\mathrm{ind}^{\alpha}_{\mathscr{L}}$

α apparent singular point: basis of power series solutions at α

$\sigma(\mathscr{L})$: non-apparent singularities of $\mathscr{L}$

$$\sigma(\mathscr{M}) \subset \sigma(\mathscr{L}) \qquad \mathrm{ind}^{\alpha}_{\mathscr{M}} \mid \mathrm{ind}^{\alpha}_{\mathscr{L}}$$

The degree of $\mathscr{M}$ is upper bounded using

. lower bounds on the val_{α} of its coefficients for $\alpha \in \sigma(\mathscr{L})$ (poles & multiplicities);

. upper bounds on val_{∞} of the coefficients (deg. numer – deg. denom.);

. upper bound on the number of apparent singularities.

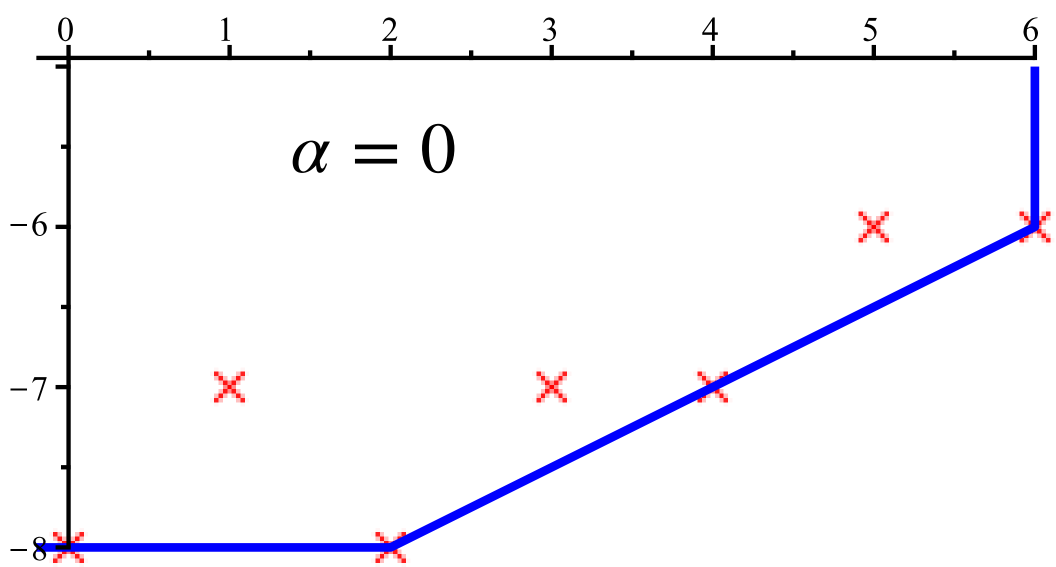
Newton Polygon

$$\mathscr{L} = \partial^r + \frac{a_{r-1}}{a_r}\partial^{r-1} + \cdots + \frac{a_0}{a_r}$$

$$\mathrm{Newt}_\alpha(\mathscr{L}) = \text{convex hull}\left(\bigcup_i \left((i, \mathrm{val}_\alpha(\frac{a_i}{a_r}) - i) + (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0})\right)\right)$$

Prop. $\mathscr{L} = \mathscr{N}\mathscr{M} \Rightarrow \mathrm{Newt}_\alpha(\mathscr{L}) = \mathrm{Newt}_\alpha(\mathscr{N}) + \mathrm{Newt}_\alpha(\mathscr{M})$.

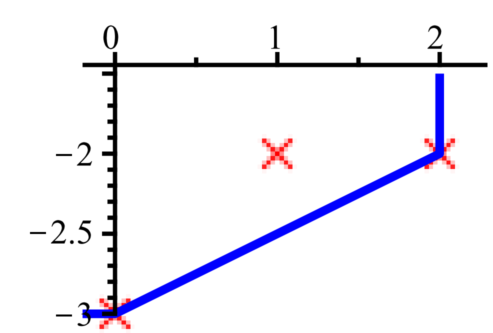
$\mathscr{L} = 1 + z^2\partial_z + z^2\partial_z^2 + z^4\partial_z^3 + z^5\partial_z^4 + z^7\partial_z^5 + z^8\partial_z^6$



possible order of $\mathscr{M}$: 2,4,6

smallest possible valuations at order 2:

$$\partial_z^2 + (z^{-1} + \cdots)\partial_z + (z^{-3} + \cdots)$$



"bien connu" in [Malgrange 79]

Fuchs Relation & Apparent Singularities

exponents at ρ

Fuchs: when all singular points are regular,

$$\sum_{\rho \in \mathrm{Sing}(\mathscr{L})} S_\rho(\mathscr{L}) = -r(r-1), \quad S_\rho(\mathscr{L}) = \sum_{i=1}^{r} e_i(\rho) - \frac{r(r-1)}{2}.$$

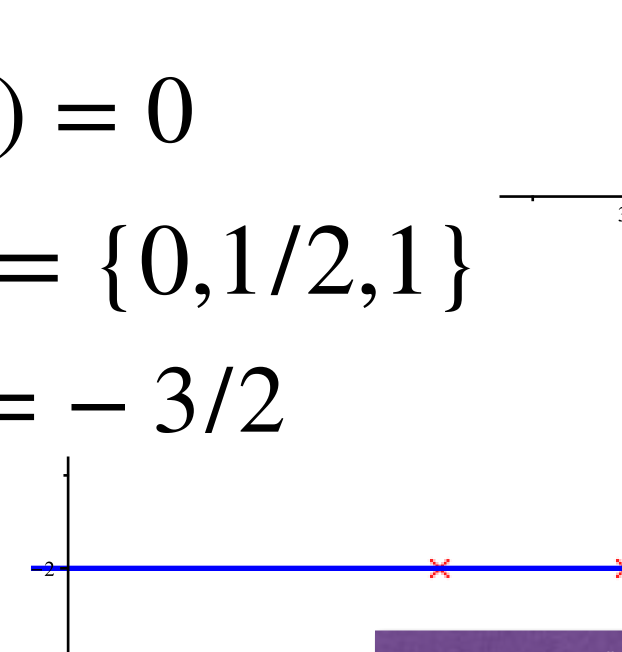
ρ apparent $\Rightarrow S_\rho \in \mathbb{N} \setminus \{0\}$

$$\#\mathrm{App}(\mathscr{L}) \le -r(r-1) - \sum_{\rho \in \sigma(\mathscr{L})} S_\rho(\mathscr{L})$$

Example: $(x^2 - 6x + 1)\, y'''(x) + 3(x-3)\, y''(x) = 0$

Exponents: $e(\infty) = \{-1, 0, 1\}$; $e(3 \pm 2\sqrt{2}) = \{0, 1/2, 1\}$

$$S_\infty(\mathscr{L}) = -3, \quad S_{3 \pm 2\sqrt{2}}(\mathscr{L}) = -3/2$$



$\mathscr{M}$ right factor of order 2 $\Rightarrow$

$$\#\mathrm{App}(\mathscr{M}) \le -2 - S_\infty(\mathscr{M}) - S_{3 \pm 2\sqrt{2}}(\mathscr{M})$$
$$\le -2 - (-1 + 0 - 1) - 2(0 + 1/2 - 1) = 1.$$

Integer linear programming

[Chudnovsky 80]

Generalized Fuchs Relation

Relation similar to Fuchs':

$$\sum_{\rho\in\mathrm{Sing}(\mathscr{L})} S_\rho(\mathscr{L}) - \frac{1}{2} I_\rho(\mathscr{L}) = -\,r(r-1), \quad S_\rho(\mathscr{L}) = \sum_{i=1}^{r} e_i(\rho) - \frac{r(r-1)}{2}.$$

Yields a bound on the number of apparent singularities

$e_i(\rho)$ generalization of a local exponent:
constant coefficient of a polynomial w_i in $z^{1/t}$ s.t.

$$\exp\left(\int \frac{w_i(1/(z-\rho))}{z-\rho}\,dz\right) S(z),\ S \in \mathbb{Q}[\log(z-\rho)][[(z-\rho)^{1/t}]],\ \mathrm{val}_\rho S = 0$$

is a formal solution of $\mathscr{L}$.

$I_\rho(\mathscr{L}) \in \mathbb{Q}$ related to Malgrange's irregularity:

$$I_\rho(\mathscr{L}) = 2 \sum_{1\le i<j\le r} \deg(w_i - w_j)\,.$$

[Bertrand-Beukers 85; Bertrand-Laumon 88]

Example

$$\text{deq} := \{z^2(z-3)y''(z) + (4z^7 + z^2 + 3z - 9)y'(z) + 4z^5(5z+3)y(z) = 0,\ y(0) = 1\}$$

Formal solutions:

at 0: $z^{0} + O(z),\ \exp(3/z)z^{2}(1 + O(z)),$

at 3: $(z-3)^{0}(1 + O(z-3)),\ (z-3)^{-972}(1 + O(z-3)),$

at ∞ : $\dfrac{1}{z^{5}}(1 + O(\dfrac{1}{z})),\ \exp\left(-\dfrac{4}{5}z^5 - 3z^4 - 12z^3 - 54z^2 - 324z\right)\dfrac{1}{z^{972}}(1 + O(\dfrac{1}{z}))\,.$

Relation for $\mathscr{L}$: $(0 + 2 - 1 - 1) + (0 - 972 - 1) + (5 + 972 - 5 - 1) = -2$

$\mathscr{M}$ right factor of order 1 $\Rightarrow$

$$\#\text{App}(\mathscr{M}) \le -(S_0(\mathscr{M}) - \frac{1}{2}\cancel{I_0(\mathscr{M})}) - (S_3(\mathscr{M}) - \frac{1}{2}\cancel{I_3(\mathscr{M})}) - (S_\infty(\mathscr{M}) - \frac{1}{2}\cancel{I_\infty(\mathscr{M})})$$

$$\le 0 + 972 - 5.$$

Demo

Bigger Example (1/2)

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!(n+k)!}{k!^4 (n-k)!^3} z^n$$

satisfies a LDE $\mathscr{L}(f) = 0$ of order 10, with singularities at 0, ∞

At 0: exponents 1,1,0,0,0,0,$\alpha_1, \ldots, \alpha_4$, α_i roots of

$$29412x^4 - 246240x^3 + 764259x^2 - 1042332x + 527381.$$

$$S_0(\mathscr{L}) = 2 + \sum_i \alpha_i - 45 = -\frac{1489}{43}$$

At ∞: formal solutions with leading terms

$$\frac{\ln z}{z^1}, \frac{1}{z^1}, \frac{1}{z^{\beta_1}}, \frac{1}{z^{\beta_2}}, \frac{1}{z^{\beta_3}}, \frac{1}{z^{\beta_4}}, \frac{e^{\gamma_1 z}}{z^{3/2}}, \frac{e^{\gamma_2 z}}{z^{3/2}}, \frac{e^{\gamma_3 z}}{z^{3/2}}, \frac{e^{\gamma_4 z}}{z^{3/2}}$$

+ irreducible pols of deg 4 for β_i, γ_i

$$S_\infty(\mathscr{L}) = 2 + \sum_i \beta_i + 4\frac{3}{2} - 45 = -\frac{1901}{43}, \quad I_\infty(\mathscr{L}) = 60$$

$$-\frac{1489}{43} - \frac{1901}{43} - 30 = -90$$

Bigger Example (2/2)

Right factor $\mathscr{M}$ of order $m \to$ bound on #App($\mathscr{M}$) :

Integer linear programming

max A subject to:

$$A = -m(m-1) - \left(c_{0,1} + c_{0,2} + c_\alpha \frac{360}{43} - \frac{m(m-1)}{2}\right)$$

$$-\left(c_{\infty,1} + c_{\infty,2} + c_\beta \frac{500}{43} + c_\gamma 4\frac{3}{2} - \frac{m(m-1)}{2}\right) + 4d_{\infty,1,\gamma} + 4d_{\infty,2,\gamma} + 16d_{\infty,\beta,\gamma} + 6d_{\infty,\gamma,\gamma} \in \mathbb{N}$$

with constraints: c_i, d_i in $\{0,1\}$,

$$c_{0,1} + \cdots + c_{0,6} + 4c_\alpha = c_{\infty,1} + c_{\infty,2} + 4c_\beta + 4c_\gamma = m,$$

$$d_{\infty,1,2} + 4d_{\infty,1,\beta} + 4d_{\infty,1,\gamma} = c_{\infty,1}(m-1),$$

$$d_{\infty,1,2} + 4d_{\infty,2,\beta} + 4d_{\infty,2,\gamma} = c_{\infty,2}(m-1),$$

$$d_{\infty,1,\beta} + 4d_{\infty,2,\beta} + 3d_{\infty,\beta,\beta} + 4d_{\infty,\beta,\gamma} = c_\beta(m-1),$$

$$d_{\infty,1,\gamma} + 4d_{\infty,2,\gamma} + 4d_{\infty,\beta,\gamma} + 3d_{\infty,\gamma,\gamma} = c_\gamma(m-1)\,.$$

m exponents at each singularity

$m-1$ pairs $w_i - w_j$

Results: for $m = 6 : A \le 4; m = 5 : A \le 2; m = 4 : A \le 0.$

no solution (= no factorization) for other values of m.

General Bounds

$$\mathscr{L} = \mathscr{N}\mathscr{M} \in \mathbb{Q}[z]\langle \partial_z \rangle$$

$$\operatorname{ord} \mathscr{L} = m,\ \deg \mathscr{L} = r,\ \operatorname{height} \mathscr{L} = H$$

If $\operatorname{ord} \mathscr{M} = q$,

$$\deg \mathscr{M} = \exp(2^{m \cdot o(2^m)})$$

$$\deg \mathscr{M} \leq r^2(q+1)\mathscr{E} + \operatorname{pol}(r, m, q)$$

$$\mathscr{E} \leq 2^{(36(q+1)m)^{9(q+1)^2 m^{3m}}} H^{(5(q+1)m)^{9(q+1)^2 m^{3m}}}$$

Ex.: $q = 1,\ m = 2$ bound larger than $10^{10^{4900}} H^{10^{2997}}$

Bounds also available in terms of $\kappa = [\mathbb{K} : \mathbb{Q}]$ when $\mathscr{L} \in \mathbb{K}[z]\langle \partial_z \rangle$

largest modulus of the generalized exponents

Not practical

[Grigoriev 90;Bertrand-Chirskii-Yebbou 04; Bostan-Rivoal-S. 21]

Summary

. Minimization of linear differential equations is algorithmic;
. general bounds are huge;
. computed bounds can be merely large;
. for E-functions, the exceptional values can be computed;
. implementation in Maple, mostly ok.

Thank you