Automatic Proofs of Identities

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I Introduction
Dynamic Dictionary of Mathematical Functions

Aim of the project

DDMF = Mathematical Handbooks + Computer Algebra + Web

1. Develop and use computer algebra algorithms to generate the formulas;
2. Provide web-like interaction with the document and the computation.

http://ddmf.msr-inria.inria.fr/
Equations Are a Good Data Structure

- **Classical:**
  polynomials represent their roots better than radicals.
  **Algorithms:** Euclidean division and algorithm, Gröbner bases.

- **Recent:**
  same for linear differential or recurrence equations.
  **Algorithms:** non-commutative analogues.

About 25% of Sloane’s encyclopedia, 60% of Abramowitz & Stegun.

\[ \text{eqn+ini. cond.} = \text{data structure} \]
Examples of Identities

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \]

\[
\int_{0}^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1-a^4)}{2\pi a^2} \]

\[
\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{3/2}} \, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !} \]

\[
\sum_{k=0}^{n} \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \]

\[
\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \]

[Strehl92]
[GlMo94]
[Doetsch30]
[Andrews74]
[Paule85].
More Identities

\[
\sum_{k=0}^{n} \binom{n}{k} i(k + i)^{k-1}(n - k + j)^{n-k} = (n + i + j)^n \quad \text{[Abel1826]}
\]

\[
\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \binom{n+1}{k+1} = \binom{n}{m}, \quad \text{[Frobenius1910]}
\]

\[
\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \quad \text{[Gessel03]}
\]

\[
\int_{0}^{\infty} x^{k-1} \zeta(n, \alpha + \beta x) \, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha),
\]

\[
\int_{0}^{\infty} x^{\alpha-1} \text{Li}_n(-xy) \, dx = \frac{\pi \left(-\alpha\right)^n y^{-\alpha}}{\sin\left(\alpha \pi\right)},
\]

\[
\int_{0}^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy) \, dx = \frac{\pi y^{-s}}{\sin((a + s)\pi)} \frac{\Gamma(s)}{\Gamma(1 - a)}.
\]
Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

- 1st slide: Zeilberger’s algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: recent generalization of previous ones (with Chyzak & Kauers).

Ideas

Confinement in finite dimension + Creative telescoping.
II Confinement in Finite Dimension
Confinement Provokes Identities

Obvious

\( k + 1 \) vectors in dimension \( k \rightarrow \text{an identity.} \)
Confinement Provokes Identities

**Obvious**

$k + 1$ vectors in dimension $k \rightarrow$ an identity.

Idea: confine a function and all its derivatives.
First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

\[
> \text{series(sin(x)^2+cos(x)^2-1,x,4)};
\]
\[
O(x^4)
\]

Why is this a proof?

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

Second algorithmic proof (same idea):

\[
> \text{fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n} \text{ od};
\]
First Algorithmic Proof: \( \sin^2 + \cos^2 = 1 \)

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First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

> series(sin(x)^2+cos(x)^2-1,x,4);

$$O(x^4)$$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
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What about $\sin' = \cos$?
First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

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> \text{series}(\sin(x)^2 + \cos(x)^2 - 1, x, 4);
\]

\[O(x^4)\]

Why is this a proof?

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. Their squares (and their sum) satisfy a 3rd order LDE;
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Second algorithmic proof (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

\[
> \text{for } n \text{ to 5 do } \text{fibonacci}(n)^2 - \text{fibonacci}(n+1)\text{fibonacci}(n-1) + (-1)^n \text{ od;}
\]
Third Proof: Contiguity of Hypergeometric Series

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1). \]

\[
\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0, \\
\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.
\]
Third Proof: Contiguity of Hypergeometric Series

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\[ \frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \xrightarrow{u_{a,n}} z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0, \]

\[ \frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_{a}F := F(a+1, b; c; z) = \frac{z}{a}F' + F. \]

Gauss 1812: contiguity relation.
\[ \dim = 2 \Rightarrow S_a F, S_a F, F \text{ linearly dependent:} \]
\[ (\text{Coordinates in } \mathbb{Q}(a, b, c, z)). \]

\[ (a+1)(z-1)S_a^2F + ((b-a-1)z+2-c+2a)S_aF + (c-a-1)F = 0. \]
Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.
1. **Monomial ordering**: order on $\mathbb{N}^k$, compatible with $+$, $0$ minimal.

2. **Gröbner basis** of a (left) ideal $\mathcal{I}$: corners of stairs.
Gröbner Basis: Euclidean Division in Several Variables

1. **Monomial ordering**: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.

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3. **Quotient mod $\mathcal{I}$**: basis below the stairs ($\text{Vect}\{\partial^\alpha f\}$).
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4. **Reduction of $P$**: Rewrite $P \mod \mathcal{I}$ on this basis.

→ An access to (finite dimensional) vector spaces
Gröbner Basis: Euclidean Division in Several Variables

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3. Quotient $\text{mod} \mathcal{I}$: basis below the stairs ($\text{Vect}\{\partial^\alpha f\}$).

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5. Dimension of $\mathcal{I}$: “size” of the quotient infinitely far.

→ An access to (finite dimensional) vector spaces
Gröbner Basis: Euclidean Division in Several Variables

1. **Monomial ordering**: order on $\mathbb{N}^k$, compatible with $+\, ,\, 0$ minimal.

2. **Gröbner basis of a (left) ideal $I$**: corners of stairs.

3. **Quotient mod $I$**: basis below the stairs ($\text{Vect}\{\partial^\alpha f\}$).

4. **Reduction of $P$**: Rewrite $P$ mod $I$ on this basis.

5. **Dimension of $I$**: “size” of the quotient infinitely far.

6. **D-finiteness**: $\dim = 0$.

   $\rightarrow$ An access to (finite dimensional) vector spaces
Examples

Binomial coeffs \( \binom{n}{k} \) wrt \( S_n, S_k \)

Hypergeometric sequences

Bessel \( J_\nu(x) \) wrt \( S_\nu, \partial_x \)

Orthogonal pols wrt \( S_n, \partial_x \)

Stirling nbs wrt \( S_n, S_k \)

Abel type wrt \( S_m, S_r, S_k, S_s \)

\[
\text{hgm}(m, k)(k + r)^k(m - k + s)^{m-k} \frac{r}{k+r}
\]

dim = 2 in space of dim 4.
Proposition

\[
\dim \text{ann}(f + g) \leq \max(\dim \text{ann} f, \dim \text{ann} g),
\]
\[
\dim \text{ann}(fg) \leq \dim \text{ann} f + \dim \text{ann} g,
\]
\[
\dim \text{ann} \partial f \leq \dim \text{ann} f.
\]

Algorithms by linear algebra.
Fourth Algorithmic Proof: Mehler’s Identity for Hermite Polynomials

\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \exp \left( \frac{4u(xy-u(x^2+y^2))}{1-4u^2} \right) \sqrt{1-4u^2} \]

1. Definition of Hermite polynomials (D-finite over \( \mathbb{Q}(x) \)): recurrence of order 2;

2. Product by linear algebra: \( H_{n+k}(x) H_{n+k}(y)/(n+k)! \), \( k \in \mathbb{N} \)
generated over \( \mathbb{Q}(x, n) \) by

\[
\frac{H_n(x)H_n(y)}{n!}, \quad \frac{H_{n+1}(x)H_n(y)}{n!}, \quad \frac{H_n(x)H_{n+1}(y)}{n!}, \quad \frac{H_{n+1}(x)H_{n+1}(y)}{n!}
\]

\rightarrow recurrence of order at most 4;

3. Translate into differential equation.
Fifth Algorithmic Proof: Moments of $K_0^n$

$$c_{n,k} = \int_0^{+\infty} t^k K_0(t)^n \, dt$$

1. Bessel $K_0$ satisfies a LDE of order 2;
2. $K_0^n$ satisfies one of order $n + 1$;
3. multiply by $t^k$ and translate into a recurrence.

**Theorem (BorweinSalvy2008)**

$$(k+1)^{n+1}c_{n,k} + \sum_{2\leq j\leq n \atop \text{$j$ even}} P_{n,j}(k)c_{n,k+j} = 0 \quad \text{with} \quad \text{deg } P_{n,j} \leq n+1-j.$$

Also, a fast algorithm.
Initially a conjecture by Bailey, Borwein$^2$, Crandall.
III Creative Telescoping
Summation by Creative Telescoping

\[ I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

**IF** one knows Pascal’s triangle:

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},
\]

then summing over \( k \) gives

\[ I_{n+1} = 2I_n. \]

The initial condition \( I_0 = 1 \) concludes the proof.
Creative Telescoping (Zeilberger 90)

\[ F_n = \sum_k u_{n,k} = ? \]

**IF** one knows \( A(n, S_n) \) and \( B(n, k, S_n, S_k) \) such that

\[
(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,
\]

then the sum “telescopes”, leading to \( A(n, S_n) \cdot F_n = 0 \).
Creative Telescoping (Zeilberger 90)

\[ I(x) = \int_{\Omega} u(x, y) \, dy = ? \]

\textbf{IF} one knows \( A(x, \partial_x) \) and \( B(x, y, \partial_x, \partial_y) \) such that

\[(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,\]

then the integral “telescopes”, leading to \( A(x, \partial_x) \cdot I(x) = 0. \)
Creative Telescoping (Zeilberger 90)

\[ I(x) = \int_{\Omega} u(x, y) \, dy =? \]

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\[
(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,
\]

then the integral “telescopes”, leading to \( A(x, \partial_x) \cdot I(x) = 0. \)

*Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*

Richard P. Feynman 1985

Creative telescoping = “differentiation” under integral + “integration” by parts
Ex.: $\int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = \frac{\pi}{2} J_0(z)$, \( zJ''_0 + J'_0 + zJ_0 = 0, \ J_0(0) = 1 \).
Ex.: \[\int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = \frac{\pi}{2} J_0(z), \quad (zJ''_0 + J'_0 + zJ_0 = 0, \ J_0(0) = 1).\]

\[I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt, \quad I'(z) = \int_0^1 -t \frac{\sin zt}{\sqrt{1 - t^2}} \, dt,\]

\[I''(z) = \int_0^1 -t^2 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = -I(z) + \int_0^1 \sqrt{1 - t^2} \cos zt \, dt,\]

\[I''(z) + I(z) = \left[\frac{\sin zt}{z} \right]_0^1 + \int_0^1 \frac{t}{\sqrt{1 - t^2}} \frac{\sin zt}{z} \, dt = -\frac{I'(z)}{z}.\]
Ex.: \[ \int_0^1 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = \frac{\pi}{2} J_0(z), \quad (zJ''_0 + J'_0 + zJ_0 = 0, \ J_0(0) = 1). \]

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\[ I''(z) = \int_0^1 -t^2 \frac{\cos zt}{\sqrt{1 - t^2}} \, dt = -I(z) + \int_0^1 \sqrt{1 - t^2} \cos zt \, dt, \]
\[ I''(z) + I(z) = \left[ \frac{\sin zt}{\sqrt{1 - t^2}} \right]_0^1 + \int_0^1 \frac{t}{\sqrt{1 - t^2}} \frac{\sin zt}{z} \, dt = -\frac{I'(z)}{z}. \]

ann \[ \frac{\cos zt}{\sqrt{1 - t^2}} \] \[ \equiv A(z, \partial_z) - \partial_t \left( \frac{t^2 - 1}{t} \partial_z \right) \]
no \( t, \partial_t \) anything
Ex.: \[ \int_{0}^{1} \frac{\cos zt}{\sqrt{1-t^2}} \, dt = \frac{\pi}{2} J_0(z), \quad (zJ''_0 + J'_0 + zJ_0 = 0, \ J_0(0) = 1). \]

\[ \text{ann} \frac{\cos zt}{\sqrt{1-t^2}} \ni A(z, \partial_z) - \partial_t \left( \frac{t^2 - 1}{t} \partial_z \right) \]

Creative Telescoping

Input: generators of (a subideal of) \( \text{ann} f \);
Output: \( A, B \) such that \( A - \partial_t B \in \text{ann} f, \ A \) free of \( t, \partial_t \).
Algorithm: sometimes. (Why would they exist?)

Telescoping of \( I \) wrt \( t \):

\[ T_t(I) := (I + \partial_t \mathbb{Q}(z, t)\langle \partial_z, \partial_t \rangle) \cap \mathbb{Q}(z)\langle \partial_z \rangle. \]
Example: Pascal’s Triangle Again

\[(S_nS_k - S_k - 1) \cdot \binom{n}{k} = 0 = (S_n - 2 + (S_k - 1)(S_n - 1)) \cdot \binom{n}{k}.\]

Sum over \(k\) \(\Rightarrow\) \((S_n - 2) \sum_k \binom{n}{k} = 0.\)
Example: Pascal’s Triangle Again

\[(S_n S_k - S_k - 1) \cdot \binom{n}{k} = 0 = (S_n - 2 + (S_k - 1)(S_n - 1)) \cdot \binom{n}{k} \cdot \binom{n}{k} .\]

Reduce all monomials of degree \( \leq s = 2 \):

\[1 \to 1, \quad S_n \to \frac{n + 1}{n + 1 - k}, \quad S_k \to \frac{n - k}{k + 1} \]

\[S_n^2 \to \frac{(n + 2)(n + 1)}{(n + 2 - k)(n + 1 - k)}, \quad S_k^2 \to \frac{(n - k - 1)(n - k)}{(k + 2)(k + 1)} \]

\[S_n S_k \to \frac{n + 1}{k + 1}.\]

Common denominator: \( D_2 = (k + 1)(k + 2)(n + 1 - k)(n + 2 - k) \).

\( D_2, D_2 S_n, D_2 S_k, D_2 S_n^2, D_2 S_k^2, D_2 S_n S_k \) confined in \( \text{Vect}_\mathbb{Q}(n)(1, k1, k^21, k^31, k^41) \).
Example: Pascal’s Triangle Again

\[(S_n S_k - S_k - 1) \cdot \binom{n}{k} = 0 = (S_n - 2 + (S_k - 1)(S_n - 1)) \cdot \binom{n}{k}.\]

Reduce all monomials of degree \(\leq s = 2\):

\[
1 \rightarrow 1, \quad S_n \rightarrow \frac{n + 1}{n + 1 - k} 1, \quad S_k \rightarrow \frac{n - k}{k + 1} 1
\]

\[
S_n^2 \rightarrow \frac{(n + 2)(n + 1)}{(n + 2 - k)(n + 1 - k)} 1, \quad S_k^2 \rightarrow \frac{(n - k - 1)(n - k)}{(k + 2)(k + 1)} 1,
\]

\[
S_n S_k \rightarrow \frac{n + 1}{k + 1} 1.
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Common denominator: \(D_2 = (k + 1)(k + 2)(n + 1 - k)(n + 2 - k)\).

\(D_2, D_2 S_n, D_2 S_k, D_2 S_n^2, D_2 S_k^2, D_2 S_n S_k\) confined in \(\text{Vect}_{\mathbb{Q}(n)}(1, k 1, k^2 1, k^3 1, k^4 1)\).

This has to happen for some degree: \(\deg D_s = O(s)\).
Polynomial Growth

**Definition (Polynomial Growth $p$)**

There exists a sequence of polynomials $P_s$, s.t. for all $(a_1, \ldots, a_k)$ with $a_1 + \cdots + a_k \leq s$, $P_s \partial_1^{a_1} \cdots \partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

**Theorem (ChyzakKauersSalvy2009)**

$$\dim T_t(\mathcal{I}) \leq \max(\dim \mathcal{I} + p - 1, 0).$$
Polynomial Growth

Definition (Polynomial Growth $p$)
There exists a sequence of polynomials $P_s$, s.t. for all $(a_1, \ldots, a_k)$ with $a_1 + \cdots + a_k \leq s$, $P_s \partial_1^{a_1} \cdots \partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

Theorem (ChyzakKauersSalvy2009)
\[ \dim T_t(I) \leq \max(\dim I + p - 1, 0). \]

Proof. Same as above. Set $q := \dim I + p$.
- In degree $s$, $\dim O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \ldots, \partial_{i_q}$: $O(s^{q+1})$;
  \[ \Rightarrow \] any $q$ variables linearly dependent $\Rightarrow \dim \leq q - 1$.

This proof gives an algorithm. Also, bounds available.
Examples (all with $p = 1$)

- Proper hypergeometric [Wilf & Zeilberger 1992]:

$$Q(n, k)\xi^k \prod_{i=1}^{u} (a_i n + b_i k + c_i)! \prod_{i=1}^{v} (u_i n + v_i k + w_i)!,$$

$Q$ polynomial, $\xi \in \mathbb{C}$, $a_i, b_i, u_i, v_i$ integers.

- Differential D-finite (definite integration);

- Stirling: ok for $n \geq 3$, e.g., Frobenius:

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ n+1 \right\} = \langle n \rangle .$$

- Abel type: $\dim = 2 \rightarrow$ ok for $n \geq 4$, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$
IV Conclusion
Conclusion

Summary:
- Linear differential/recurrence equations as a data structure;
- Confinement in vector spaces + creative telescoping → identities.

Also:
- Fast algorithms: Zeilberger 1990 (hypergeom); Chyzak 2000 (D-finite)
  Us 2009 (non-D-finite).
- Bounds → identities;
- Fast algorithms for special classes;
- Efficient numerical evaluation.

Open questions:
- Replace polynomial growth by something intrinsic;
- Exploit symmetries;
- Structured Padé-Hermite approximants;
- Understand non-minimality.
Conclusion

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  - Bounds $\rightarrow$ identities;
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*THE END*