# Gfun — 15 Years Later

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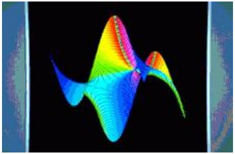

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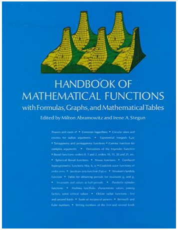
# I Introduction

# Framework: D-finite Series & Sequences

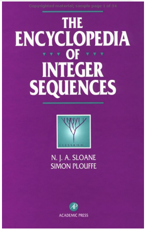
**Code: gfun (S. & Zimmermann 94)**

Maple package to guess, manipulate and prove D-finite identities.




About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

| eqn+ini. cond.=data structure |
|---|




**Definition**

A *series* $f(x) \in \mathbb{K}[[x]]$ is D-finite over $\mathbb{K}$ when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$. (LDE)
A *sequence* $u_n$ is D-finite over $\mathbb{K}$ when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{K}(n)$. (LRE)

References: Stanley vol. 2, $A = B$ (Petkovšek, Wilf, Zeilberger 96)

# II Guessing Identities

# Mehler's Identity on Hermite Polynomials (1866)

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = ?$$

Answer: compute the first 10 polynomials and guess!

```
> L:=[seq(orthopoly[H](n,x)*orthopoly[H](n,y),n=0..9)];
```

$$L := [1, 4xy, (-2+4x^2)(-2+4y^2), (8x^3-12x)(8y^3-12y), (12+16x^4-48x^2)(12+16y^4-48y^2), (32x^5-160x^3+120x)(32y^5-160y^3+120y), (-120+64x^6-480x^4+720x^2)(-120+64y^6-480y^4+720y^2), (128x^7-1344x^5+3360x^3-1680x)(128y^7-1344y^5+3360y^3-1680y), (1680+256x^8-3584x^6+13440x^4-13440x^2)(1680+256y^8-3584y^6+13440y^4-13440y^2), (512x^9-9216x^7+48384x^5-80640x^3+30240x)(512y^9-9216y^7+48384y^5-80640y^3+30240y)]$$

```
> deq:=gfun[listtodiffeq](L,F(u),['egf']);
```

$$deq := \left[\left\{(1-8u^2+16u^4)\left(\frac{d}{du}F(u)\right)+(-4xy+8ux^2-4u+8uy^2-16u^2xy+16u^3)F(u), F(0)=1\right\}, egf\right]$$

```
> dsolve(deq[1],F(u)) assuming 0<u,u<1/2;
```

$$F(u) = \frac{e^{\frac{-4xyu+x^2+y^2}{(2u+1)(2u-1)}}\sqrt{\frac{1}{(-2u+1)(2u+1)}}}{e^{-x^2-y^2}}$$

```
>
```

# Guess = Good Approximant

## Definition (Padé-Hermite Approximant)

The vector of polynomials $(P_1,\dots,P_k)$ with $\deg P_i \le d_i$ is a Padé-Hermite approximant of type $(d_1,\dots,d_k)$ for a vector of power series $(f_1,\dots,f_k)$ when

$$P_1 f_1 + \cdots + P_k f_k = O(x^{d_1+\cdots+d_k+k-1}).$$

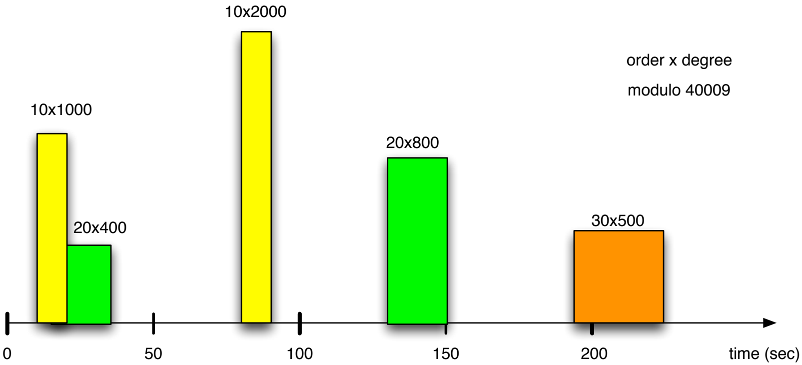
Special cases: (given one series $y$)

- $k=2$, $f_1=-1$, $f_2=y$: Padé-approximant;
- $f_i = y^{i-1}$, $i=1,\dots,k$: algebraic approximants (Guttmann);
- $f_i = y^{(i-1)}$, $i=1,\dots,k$: differential approximants.

Algorithms and complexity $(D = d_1+\cdots+d_k)$:

- Linear algebra: $O(D^\omega)$ ops
  ($\omega \le 3$ complexity of matrix product);
- minimal basis of approximants in $O(k^\omega D^{1+\epsilon})$ ops
  (Beckermann & Labahn 94);
- genset in $O(k^\omega (D/k)^{1+\epsilon})$ ops (Storjohann 06)

# Example from a Study of Susceptibility of an Ising Model




- Direct linear algebra runs out of memory;
- $30 \times 1,000$ coefficients in about 10 min.;
- Work in progress (Bostan, Maillard *et alii*);

## Code

Magma code available from Alin Bostan.

# III Computing Identities

# Bound = Proof = Algorithm

$> \texttt{series}(\texttt{sin}(\texttt{x})^2 + \texttt{cos}(\texttt{x})^2, \texttt{x}, 4);$

$$1 + O(x^4)$$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. it is not singular at 0, Cauchy's theorem concludes.

# Proof of Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x,n)$ by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\longrightarrow$ recurrence of order at most 4;
3. Translate into differential equation.

# I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\}$ :

> $R_2 := subs(H=H_2, x=y, R_1)$;

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

# II. Product

> $R_3 := gfun:\text{-}poltorec(H(n)\cdot H_2(n)\cdot v(n), [R_1, R_2, \{v(n+1)\cdot(n+1) = v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n))$;

$$R_3 := \Big\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xyc(n+1)+(-8n-20+8y^2+8x^2)c(n+2)-4xc(n+3)y+(n+4)c(n+4)\Big\}$$

# III. Differential Equation

> $gfun:\text{-}rectodiffeq(R_3, c(n), f(u))$;

$$\left\{(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy)f(u)+(16u^4-8u^2+1)\left(\frac{d}{du}f(u)\right), f(0)=1\right\}$$

> $dsolve(\%, f(u))$;

$$f(u) = \frac{I\,e^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{e^{(-y^2-x^2)}\sqrt{2u+1}\sqrt{2u-1}}$$

>

# Closure Properties

**Theorem (XIXth century)**

- *D-finite series and sequences over* $\mathbb{K}$ *form* $\mathbb{K}$*-algebras;*
- $y$ *algebraic,* $f$ *is D-finite* $\Longrightarrow y, f \circ y, \exp \int y$ *D-finite.*

Proof.

Linear algebra in finite dimension. □

**Corollary**

*D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf↔egf).*

All implemented in `Gfun`.

# Example: Recursion for Bessel Moments

**Proposition (Borwein & S. 08)**

$$C_{n,k} := \frac{1}{n!}\int_0^\infty \cdots \int_0^\infty \frac{dx_1 \cdots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}$$

*satisfies a (computable) linear recurrence wrt $k$.*

Proof.

1. $c_{n,k} := k!n!2^{-n}C_{n,k} = \int_0^\infty t^k K_0(t)^n\,dt$;
2. $K_0$ satisfies a linear diff. eqn. of order 2;
3. $K_0^n$ satisfies a linear diff. eqn. of order $n+1$;
4. this translates into a linear recurrence for the $c_{n,k}$.

Step 3 can be optimized + more precise form of the recurrence. □

# Linear Recurrences for Algebraic Series

**Proposition (Bostan-Chyzak-Lecerf-S.-Schost 07)**

*If $P(x, y)$ has degree at most $D$ then the coefficients of its series solutions obey a recurrence of order at most $D^2 + 1$.*

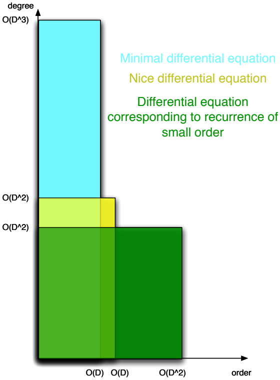
(First observed experimentally, then proved rigorously)

minimal$\neq$nice;

Guess + Bound + Algebraic

= Proof

= Fast Algorithm.



# IV Multivariate Functions & Sequences

# D-finiteness in Several Variables

## Definition

A series $F(x, y) \in \mathbb{K}[[x, y]]$ is D-finite when its derivatives generate a finite-dimensional space over $\mathbb{K}(x, y)$.

More generally, finite dimensional quotients by ideals of operators in suitable Ore algebras $\Rightarrow$ closure properties (Chyzak & S. 98).

Example: Contiguity of Hypergeometric Series (Gauss 1812)

# D-finiteness in Several Variables

More generally, finite dimensional quotients by ideals of operators in suitable Ore algebras ⇒ closure properties (Chyzak & S. 98).

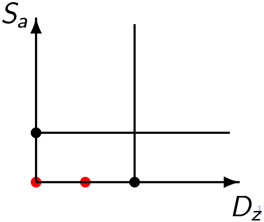
Example: Contiguity of Hypergeometric Series (Gauss 1812)

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

dim=2 ⇒ $S_a^2F, S_aF, F$ linearly dependent



# Creative Telescoping (Zeilberger 90)

$$F_n = \sum_k u_{n,k} = ?$$

**IF** one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to $A(n, S_n) \cdot F_n = \text{simple.}$

# Creative Telescoping (Zeilberger 90)

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

**IF** one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = \text{simple}.$

> *Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*
>
> Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

# Creative Telescoping (Zeilberger 90)

- General case: Find annihilators of

$$I(x_1,\dots,x_{n-1}) = \partial_n^{-1}\big|_{\Omega}\, u(x_1,\dots,x_n)$$

  knowing generators of $\mathrm{Ann}_u$ in
  $\mathbb{O}_n = \mathbb{K}(x_1,\dots,x_n)[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$;
- Crucial step: compute $(\underbrace{\mathbb{O}_n\,\mathrm{Ann}_u}_{\text{left ideal}} + \underbrace{\partial_n\mathbb{O}_n}_{\text{right ideal}}) \cap \mathbb{O}_{n-1}$.

  Algorithms: Zeilberger 91 (dim=1), Chyzak 00 (gl case).

# Applications of Creative Telescoping

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \quad \text{(Strehl 92)}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{(GlMo 94)}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \quad \text{(Doetsch30}$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{(Andrews74}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{(Paule 85).}$$

**Code**

All $<$ 1min. with Frédéric Chyzak's `Mgfun` (98).

# Example: Nice Bessel Integral (Glasser & Montaldi 94)

$$\int_0^\infty xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx = \frac{1}{2\pi a^2}\ln\frac{1}{1-a^4}.$$

```
> libname:="/Users/salvy/lib/maple/Algolib",libname:
> f:=x*BesselJ(1,a*x)*BesselI(1,a*x)*BesselY(0,x)*BesselK(0,x);
```

$$f := x\,\mathrm{BesselJ}(1, a\,x)\,\mathrm{BesselI}(1, a\,x)\,\mathrm{BesselY}(0, x)\,\mathrm{BesselK}(0, x)$$

```
> sys:=Mgfun:-dfinite_expr_to_sys(f,y(x::diff,a::diff));
```

$$sys := \Big\{(-4a^4x^4+4x^4+3)\,y(x,a) + 12x^2a\left(\frac{\partial^3}{\partial x^2\partial a}y(x,a)\right) - 4a^3x\left(\frac{\partial^4}{\partial x\,\partial a^3}y(x,a)\right) - 3x\left(\frac{\partial}{\partial x}y(x,a)\right)$$
$$+26a\left(\frac{\partial}{\partial a}y(x,a)\right) - 26ax\left(\frac{\partial^2}{\partial x\,\partial a}y(x,a)\right) + 40a^2\left(\frac{\partial^2}{\partial a^2}y(x,a)\right) + 6a^2x^2\left(\frac{\partial^4}{\partial x^2\partial a^2}y(x,a)\right)$$
$$+x^2\left(\frac{\partial^2}{\partial x^2}y(x,a)\right) - 24a^2x\left(\frac{\partial^3}{\partial x\,\partial a^2}y(x,a)\right) + 8a^3\left(\frac{\partial^3}{\partial a^3}y(x,a)\right) - 4x^3a\left(\frac{\partial^4}{\partial x^3\partial a}y(x,a)\right) + x^4\left(\frac{\partial^4}{\partial x^4}y(x,a)\right),$$
$$4x^4a^3y(x,a) + a^3\left(\frac{\partial^4}{\partial a^4}y(x,a)\right) + 3\left(\frac{\partial}{\partial a}y(x,a)\right) - 3a\left(\frac{\partial^2}{\partial a^2}y(x,a)\right) + 4a^2\left(\frac{\partial^3}{\partial a^3}y(x,a)\right)\Big\}$$

```
> deq:=Mgfun:-int_of_sys(sys,x=0..infinity,_takayama_algo);
```

$$deq := \Big\{32a^3y(a) + (16a^6-4a^2)\left(\frac{d^3}{da^3}y(a)\right) + (-a^3+a^7)\left(\frac{d^4}{da^4}y(a)\right) + (73a^5+3a)\left(\frac{d^2}{da^2}y(a)\right)$$
$$+(103a^4-3)\left(\frac{d}{da}y(a)\right)\Big\}$$

```
> normal(eval(deq,y(a)=1/2/Pi/a^2*ln(1/(1-a^4))));
```

$$\{0\}$$

```
> sol:=subs(dsolve(deq,y(a)),y(a)) assuming a>0,a<1;
```

$$sol := \frac{\_C1}{a^2} + \frac{\_C2\ln\left((-1+a)(a+1)(a^2+1)\right)}{a^2} + \frac{\_C3\left(\ln(a+1)+\ln(-1+a)-\ln(a^2+1)\right)}{a^2}$$
$$+\frac{1}{a^2}\Big(\_C4\big(2\,\mathrm{dilog}(a+1) + 2\ln(a)\ln(a+1) - 2\,\mathrm{dilog}(a) + 2\ln(a)\ln(I(-I+a)) + 2\ln(a)\ln(-I(I+a))$$
$$+2\,\mathrm{dilog}(I(-I+a)) + 2\,\mathrm{dilog}(-I(I+a)) - \ln\left((-1+a)(a+1)(a^2+1)\right)\big)\Big)$$

# V D-finiteness in Infinitely Many Variables

# D-finite Symmetric Series (Gessel 90)

Algebra of symmetric functions: $\Lambda := \mathbb{K}[[p_1, p_2, \dots]]$

| | |
|---|---|
| power $p_k$ | $p_3 = x_1^3 + x_2^3 + x_3^3 + \cdots$ |
| homogeneous $h_k$ | $h_3 = x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + \cdots + x_1 x_2 x_3 + \cdots$ |
| monomial $m_\lambda$ | $m_{(3,2,1)} = x_1^3 x_2^2 x_3 + x_2^3 x_1^2 x_3 + \cdots$ |

## Definition

$F \in \Lambda[[t]]$ D-finite if for any $n$, $F(p_1, \dots, p_n, 0, \dots; t)$ D-finite.

## Theorem (Gessel 90)

- *Closed under $+$, $\times$, $\partial/\partial p_i$, algebraic substitution.*
- *[Technical conds] closed under plethysm and scalar product.*

Scalar product: $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$

Adjoint: $\langle \phi F, G \rangle = \langle F, \phi^\perp G \rangle$, with $p_k^\perp = k \frac{\partial}{\partial p_k}$, $\left(\frac{\partial}{\partial p_k}\right)^\perp = \frac{p_k}{k}$.

# $k$-uniform Young Tableaux

| | | | | | |
|---|---|---|---|---|---|
| 4 | 4 | | | | |
| 3 | 3 | 5 | | | |
| 2 | 2 | 3 | 4 | | |
| 1 | 1 | 1 | 2 | 5 | 5 |

Question: Asymptotic number of semi-standard Young tableaux filled with $k$ 1's, $k$ 2's,..., $k$ n's?

**Result (proof for $k = 1, 2$, partial for $k = 3, 4$, conj. beyond)**

$$\sim \frac{1}{\sqrt{2}} \left( \frac{e^{k-2}}{2\pi} \right)^{k/4} n!^{k/2-1} \left( \frac{k^{k/2}}{k!} \right)^{n} \frac{\exp \sqrt{kn}}{n^{k/4}}, \qquad n \to \infty.$$

Method:

Combinatorics → Symmetric functions → LDE → Asymptotics.

# Algorithm (Chyzak, Mishna, S. 05)

$$\Big\langle \underbrace{\exp\left(\sum_{n=1}^{\infty}\frac{p_n^2}{2n}+\sum_{n\ \text{odd}}^{\infty}\frac{p_n}{n}\right)}_{\text{All semi-standard Young tableaux}}, \underbrace{\sum_{n\geq 0} h_k^n t^n}_{\text{Extract coeffs corresponding to } k\text{-uniform}} \Big\rangle$$

$$= \left\langle \exp\left(\sum_{n=1}^{k}\frac{p_n^2}{2n}+\sum_{n\ \text{odd}}^{k}\frac{p_n}{n}\right), \sum_{n\geq 0} h_k^n t^n \right\rangle =: \langle F, G\rangle$$

Wanted: $\big( \underbrace{\operatorname{Ann}_F^{\perp}}_{\text{right ideal}} + \underbrace{\operatorname{Ann}_G}_{\text{left ideal}} \big) \cap \mathbb{K}[t,\partial_t]$.

## Code

`ScalarProduct` available from Marni Mishna.

# VI Conclusion

# Work in Progress

- Minimal recurrences and differential equations by removing apparent singularities (ChDuLeMaMiSa08) code available;
- Better/faster guessing in `gfun` (also ask M. Rubey);
- Automatic bounds and fast numerical evaluation (M. Mezzarobba) code available;
- Fast change of bases (orthogonal polynomials, Bessel functions,...) (A. Benoit);
- `DDMF`: Dynamic Dictionary of Mathematical Functions;
- Faster `Mgfun`.

Do not hesitate to ask for more, or provide code!