Fast compact solutions of linear differential or difference equations

Variants of some of S. Abramov's algorithms exploiting fast multiplication

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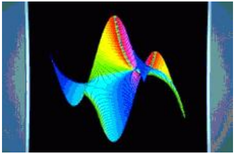

Joint work with A. Bostan, F. Chyzak and T. Cluzeau (ISSAC'05,06 and more)

Waterloo Abramov Workshop

I Introduction

Timings (I)

$$(2x+1)^3 y'' + (2x+1)(8x+3)y' + 2N((4x+2)N - 4x - 1)y = 0$$

has no polynomial solution ... but it takes time to detect this.

N	ABP	New	New
2^{12}	62.59	0.44	0.03
2^{14}	4597.2	2.40	0.07
2^{16}	$>$ 4Gb	14.67	0.19
2^{22}		3060.1	2.54
	$\tilde{O}(N^2)$	$\tilde{O}(N)$	$\tilde{O}(\sqrt{N})$
	deterministic		proba.

Timings (II)

$$S_m = \sum_n \binom{2n+m+N}{N} \binom{2m}{2n} \binom{m}{n} \quad \rightarrow \text{3rd order rec}$$

	Classical Zeilberger				Compact Z., random m			
N	$r=0$	$r=1$	$r=2$	$r=3$	$r=0$	$r=1$	$r=2$	$r=3$
2^4	0.1	0.2	0.3	0.6	0.1	0.3	0.9	2.5
2^5	0.3	0.7	1.5	3.4	0.1	0.5	1.4	5.1
2^6	2.9	6.8	12.	34.3	0.2	0.7	2.6	7.3
2^7	44.	131.	276.	1203.	0.3	1.5	5.0	15.2
2^8	1793.	$>$ 2Gb			0.5	2.7	11.3	35.5

deterministic — proba.

$\tilde{O}(N^4)$?

$\tilde{O}(N)$

symbolic m, det.: $\tilde{O}(N^2)$

II Matrix Factorials

Binary Splitting & Bit Complexity

Problem (Fast computation of $N! = 1 \times \cdots \times N$)

Naïve way: complexity $\tilde{O}(N^2)$

- Binary Splitting:

$$N! = \underbrace{(1 \times \cdots \times \lfloor N/2 \rfloor)}_{\text{size } \frac{1}{2} N \log N} \times \underbrace{((\lfloor N/2 \rfloor + 1) \times \cdots \times N)}_{\text{size } \frac{1}{2} N \log N}$$

  and recurse. Complexity $\tilde{O}(N)$.

- Extends to matrix factorials $C(N)C(N-1)\cdots C(1)$, same complexity $\rightarrow$ recurrences of arbitrary order.

[Hakmem72, Brent75, Chudnovsky^{2}88]

Baby-Steps/Giant-Steps

Problem ($N!$ mod p in less than N operations)

Naïve: N arithmetic operations

- Strassen 76: $\tilde{O}(\sqrt{N})$.
  1. Compute $Q(x) = (x+1)\cdots(x+k)$ — $\tilde{O}(k)$;
  2. Evaluate $Q(0), Q(k), \ldots, Q(N-k)$ — $\tilde{O}(N/k)$;
  3. Compute their product — $O(N/k)$.
- Extends to matrix factorials in $\tilde{O}(\sqrt{N})$ [Chudnovsky^{2}88];
- Fast extrapolation saves a log [BoGaSc03].

Application: dimension of kernel of $C(N)C(N-1)\cdots C(1)$ over $\mathbb{Q}$

Probabilistic algorithm in $\tilde{O}(\sqrt{N})$ bit operations [BoClSa05].

III Compact Representations

Solutions with Finite Support

Problem (Solutions such that $u_n = 0$ for all large n)

$$a_r(n)u_{n+r} + \cdots + a_0(n)u_n = 0.$$

- Upper end of finite support: largest integer root N of a_0;
- Undetermined coefficients $\rightarrow$

  $$u_0, u_1, \ldots, u_N, u_{N+1}, \ldots, u_{N+r-1} \text{ in } \tilde{O}(N^2) \text{ bit ops;}$$

- Matrix factorial $\rightarrow$ initial conditions in $\tilde{O}(N)$ bit ops and probabilistic test in $\tilde{O}(\sqrt{N})$.

Works for regular case ($0 \notin a_r(\mathbb{N})$) and irregular case (more technical).

Application: polynomial solutions

- Linear differential equation [BoClSa05]: linear recurrence on the coefficients;
- Linear difference equation [BoChClSa06]: same in the binomial basis [Boole1872,AbBrPe95].

Operations on the Compact Representation

Definition (Compact Representation)

Recurrence + initial conditions + bound N on support.

Data-structure of size $\tilde{O}(N)$ for a polynomial of size $\tilde{O}(N^2)$.

Lemma (Classical)

If u_n and v_n satisfy linear recurrences, so does $w_N := \sum_{n=0}^{N} u_n v_n$.

Corollary

P polynomial in compact representation, a algebraic number, then $P^{(k)}(a)$ can be computed in $\tilde{O}(N)$ bit operations (with $k = O(N)$).

Applications:

- $\exp(1), \gamma, \pi, \ldots$ [Hakmem,Brent,ChCh].
- Rational solutions of linear differential equations (gcd with $(x-a_1)^{N_1} \cdots (x-a_k)^{N_k}$) $\rightarrow$ quasi-optimal wrt solution.

IV Hypergeometric Summation

Abramov's Algorithm using Compact Representation (I)

Definition (Gosper-Petkovšek normal form)

$$\frac{P(n)}{Q(n)} = \frac{A(n)}{B(n)}\frac{C(n+1)}{C(n)}, \qquad \begin{cases} A(n) \wedge C(n) = B(n) \wedge C(n+1) = 1, \\ A(n) \wedge B(n+h) = 1 \quad (h \in \mathbb{N}). \end{cases}$$

deg C potentially exponential. Compact form via [Abramov95]:

$$\frac{C(n+1)}{C(n)} = \frac{g_1(n)}{g_1(n-h_1)} \cdots \frac{g_s(n)}{g_s(n-h_s)},$$

h_i's roots of the dispersion polynomial, g_i factors of P.

Theorem (Abramov 95)

The polynomial $C(n)$ of the Gosper-Petkovšek form of $(a_r(n-r+1), a_0(n))$ is a multiple of the denominator of all rational solutions of $a_r(n)u_{n+r} + \cdots + a_0(n)u_n = f(n)$.

Abramov's Algorithm using Compact Representation (II)

Rational solutions of LRE

$$a_r(n)u(n+r) + \cdots + a_0(n)u(n) = f(n).$$

1. Compute a compact representation of $C(n)$.
2. Change unknown: $u(n) = v(n)/C(n)$.
3. Homogeneous case ($f = 0$):
   reduce to same denominator and find polynomial solutions;
4. Non-homogeneous case (f non-zero polynomial):
   make homogeneous first. (Otherwise, rhs of expl degree.)

Corollary: Compact Gosper algorithm for indefinite summation.

Zeilberger's Algorithm using Compact Representation

Input: two rational functions $t_{n+1,m}/t_{n,m}$ and $t_{n,m+1}/t_{n,m}$
Output: linear recurrence $\sum_{i=0}^{r} \lambda_i(m) U_{m+i} = 0$ for $U_m = \sum_n t_{n,m}$.
For $r = 0, 1, 2, \dots$ while not found do

1. Construct (E_r) : $u_{n+1,m}\dfrac{t_{n+1,m}}{t_{n,m}} - u_{n,m} = \displaystyle\sum_{i=0}^{r} \lambda_i(m)\dfrac{t_{n,m+i}}{t_{n,m}}$;
2. Find if $\exists \lambda_1, \dots, \lambda_r \in \mathbb{Q}(m)$ s.t. solution $u_{n,m} \in \mathbb{Q}(n, m)$ via a linear system of size N in the λ_i's and the coeffs of numer(u).

Compact Version of Step 2

1. Take numerator of (E_r) and compute $C(n)$ compact multiple of denominator of sols;
2. Homogenize and change variable $u = v/C$ in this order;
3. This is not linear in the λ_i's. Linearity retrieved on subspace.
4. Compute system by matrix factorials. $(\to \tilde{O}(N^2))$

Can also be used for random $m \in \mathbb{Q}$ $(\to \tilde{O}(\sqrt{N})$ proba).

V Conclusion

Final Comments

- Thresholds to be found.
- Extensions to systems, to more general creative telescoping.
- In the difference case, gcd in compact representation?
- Presence of polynomial solutions in subexponential complexity?