Automatic Proofs of Special Functions or Combinatorial Identities

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I Introduction
Theorem (Richardson 68)

In the class obtained from $\mathbb{Q}(x)$, $\pi$, log 2 by the operations $+,-,\times$ and composition with exp, sin and $| \cdot |$, testing for zero-equivalence is undecidable.

Consequences:

1. “Simplification” is always difficult;
2. Automatic proofs cannot cover very general classes of identities;
3. Computer algebra isolates classes for which it can provide algorithms.
$x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.$
\[ x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}. \]

\[ \mathbb{Q}(\exp(i\pi/7)) \text{ has dim 6 over } \mathbb{Q} \]
$x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.$

$\mathbb{Q}(\exp(i\pi/7))$ has dim 6 over $\mathbb{Q}$

Coordinates of $x$

Algebraic Numbers and Finite Dimension

\[ x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}. \]

\( \mathbb{Q}(\exp(i\pi/7)) \) has dim 6 over \( \mathbb{Q} \)

Coordinates of \( x^2 \)

**Tools:** Euclidean division, (extended) Euclidean algorithm, linear algebra.
Algebraic Numbers and Finite Dimension

\[ x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}. \]

\( \mathbb{Q}(\exp(i\pi/7)) \) has dim 6 over \( \mathbb{Q} \)

Coordinates of \( x^2 \)

**Definition**

A number \( x \in \mathbb{C} \) is *algebraic* when its powers generate a finite-dimensional vector space over \( \mathbb{Q} \).

**Tools**: Euclidean division, (extended) Euclidean algorithm, linear algebra.
> x := sin(2*Pi/7)/sin(3*Pi/7)^2-sin(Pi/7)/sin(2*Pi/7)^2+sin(3*Pi/7)/sin(Pi/7)^2;

\[
\begin{align*}
\sin\left(\frac{2}{7}\pi\right) & \quad \sin\left(\frac{1}{7}\pi\right)^2 \\
\sin\left(\frac{3}{7}\pi\right)^2 & \quad \sin\left(\frac{2}{7}\pi\right) \\
\end{align*}
\]

> convert(x, exp);

\[
\begin{align*}
2 \ln\left(e^{\frac{2}{7}\pi} - \frac{1}{e^{\frac{2}{7}\pi}}\right) & \quad 2 \ln\left(e^{\frac{1}{7}\pi} - \frac{1}{e^{\frac{1}{7}\pi}}\right) & \quad 2 \ln\left(e^{\frac{3}{7}\pi} - \frac{1}{e^{\frac{3}{7}\pi}}\right) \\
\left(e^{\frac{2}{7}\pi} - \frac{1}{e^{\frac{3}{7}\pi}}\right)^2 & \quad \left(e^{\frac{1}{7}\pi} - \frac{1}{e^{\frac{2}{7}\pi}}\right)^2 & \quad \left(e^{\frac{3}{7}\pi} - \frac{1}{e^{\frac{1}{7}\pi}}\right)^2
\end{align*}
\]

> R := subs([seq(exp(I*Pi/7*j) = X^j, j = 1..6)], %);

\[
\begin{align*}
R := 2 \ln\left(\frac{X^2 - \frac{1}{X^2}}{X^3 - \frac{1}{X^3}}\right) & \quad 2 \ln\left(\frac{X - \frac{1}{X}}{X^2 - \frac{1}{X^2}}\right) & \quad 2 \ln\left(\frac{X^3 - \frac{1}{X^3}}{X^2 - \frac{1}{X^2}}\right) \\
\end{align*}
\]

> minpol := normal((X^7+1)/(X+1));

\[
\text{minpol} := X^6 - X^5 + X^4 - X^3 + X^2 - X + 1
\]

> gcdex(denom(R), minpol, X, 'u', 'v');

> rem(numer(R)*u, minpol, X);

\[-2 - 4 I X^4 - 4 I X^2 + 4 I X\]

> rem(\%^2, minpol, X);

\[28\]
Automatic Univariate Identities

\[ F_n F_{n+2} - F_{n+1}^2 = (-1)^n \]

\[ \frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1} \]

\[ 2F_1 \left( \begin{array}{c} a, b \\ a + b + 1/2 \end{array} \bigg| z \right) = 2F_1 \left( \begin{array}{c} 2a, 2b \\ 2 \end{array} \bigg| \frac{1 - \sqrt{1 - z}}{2} \right) \]

\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \exp \left( \frac{4u(xy - u(x^2 + y^2))}{1 - 4u^2} \right) \frac{1}{\sqrt{1 - 4u^2}} \]

Cassini
Catalan numbers
Legendre
Mehler

Bruno Salvy
Special Functions and Combinatorial Identities
Automatic Multivariate Identities

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3,
\]

\[
\sum_{n=0}^{+\infty} P_n(x) y^n = \frac{1}{\sqrt{1 - 2xy + y^2}}, \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 x^k
\]

\[
\int_{0}^{+\infty} xJ_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1 - a^4)}{2\pi a^2},
\]

\[
\int_{0}^{\phi} \frac{(1 + 2xy + 4y^2) \exp \left( \frac{4x^2y^2}{1+4y^2} \right)}{y^{n+1}(1 + 4y^2)^{3/2}} \, dy = \frac{n! H_n(x)}{\lfloor n/2 \rfloor!},
\]

\[
\sum_{k=0}^{n} \frac{q^{k^2}}{(q; q)_{k}(q; q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}}
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j}(q; q)i(q; q)_{j}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k}(q; q)_{n-k}}.
\]
II D-finiteness in One Variable
Definition

A series $f(x) \in K[[x]]$ is D-finite over $K$ when its derivatives generate a finite-dimensional vector space over $K(x)$. (LDE)

A sequence $u_n$ is D-finite over $K$ when its shifts $(u_n, u_{n+1}, \ldots)$ generate a finite-dimensional vector space over $K(n)$. (LRE)

About 25% of Sloane’s encyclopedia, 60% of Abramowitz & Stegun.

Tools: right Euclidean division; right (extended) Euclidean algorithm; linear algebra; equivalence via generating series. Implemented in gfun [SaZi94].
Examples

exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arctan,
arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, \( pF_q \)
(includes Bessel \( J, Y, I \) and \( K \), Airy \( Ai \) and \( Bi \) and
polylogarithms), Struve, Weber and Anger fcns, the large class of
algebraic functions, . . .
First Proof: $\sin^2 + \cos^2 = 1$

$$> \text{series}(\sin(x)^2+\cos(x)^2,x,4);$$

$$1 + O(x^4)$$

Why is this a proof?

---

Second proof (same idea):

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$$

for $n$ to 5 do

$$\text{fibonacci}(n)^2 - \text{fibonacci}(n+1)\text{fibonacci}(n-1) + (-1)^n$$

od;
First Proof: \( \sin^2 x + \cos^2 x = 1 \)

\[ > \text{series}((\sin(x))^2 + (\cos(x))^2, x, 4); \]

\[ 1 + O(x^4) \]

Why is this a proof?

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: \( y' = 0 \);
4. \( \sin^2 + \cos^2 - 1 \) satisfies a LDE of order at most 4;
5. it is not singular at 0;

Second proof (same idea): \( F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \)

\[ > \text{for } n \text{ to 5 do} \]

\[ \text{fibonacci}(n)^2 - \text{fibonacci}(n+1)\text{fibonacci}(n-1) + (-1)^n \text{ od}; \]
Third Proof: Mehler’s Identity for Hermite Polynomials

\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \exp \left( \frac{4u(xy-u(x^2+y^2))}{1-4u^2} \right) / \sqrt{1-4u^2} \]

1. Definition of Hermite polynomials (D-finite over \( \mathbb{Q}(x) \)): recurrence of order 2

2. Product by linear algebra: \( H_{n+k}(x) H_{n+k}(y)/(n+k)! \), \( k \in \mathbb{N} \)
   generated over \( \mathbb{Q}(x, n) \) by
   \[ \frac{H_n(x) H_n(y)}{n!}, \frac{H_{n+1}(x) H_n(y)}{n!}, \frac{H_n(x) H_{n+1}(y)}{n!}, \frac{H_{n+1}(x) H_{n+1}(y)}{n!} \]
   \[ \rightarrow \text{recurrence of order at most 4}; \]

3. Translation into a differential equation
I. Definition

\[
R_1 := \{ H(n + 2) = (-2n - 2)H(n) + 2H(n + 1)x, H(0) = 1, H(1) = 2x \} : \\
R_2 := \text{subs}(H = H_2, x = y, R_1); \\
R_2 := \{ H_2(0) = 1, H_2(n + 2) = (-2n - 2)H_2(n) + 2H_2(n + 1)y, H_2(1) = 2y \}
\]

II. Product

\[
R_3 := \text{gfun}:-\text{poltorec}(H(n) \cdot H_2(n) \cdot v(n), [R_1, R_2, \{ v(n + 1) \cdot (n + 1) = v(n), v(1) = 1 \}], [H(n), H_2(n), v(n)], c(n)); \\
R_3 := \left\{ c(0) = 1, c(1) = 4xy, c(2) = 8x^2y^2 + 2 - 4y^2 - 4x^2, c(3) = \frac{32}{3}x^3y^3 + 24xy - 16x^3y - 16x^3y, (16n + 16)c(n) - 16xy c(n + 1) + (-8n - 20 + 8y^2 + 8x^2)c(n + 2) - 4xc(n + 3)y + (n + 4)c(n + 4) \right\}
\]

III. Differential Equation

\[
R_3, c(n), f(u) \}; \\
(16u^3 - 16u^2yx - 4u + 8uy^2 + 8ux^2 - 4xy)f(u) + (16u^4 - 8u^2 + 1)\left( \frac{d}{du}f(u) \right), f(0) = 1
\]

\[
\text{dsolve}(%, f(u)); \\
f(u) = \frac{\left( -4xyu + y^2 + x^2 \right)}{\left( \frac{2u - 1}{2u + 1} \right)} \cdot \frac{1}{e^{-y^2 - x^2}} \sqrt{2u + 1} \sqrt{2u - 1}
\]
Euclidean Division & Finite Dimension

**Theorem (XIXth century)**

_D-finite series and sequences over \( K \) form \( K \)-algebras._

**Proof.**

Linear algebra

**Corollary**

_D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf\(\leftrightarrow\)egf)._
Euclidean Division & Finite Dimension

**Theorem (Tannery 1874)**

D-finite series composed with algebraic power series are D-finite.

**Proof.**

\[ P(x, y) = 0 \text{ and } AP + BP_y = 1 \Rightarrow y' = -\frac{P_x}{P_y} = -BP_x \mod P \]

\[ \Rightarrow y^{(k)} \in \bigoplus_{i < \deg_y P} \mathbb{K}(x)y^i. \]

\[ (f \circ y)^{(p)} \text{ linear combination of } (f^{(j)} \circ y)y^k. \]

Also, \( \exp \int y \).
Example: Airy Ai at Infinity

\[ \text{Ai}(z) = \frac{\sqrt{z} e^{-\xi}}{2\pi} \int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]} \, dv, \quad \xi = \frac{2}{3} z^{3/2}, \quad u = \sqrt{1 + \frac{v^2}{3}} \]

\[ \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\xi} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{\Gamma(3n + \frac{1}{2})}{54^n n! \Gamma(n + \frac{1}{2})}. \]
Example: Airy Ai at Infinity

\[
Ai(z) = \frac{\sqrt{z}e^{-\xi}}{2\pi} \int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]} \, dv, \quad \xi = \frac{2}{3} z^{3/2}, \quad u = \sqrt{1 + \frac{v^2}{3}}
\]

\[
\sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\xi} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{\Gamma(3n + \frac{1}{2})}{54^n n! \Gamma(n + \frac{1}{2})}.
\]

Computation:

1. **algebraic** change of variables \( t^2 = (u - 1)(4u^2 + 4u + 1); \)

\[
\rightarrow \int_{-\infty}^{\infty} e^{-\xi t^2} f(t) \, dt, \quad f(t) = \frac{dv}{dt},
\]

2. recurrence satisfied by the coefficients of \( f \) (**generating series**);

3. termwise integration (**Hadamard product**).
I. Algebraic change of variables

\[ e_{u} := u^2 - \left(1 + \frac{v^2}{3}\right) : \]
\[ e_{t} := t^2 - (u - 1) \cdot \left(4 \cdot u^2 + 4 \cdot u + 1\right) : \]
\[ res := \text{resultant}(e_{u}, e_{t}, u); \]
\[ t^4 + 2 t^2 - 3 v^2 - \frac{8}{3} v^4 - \frac{16}{27} v^6 \]

\[ gfun := \text{algeqtodiffeq}\left(res, v(t), \left\{ v(0) = 0, D(v)(0) = \sqrt{\frac{2}{3}} \right\}\right); \]
\[ \left\{ -4 v(t) + 9 t \left(\frac{d}{dt} v(t)\right) + \left(9 t^2 + 18\right) \left(\frac{d^2}{dt^2} v(t)\right), v(0) = 0, (D(v))(0) = \frac{1}{3} \sqrt{6} \right\} \]

II. Recurrence satisfied by the coefficients of \( f \)

\[ gfun := \text{poltodiffeq}(\text{diff}(v(t), t), [\%], [v(t)], f(t)); \]
\[ \left\{ 5 f(t) + 27 t \left(\frac{d}{dt} f(t)\right) + \left(9 t^2 + 18\right) \left(\frac{d^2}{dt^2} f(t)\right), f(0) = \frac{1}{3} \sqrt{6}, (D(f))(0) = 0 \right\} \]
\[ R_{f} := gfun := \text{diffeqtorec}(\%, f(t), c(n)); \]
\[ \left\{ (5 + 18 n + 9 n^2) c(n) + \left(18 n^2 + 54 n + 36\right) c(n + 2), c(0) = \frac{1}{3} \sqrt{6}, c(1) = 0 \right\} \]
**III. Hadamard product**

\[ \int_{-\infty}^{\infty} e^{-\frac{x^2 \xi}{2}} x^n dx = - \int_{-\infty}^{\infty} \frac{2 \xi - t e^{\frac{-(\xi - t)^2}{2}}} {n + 1} (n + 1) dt \]

\[ R_i := \left\{ c(n) = \frac{2 \cdot \xi} {n + 1} \cdot c(n + 2), \quad c(0) = \text{value}(\text{eval}(s, n = 0)), \quad c(1) = \text{value}(\text{eval}(s, n = 1)) \right\} \]

\[ \{ c(n) = \frac{2 \xi \cdot c(n + 2)} {n + 1}, \quad c(0) = \frac{\sqrt{\pi}} {\sqrt{\xi}}, \quad c(1) = 0 \} \]

\[ \text{FinalRec} := \text{gfun}:-\text{rec}^\ast\text{rec}^{-}\left(R[i], R[f], c(n)\right); \]

\[ \text{FinalRec} := \left\{ 5 + 18 n + 9 n^2 \right\} c(n) + (36 \xi n + 72 \xi \xi) c(n + 2), \quad c(1) = 0, \quad c(0) = \frac{1} {3} \frac{\sqrt{\pi} \sqrt{6}} {\sqrt{\xi}} \right\} \]

\[ \text{Sol} := \text{rsolve}(\text{FinalRec}, c(n)); \]

\[ \left\{ \frac{1} {3} (\xi)^n \left( \frac{1} {2} \right)^n \frac{\Gamma\left( \frac{1} {2} n + \frac{5} {6} \right) \Gamma\left( \frac{1} {2} n + \frac{1} {6} \right) \xi \left( -\frac{1} {2} n \right)} {\sqrt{\pi} \Gamma\left( \frac{1} {2} n + 1 \right) \sqrt{\xi}} \right\} n:\text{even} \]

\[ 0 \quad n:\text{odd} \]
Other Uses of the LDE as a Data-Structure

ESF: http://algo.inria.fr/esf [MeSa03]
III Definite Hypergeometric Summation
Creative Telescoping

\[ I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

**IF** one knows Pascal’s triangle:

\[
\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k},
\]

then summing over \( k \) gives

\[ I_{n+1} = I_n + I_n = 2I_n. \]

The initial condition \( I_0 = 1 \) concludes the proof.
Zeilberger’s idea: look for rational $r(n, k)$ and $\lambda(n)$ such that

$$v_{n, k+1} - v_{n, k} = u_{n+1, k} + \lambda(n)u_{n, k}, \quad \text{with } v_{n, k} := r(n, k)u_{n, k}$$

When summing over $k$, the left-hand side telescopes.
Creative Telescopying

$$I_n := \sum_{k=0}^{n} \binom{n}{k} u_{n,k} = 2^n.$$ 

Zeilberger’s idea: look for rational $r(n, k)$ and $\lambda(n)$ such that

$$v_{n,k+1} - v_{n,k} = u_{n+1,k} + \lambda(n) u_{n,k}, \quad \text{with } v_{n,k} := r(n, k) u_{n,k}$$

When summing over $k$, the left-hand side telescopes.

1. Dividing out by $u_{n,k}$ gives a recurrence for $r$:

$$\frac{n-k}{k+1} r(n, k+1) - r(n, k) = \frac{n+1}{n+1-k} + \lambda(n).$$

2. discussion on poles and degree numerator $\rightarrow r = \frac{a(n)k+b(n)}{n-k};$

3. normalize and extract coefficients of $k \rightarrow$ linear system

4. solution: $a = -1, b = 0, \lambda = -2.$

The rational function $r(n, k) = k/(k - n)$ is the certificate.
Zeilberger’s Algorithm

Input: a hypergeometric term \( u_{n,k} \), i.e., \( u_{n+1,k}/u_{n,k} \) and \( u_{n,k+1}/u_{n,k} \) rational functions;
Output: a linear recurrence satisfied by \( \sum_k u_{n,k} \) and a certificate.
For \( m = 1, 2, 3, \ldots \)

1. Set up the recurrence for \( v_{n,k} = r(n,k)u_{n,k} \)

\[
v_{n,k+1} - v_{n,k} = u_{n+m,k} + \lambda_1(n)u_{n+m-1,k} + \cdots + \lambda_{m-1}(n)u_{n,k}
\]
with unknown \( r \) and \( \lambda_i \);

2. Discuss denominator of \( r \) (Gosper’s or Abramov’s algorithm);

3. Look for numerator \( \rightarrow \) linear system in its coefficients and the \( \lambda_i \)’s;

4. If a solution is found, break.
Example: $\zeta(3)$ is Irrational \cite{Apéry78}

\[ a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n = a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^{m+1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}. \]

1. \( b_n/a_n \to \zeta(3), \ n \to \infty; \ d_n^3 b_n \in \mathbb{Z}, \text{ where } d_n = \text{lcm}(1, \ldots, n); \)
2. By creative telescoping, both \( a_n \) and \( b_n \) satisfy

\[(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1; \]

"Neither Cohen nor I had been able to prove [this] in the intervening two months." \cite{Van der Poorten}

3. \( 0 < \zeta(3) - \frac{b_n}{a_n} = \sum_{k \geq n+1} \frac{b_k}{a_k} - \frac{b_{k-1}}{a_{k-1}}: \ b_k a_{k-1} - b_{k-1} a_k = \frac{6}{k^3}; \)
4. \( \lambda a_n + \mu b_n \approx \alpha_n^2, \text{ with } \alpha_\pm^2 = 34\alpha_\pm - 1; \)
5. Conclusion: \( 0 < a_n d_n^3 \zeta(3) - d_n^3 b_n \approx \alpha_n^2 e^{3n} \to 0. \)
Algolib can be downloaded from http://algo.inria.fr/libraries.

> libname := "/Users/salvy/lib/maple/Algolib", libname:
> a := binomial(n, k)^2 * binomial(n + k, k)^2;

\[ a := \binom{n}{k}^2 \binom{n + k}{k}^2 \]

> Mgfun[creative_telescoping](a, n :: shift, k :: shift);

\[
\left[ ( -3 n^2 - 3 n - 1 ) f(n, k) + \left( 34 n^3 + 153 n^2 + 231 n + 117 \right) f(n + 1, k) + \left( -n^3 - 6 n^2 - 12 n - 8 \right) f(n + 2, k) \right] - \frac{4 k^4 \left( 4 n^2 + 12 n + 8 + 3 k - 2 k^2 \right) \left( 2 n + 3 \right) f(n, k)}{4 + 12 n - 12 k - 4 n k^3 + 13 n^2 + 13 k^2 + k^4 - 26 n k + n^4 + 6 n^3 - 6 k^3 + 6 n^2 k^2 - 18 n^2 k + 18 n k^2 - 4 n^3 k} \]

Neither Cohen nor I had been able to prove [this] in the intervening two months. [Van der Poorten]
IV D-finiteness in Several Variables
Example: Contiguity of Hypergeometric Series

\( F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \), \hspace{1cm} (x)_n := x(x+1) \cdots (x+n-1). \)

\[
\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,
\]

\[
\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.
\]
Example: Contiguity of Hypergeometric Series

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1). \]

\[ \frac{u_{a,n+1}}{u_{a,n}} = \frac{(a + n)(b + n)}{(c + n)(n + 1)} \to z(1 - z)F'' + (c - (a + b + 1)z)F' - abF = 0, \]

\[ \frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \to S_a F := F(a + 1, b; c; z) = \frac{z}{a} F' + F. \]

dim=2 \Rightarrow S_a^2 F, S_a F, F linearly dependent [Gauss1812]

Also:

- \( S_a^{-1} \) in terms of Id, \( D_z \);
- relation between any three polynomials in \( S_a, S_b, S_c \);
- generalizes to any \( pFq \) and multivariate case [Takayama89].
Skew polynomial ring: \( \mathbb{A}[\partial; \sigma, \delta] \), \( \mathbb{A} \) integral domain and commutation \( \partial P = \sigma(P)\partial + \delta(P), \ P \in \mathbb{A} \) (ex. \( \partial_x P(x) = P(x)\partial_x + P'(x), S_n P(n) = P(n + 1)S_n \)). Technical conditions on \( \sigma, \delta \) to make product associative.

Ore algebra: \( \mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n] \), \( \sigma, \delta \) s. t. \( \partial_i \partial_j = \partial_j \partial_i \).

Aim [ChSa98]: manipulate (solutions of) systems of mixed linear (\( q \)-)differential or (\( q \)-)difference operators.

Main property: the leading term of a product is (up to a cst) the product of leading terms.
Ore Polynomials & Ore Algebras

- **Skew polynomial ring**: $\mathbb{A}[\partial; \sigma, \delta]$, $\mathbb{A}$ integral domain and commutation $\partial P = \sigma(P)\partial + \delta(P)$, $P \in \mathbb{A}$
  
  (ex. $\partial_x P(x) = P(x)\partial_x + P'(x)$, $S_n P(n) = P(n+1)S_n$).

  Technical conditions on $\sigma, \delta$ to make product associative.

- **Ore algebra**: $\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$, $\sigma, \delta$ s. t. $\partial_i \partial_j = \partial_j \partial_i$.

- **Main property**: the leading term of a product is (up to a cst) the product of leading terms.

- **Consequences**:
  1. **Univariate**: Right Euclidean division and extended Euclidean algorithm [Ore 33];
  2. **Multivariate**: Buchberger’s algorithm for Gröbner bases works in Ore algebras [Kredel93].
0-dimensionality & D-finiteness

<table>
<thead>
<tr>
<th>Polynomial algebra</th>
<th>Ore algebra</th>
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Exs: Orthogonal polynomials, hypergeometric series, their \( q \)-analogues, ...
0-dimensionality & D-finiteness

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<td>polynomial expressions are algebraic</td>
<td>polynomials and ∂’s are D-finite</td>
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</tbody>
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Tools: linear algebra, Gröbner bases. Implemented in Mgfun [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their $q$-analogues, ...

system+ini. cond.=data structure
Example: Binomial Coefficients and Pascal’s Triangle

- Algebra: \( A = \mathbb{Q}(n, k)[S_n; S_n, 0][S_k; S_k, 0] \)
- \( \binom{n}{k} \) is annihilated by \( S_n - \frac{n+1}{n+1-k} \) and \( S_k - \frac{n-k}{k+1} \)
- They generate a left ideal \( \mathcal{I} \).
- The quotient has dimension 1 (≡ hypergeometric).
- Pascal’s triangle is \( P = S_nS_k - S_k - 1 \in \mathcal{I} \).
- Creative telescoping is obtained by left division by \( S_k - 1 \):

\[
P = (S_k - 1)(S_n - 1) + S_n - 2
\]

\(\text{certificate} \quad \text{result}\)
General Creative Telescoping [Zeilberger 90]

\[ F_n = \sum_k u_{n,k} = ? \]

If one knows \( A(n, S_n) \) and \( B(n, k, S_n, S_k) \) such that

\[ (A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0, \]

then the sum “telescopes”, leading to \( A(n, S_n) \cdot F_n = 0 \).
General Creative Telescoping [Zeilberger 90]

\[ I(x) = \int_{\Omega} u(x, y) \, dy = ? \]

**IF** one knows \( A(x, \partial_x) \) and \( B(x, y, \partial_x, \partial_y) \) such that

\[
(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,
\]

then the integral “telescopes”, leading to \( A(x, \partial_x) \cdot I(x) = 0 \).
General Creative Telescoping [Zeilberger 90]

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then the integral “telescopes”, leading to \( A(x, \partial_x) \cdot I(x) = 0 \).

*Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*

Richard P. Feynman 1985

Creative telescoping = “differentiation” under integral + “integration” by parts
General Creative Telescoping [Zeilberger 90]

\[ I(x) = \int_{\Omega} u(x, y) \, dy = ? \]

**IF** one knows \( A(x, \partial_x) \) and \( B(x, y, \partial_x, \partial_y) \) such that

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then the integral “telescopes”, leading to \( A(x, \partial_x) \cdot I(x) = 0 \).

**Creative telescoping= “differentiation” under integral + “integration” by parts**

- **General case:** Find annihilators of

  \[ I(x_1, \ldots, x_{n-1}) = \partial_n^{-1} \big|_{\Omega} f(x_1, \ldots, x_n) \]

  knowing generators of \( \text{Ann}_f \) in

  \( \mathcal{O}_n = \mathbb{K}(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]; \)

- **Crucial step:** compute \( (\mathcal{O}_n \text{Ann}_f + \partial_n \mathcal{O}_n) \cap \mathcal{O}_{n-1}. \)
Applications of Creative Telescoping

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \]  

\[ \int_{0}^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1-a^4)}{2\pi a^2} \]  

\[ \frac{1}{2\pi i} \int_{C} \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1 + 4y^2)^{3/2}} \, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !} \]

\[ \sum_{k=0}^{n} q^k = \frac{(q; q)_{n-k}}{(q; q)_{n+k}} \]  

\[ \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j}(q; q)_i(q; q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}} \]  

\[ \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k}(q; q)_{n-k}} \]  

[Strehl92]  
[GlMo94]  
[Doetsch30]  
[Andrews74]  
[Paule85].
(Partial) Algorithms for Creative Telescoping

Aim: \( \mathcal{I} = (\mathbb{O}_n \text{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1} \)

- By Gröbner bases, eliminate \( x_n \) and set \( \partial_n \) to 0 [ChSa98]
  \[ \rightarrow (\mathbb{O}_n \text{Ann}_f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I} \]

- Differential case: algorithms from \( \mathcal{D} \)-module theory [SaStTa00, Tsai00], Gröbner bases with negative weights.

- Shift case, \( n = 2, \text{dim } 1 \) (= hypergeometric): [Zeilberger91]
  For increasing \( k \), search for \( a_i \) and \( B 
  \mathbb{O}_{n-1} \ni \sum_{i=0}^{k} a_i \partial_{n-1}^i f = \partial_n B f 
  
  Termination [Abramov03].

- Arbitrary \( n \) and \( \mathbb{O}_n \): [Chyzak00]
  \( \mathbb{O}_{n-1} \ni \sum_{\lambda} a_\lambda \partial_\lambda = \partial_n B \mod \text{Ann}_f 
  
  \( B \) is given by rational solutions of a linear system in \( \sigma_n, \delta_n \).
Open Problems

Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93, GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants.

Understand non-minimality

- Remove apparent singularities by Ore closure, a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (extend [Paule94]).

Easy-to-use Implementations

- Improve gfun and Mgfun. Make the ESF interactive.
Bibliographie

