Farkas Lemma made easy
Tool Demonstration

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Abstract
In this paper, we present fkcc, a scripting tool to prototype program analyses and transformations exploiting the affine form of Farkas lemma. Our language is general enough to prototype in a few lines sophisticated termination and scheduling algorithms. The tool is freely available and may be tried online via a web interface. We believe that fkcc is the missing chain to accelerate the development of program analyses and transformations exploiting the affine form of Farkas lemma.

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1 Introduction
Many program analyses require to handle universally quantified constraints such as \( \forall x \in D : \Phi(x) \), where \( D \) is a convex polyhedron and \( \Phi \) is a conjunction of affine constraints. For instance, this occurs in loop scheduling [5, 6], loop tiling [3], program termination [2] or generation of loop invariants [4]. Farkas lemma – affine form – provides a way to get rid of that universal quantification, at the price of introducing quadratic terms. It is even possible to use Farkas lemma to turn universally quantified quadratic constraints into existentially quantified affine constraints [5, 6]. However, this requires tricky algebraic manipulations, notoriously difficult to experiment by hand and to implement.

In this tool demonstration, we present a scripting tool, fkcc [1] which makes it possible to manipulate easily Farkas lemma to benefit from those nice properties. Specifically, we will discuss the following points:

- A general formulation for the resolution of equations \( \forall x : S(\bar{x}) = 0 \) where \( S \) is summation of affine forms including Farkas terms. So far, this resolution was applied for specific instances of Farkas summation. This result is the basic engine of the fkcc scripting language.
- A scripting language to apply and exploit Farkas lemma; among polyhedra, affine functions and affine forms.
- Our tool, fkcc, implementing these principles, is available at http://foobar.ens-lyon.fr/fkcc. fkcc may be downloaded and tried online via a web interface. fkcc comes with many examples, making it possible to adopt the tool easily.

This tool demonstration is structured as follows. Section 2 presents the affine form of Farkas lemma, our resolution theorem, and explains how it applies to compute scheduling functions. Then, Section 3 defines the syntax and outlines informally the semantics of the fkcc language. Finally, Section 4 concludes this paper and draws future research perspectives, then Annex A gives a real-life example of fkcc script to compute a Pluto-style tiling [3] for the Jacobi 1D kernel.

2 Farkas lemma in polyhedral compilation
This section presents the theoretical background of this tool demonstration. We first introduce the affine form of Farkas lemma. Then, we present our theorem to solve equations \( \forall \bar{x} : S(\bar{x}) = 0 \) where \( S \) is a summation of affine forms including Farkas terms. This formalization will then be exploited to design the fkcc language.

Lemma 2.1 (Farkas Lemma, affine form). Consider a non-empty convex polyhedron \( P = (\bar{x}, A\bar{x} + \vec{b} \geq 0) \subseteq \mathbb{R}^n \) and an affine form \( \phi : \mathbb{R}^n \to \mathbb{R} \) such that \( \phi(\bar{x}) \geq 0 \forall \bar{x} \in P \).

Then: \( \exists \vec{\lambda} \geq 0, \lambda_0 \geq 0 \) such that:

\[
\phi(\bar{x}) = \vec{f}^\top \vec{\lambda} (A\bar{x} + \vec{b}) + \lambda_0 \quad \forall \bar{x}
\]

Hence, Farkas lemma makes it possible to remove the quantification \( \forall \bar{x} \in P \) by encoding directly the positivity over \( P \) into the definition of \( \phi \), thanks to the Farkas multipliers \( \vec{\lambda} \) and \( \lambda_0 \). In the remainder, Farkas terms will be denoted by: \( \vec{\phi}(\lambda_0, \vec{\lambda}, A, \vec{b})(\bar{x}) = \vec{f}^\top \vec{\lambda} (A\bar{x} + \vec{b}) + \lambda_0 \). We now recall our theorem [1] to solve equations \( \forall \bar{x} : S(\bar{x}) = 0 \) where \( S \) involves Farkas terms. The result is expressed as a conjunction of affine constraints, which is suited for integer linear programming:

Theorem 2.2 (solve). Consider a summation \( S(\bar{x}) = \vec{u} \cdot \bar{x} + \nu + \sum_i \vec{b}_i(\lambda_{i0}, \vec{\lambda}_i, A_i, \vec{b}_i)(\bar{x}) \) of affine forms, including Farkas terms. Then:

\[
\forall \bar{x} : S(\bar{x}) = 0 \iff \begin{cases}
\vec{u} + \sum_i \vec{A}_i \vec{\lambda}_i = \vec{0} & \land \\
\nu + \sum_i (\vec{A}_i \cdot \vec{b}_i + \lambda_{i0}) = 0
\end{cases}
\]
Application to scheduling  Figure 1 depicts an example of a program (a) computing the product of two polynomials specified by their arrays of coefficients a and b, and the iteration domain with the data dependence across iterations (b) and an example schedule \( \theta(i,j) \) = \( i \) prescribing a parallel execution by vertical waves. This paragraph reformulates the technique presented in [5] with our theorem 2.2. This formulation will directly inspire the \texttt{FKCC} syntax presented in the next section.

A schedule must be positive everywhere on its iteration domain:

\[ \theta(i, j, N) \geq 0 \quad \forall (i, j) \in D_N \quad (1) \]

Applying Farkas lemma, this translates to:

\[ \exists \lambda_0 \geq 0, \bar{\lambda} \geq 0 \quad \text{s.t.} \quad \theta(i, j, N) = \bar{\gamma}(\lambda_0, \bar{\lambda}, A, \bar{b})(i, j, N) \quad (2) \]

Moreover, a schedule must satisfy the data dependences \((i, j) \rightarrow (i', j')\) abstracted by a dependence polyhedron \( \Delta_N \):

\[ \theta(i', j', N) > \theta(i, j, N) \quad \forall (i, j, i', j') \in \Delta_N \quad (3) \]

This is equivalently written as the positivity of the affine form \((i, j, i', j', N) \rightarrow \theta(i', j', N) - \theta(i, j, N) - 1 \) over the convex polyhedron \( \Delta_N \). Applying Farkas lemma:

\[ \exists \mu_0 \geq 0, \bar{\mu} \geq 0 \quad \text{such that} \quad \theta(i', j', N) - \theta(i, j, N) - 1 = \bar{\gamma}(\mu_0, \bar{\mu}, C, \bar{d})(i, j, i', j', N) \]

Substituting \( \theta \) using Equation (2), this translates to solving \( \forall (i, j, i', j', N) : S(i, j, i', j', N) = 0 \), where \( S(i, j, i', j', N) \) is the summation:

\[ \bar{\gamma}(\lambda_0, \bar{\lambda}, A, \bar{b})(i', j', N) - \bar{\gamma}(\lambda_0, \bar{\lambda}, A, \bar{b})(i, j, N) - 1 \]

\[-\bar{\gamma}(\mu_0, \bar{\mu}, C, \bar{d})(i, j, i', j', N) \]

Since \(-\bar{\gamma}(\lambda_0, \bar{\lambda}, A, \bar{b}) = \bar{\gamma}(-\lambda_0, -\bar{\lambda}, A, \bar{b})\), we may apply theorem 2.2 to obtain a system of affine constraints with \( \lambda \), \( \bar{\lambda} \), \( \mu \), \( \bar{\mu} \). Linear programming may then be applied to find out the desired schedule [3, 6].

3 FKCC at a glance

This section outlines briefly the input syntax of \texttt{FKCC} on our motivating example. For a detailed description, the reader is referred to [1].

Program, instructions, polyhedra  An \texttt{FKCC} program consists of a sequence of instructions. There is no other control structure than the sequence. An instruction may assign an \texttt{FKCC} object (polyhedron, affine form or affine function) to an \texttt{FKCC} identifier, or may be an \texttt{FKCC} object alone. In the latter case, the \texttt{FKCC} object is streamed out to the standard output. \texttt{FKCC} objects are expressed with the same syntax as \texttt{ISCC}[7]:

\[
\text{parameters} := (M, \epsilon) \; ; \\
\text{parametrized_iterations} := [M] \rightarrow \{ [i] : \emptyset \leq i \leq M \} ;
\]

Parameters must be declared with the parameters construct. The parameters of a polyhedron may optionally be declared on preceding brackets \([M] \rightarrow \ldots\). The set intersection of two polyhedra \( P \) and \( Q \) is obtained with \( P \cap Q \).

\[
\text{for } i := 0 \text{ to } N \\
\text{for } j := 0 \text{ to } N \\
c[i+j] := c[i+j] + a[i]^*b[j] ;
\]

(a) Product of polynomials

(b) Iterations and schedule

Figure 1. Motivating example

Affine forms  An affine form may be defined as a Farkas term:

\[
\text{iterations} := [] \rightarrow \{ [i,j,N] : \emptyset \leq i \leq N \; \text{and} \; \emptyset \leq j \leq N \} ; \\
\text{theta} := \text{positive_on iterations} ;
\]

If \( \text{iterations} \) is \( \{ x \mid A^T \cdot x \geq 0 \} \), then \( \text{theta} \) is defined as \( \bar{\gamma}(\lambda_0, \bar{\lambda}, A, \bar{b}) \) where \( \lambda_0 \) and \( \bar{\lambda} \) are fresh positive variables. In this case, the polyhedron is never parametrized: the parameters must be handled as variables. Affine forms may be summed, scaled and composed with affine functions, typically to adjust the input dimension:

\[
\text{dependence} := [] \rightarrow \{ [i,j,i',j',N] : \emptyset \leq i \leq N \; \text{and} \; \emptyset \leq j \leq N \; \text{and} \; \emptyset \leq i' \leq N \; \text{and} \; \emptyset \leq j' \leq N \; \text{and} \; i+j = i'+j' \; \text{and} \; i<i' \} ; \\
\text{to_target} := \{ [i,j,i',j',N] \rightarrow [i',j',N] ; \\
\text{to_source} := \{ [i,j,i',j',N] \rightarrow [i,j,N] ; \\
\text{sum} := (\text{theta} \cdot \text{to_target}) - (\text{theta} \cdot \text{to_source}) - 1 \; \text{positive_on dependence} ;
\]

In a summation of affine forms, affine forms must have the same input dimension. Also, a constant \(-1\) is automatically interpreted as an affine form \([i,j,i',j',N] \rightarrow -1\). The terms of the summation are simply separated with + and −, no parenthesis are allowed. Affine forms may also be stated explicitly:

\[
\text{sum} := \lambda \cdot \text{to_target} - \lambda \cdot \text{to_source} + [i,j,i',j',N] \rightarrow -1 + \epsilon \text{ positive_on dependence} ;
\]

Resolution  The main feature of \texttt{FKCC} is the resolution of equations \( \forall x : S(x) = 0 \) where \( S \) is a summation of affine forms including Farkas terms. This is obtained with the instruction solve:

\[
\text{solve} \quad \text{sum} = 0 ;
\]

The result is a polyhedron with Farkas multipliers (obtained after applying Theorem 2.2 (solve)).

\[
\{ \text{lambda}_0, \text{lambda}_1, \text{lambda}_2, \text{lambda}_3, \text{lambda}_4, \text{lambda}_5, \text{lambda}_6, \text{lambda}_7, \text{lambda}_8, \text{lambda}_9, \text{lambda}_10, \text{lambda}_11, \text{lambda}_12, \text{lambda}_13, \text{lambda}_14, \text{lambda}_15, \text{lambda}_16 \} : \\
\#\#\# \text{ constraints} ;
\]
When the summation contains affine forms with parameters (as \( \text{sum}_\text{eps} \)), the resolution interprets parameters as constants. In particular, this makes it possible to tune dependence satisfaction: \( \theta(i', j') \geq \theta(i, j) + e_\psi \) with \( 0 \leq e_\psi \leq 1 \).

At this point, we need to recover the coefficients of our affine form \( \theta \) in terms of \( \lambda \) (\( \lambda_0, \ldots, \lambda_3 \)) and \( \lambda_0 \). Observe that \( \theta(x) = \overline{\theta}(\lambda_0, \lambda, A, b)(x) \), in turn equal to \( \overline{\lambda} A x + \overline{\lambda} b + \lambda_0 \). If the coefficients of \( \theta \) are written: \( \theta(x) = \overline{\tau} \cdot x + \tau_0 \), we simply have: \( \overline{\tau} = \overline{\lambda} A \) and \( \tau_0 = \overline{\lambda} \cdot b + \lambda_0 \). This is obtained with define:

\[ \text{define theta with tau;} \]

The result is a conjunction of definition equalities, gathered in a polyhedron:

\[ [] \rightarrow \{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4), \]
\[ \tau_0, \tau_1, \tau_2, \tau_3, \text{tau}_N \} : \]
\[ \lambda_0 = \text{tau}_0 \text{ and } \lambda_1 = \text{tau}_1 \text{ and } \lambda_2 = \text{tau}_2 \text{ and } \lambda_3 = \text{tau}_3 ; \]

The first coefficients \( \tau_k \) define \( \overline{\lambda} \), the last one defines the constant \( \tau_0 \). On our example, \( \theta(i, j, N) = \text{tau}_0 \cdot i + \text{tau}_1 \cdot j + \text{tau}_2 \cdot N + \text{tau}_3 \). Now we may gather the results and eliminate the \( \lambda \) to keep only \( \overline{\tau} \) and \( \tau_0 \):

\[ \text{keep } \tau_0, \tau_1, \tau_2, \tau_3 \text{ in } \]
\[ \text{(solve sum = 0)\text{def} theta with tau}; \]

The result is a polyhedron with all the valid schedules:

\[ [] \rightarrow \{ \theta_0, \theta_1, \theta_2, \theta_3 \} : \]
\[ \theta_3 >> 0 \text{ and } (\theta_0 \text{tau}_1) \cdot \theta_2 >> 0 \text{ and } \]
\[ (\text{\text{-}1} \cdot \text{tau}_0) \cdot (\text{\text{-}1} \cdot \text{tau}_1) >> 0 \text{ and } \]
\[ \text{tau}_1 \cdot \text{tau}_2 >> 0 \text{ and } \text{tau}_2 >> 0 ; \]

All these steps may be applied at once with the find command:

\[ \text{find theta s.t. sum = 0;} \]

The coefficients are automatically named \( \text{theta}_0, \text{theta}_1, \text{etc} \) with the same convention as \( \text{define theta with tau;} \). This will allow \( \text{find theta with tau;} \) whereas \( \text{find theta with tau;} \) whereas \( \text{find theta with tau;} \).

We believe that scripting tools are mandatory to evaluate rapidly research ideas. So far, Farkas lemma-based approaches were locked by two facts: applying by hand Farkas Lemma is a pain; and implementing an analysis with Farkas lemma is notoriously time consuming and bug prone. With FKKC, computer scientists are now freed from these constraints.

Let the power of \( \text{FCC} \) be with you!

References


A Pluto-style tiling with fkcc

We give here the fkcc script discussed in the demo session to compute a Pluto-style tiling [3] for the Jacobi-1D kernel. Note the sections expressing successively the data dependences (→), the correctness condition (∥S, i∥ → ∥T, j∥ ⇒ ϕT(j) − ϕS(i) ≥ 0), the laziness (minimal dependence distance δST = ϕT(j) − ϕS(i)), and finally the computation of non-null and linearly independent affine tiling hyperplanes in ϕS and ϕT. Applying fkcc with the -pretty option, we obtain:

$ fkcc -pretty < pluto.fk$

\[
\begin{align*}
\phi_S_{0} &= 1 \\
\phi_S_{1} &= 0 \\
\phi_S_{2} &= 0 \\
\phi_S_{3} &= 0 \\
\phi_S_{4} &= 0 \\
\phi_T_{0} &= 1 \\
\phi_T_{1} &= 0 \\
\phi_T_{2} &= 0 \\
\phi_T_{3} &= 0 \\
\phi_T_{4} &= 1 \\
\text{latency}_0 &= 0 \\
\text{latency}_1 &= 0 \\
\text{latency}_2 &= 1 \\
\text{latency}_3 &= 0 \\
\text{latency}_4 &= 0
\end{align*}
\]

Which corresponds to the affine tiling \( \phi_S(t, i) = (t, 2t + i) \) and \( \phi_T(t, i) = (t, 2t + i + 1) \) with a maximum dependence distance 1 for the first tiling hyperplane and 2 for the second tiling hyperplane.

# jacobi-1D non-perfect

# for (t = 1; t <= T; t++)
# {
#   # for (i = 1; i < N - 1; i++)
#   # for (i = 1; i < N - 1; i++)
#   #   A[i] = B[i]; //T
#   # }

D_S := [] -> { [t,i,T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1};
D_T := [] -> { [t,i,T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1};
phi_S := positive_on D_S;
phi_T := positive_on D_T;

#S --> T
Delta_ST := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t=t' and i=i'};

#S --> T (anti read(a[i-1]) --> write(a[i]))
Delta_ST_anti_1 := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t+1=t' and i-1=i'};

#S --> T (anti read(a[i+1]) --> write(a[i]))
Delta_ST_anti_2 := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t+1=t' and i+1=i'};

#T --> S, read a[i-1]
Delta_TS_1 := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t+1=t' and i-1=i'};

#T --> S, read a[i]
Delta_TS_2 := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t+1=t' and i=i'};

#T --> S, read a[i+1]
Delta_TS_3 := [] -> { [t,i,t',i',T,N]: 1 <= t and t <= T and 1 <= i and i <= N-1 and
t+1=t' and i+1=i'};
Correctness: $s \rightarrow t \implies \phi(s) \leq \phi(t)$

```
to_target := \{[t,i,t',i',T,N] \rightarrow [t',i',T,N]\};
to_source := \{[t,i,t',i',T,N] \rightarrow [t,i,T,N]\};

\phi_correct :=
\# S \rightarrow T
(find \phi_S,\phi_T s.t. (\phi_T \cdot to_target) - (\phi_S \cdot to_source) - positive_on Delta_ST = 0) *
\# S \rightarrow T, \text{ (anti read(a[i-1]) \rightarrow write(a[i])})
(find \phi_S,\phi_T s.t. (\phi_T \cdot to_target) - (\phi_S \cdot to_source) - positive_on Delta_ST_anti_1 = 0) *
\# S \rightarrow T, \text{ (anti read(a[i+1]) \rightarrow write(a[i])})
(find \phi_S,\phi_T s.t. (\phi_T \cdot to_target) - (\phi_S \cdot to_source) - positive_on Delta_ST_anti_2 = 0) *
\# T \rightarrow S, \text{ read a[i-1]}
(find \phi_S,\phi_T s.t. (\phi_S \cdot to_target) - (\phi_T \cdot to_source) - positive_on Delta_ST_1 = 0) *
\# T \rightarrow S, \text{ read a[i]}
(find \phi_S,\phi_T s.t. (\phi_S \cdot to_target) - (\phi_T \cdot to_source) - positive_on Delta_ST_2 = 0) *
\# T \rightarrow S, \text{ read a[i+1]}
(find \phi_S,\phi_T s.t. (\phi_S \cdot to_target) - (\phi_T \cdot to_source) - positive_on Delta_ST_3 = 0);

\#
\# Efficiency: $s \rightarrow t \implies \phi(t) - \phi(s) \leq latency(N)$, then min latency(N)
\#

\# L(N) >= 0 on the parameter domain
latency := positive_on ([] \rightarrow \{[T,N]: T \geq 0 \text{ and } N \geq 0\});
latency_def := define latency with latency;
to_param := \{[t,i,t',i',T,N] \rightarrow [T,N]\};

\phi_bounded :=
\# S \rightarrow T
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_T \cdot to_target) + (\phi_S \cdot to_source) - positive_on Delta_ST = 0) *
\# S \rightarrow T, \text{ (anti read(a[i-1]) \rightarrow write(a[i])})
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_T \cdot to_target) + (\phi_S \cdot to_source) - positive_on Delta_ST_anti_1 = 0) *
\# S \rightarrow T, \text{ (anti read(a[i+1]) \rightarrow write(a[i])})
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_T \cdot to_target) + (\phi_S \cdot to_source) - positive_on Delta_ST_anti_2 = 0) *
\# T \rightarrow S, \text{ read a[i-1]}
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_S \cdot to_target) + (\phi_T \cdot to_source) - positive_on Delta_ST_1 = 0) *
\# T \rightarrow S, \text{ read a[i]}
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_S \cdot to_target) + (\phi_T \cdot to_source) - positive_on Delta_ST_2 = 0) *
\# T \rightarrow S, \text{ read a[i+1]}
(find latency,\phi_S,\phi_T s.t. latency \cdot to_param - (\phi_S \cdot to_target) + (\phi_T \cdot to_source) - positive_on Delta_ST_3 = 0);

\#
\# First hyperplane: avoid the null solution
\#

\phi_filter_level_1 := [] \rightarrow \{[\phi_S_0,\phi_S_1,\phi_S_2,\phi_S_3,\phi_S_4,\phi_T_0,\phi_T_1,\phi_T_2,\phi_T_3,\phi_T_4]\:

\#phi_S: positive coefficients + no parameters
phi_S_0 \geq 0 \text{ and } phi_S_1 \geq 0 \text{ and } phi_S_2 = 0 \text{ and } phi_S_3 = 0 \text{ and } phi_S_4 \geq 0 \text{ and }

\#phi_T: positive coefficients + no parameters
phi_T_0 \geq 0 \text{ and } phi_T_1 \geq 0 \text{ and } phi_T_2 = 0 \text{ and } phi_T_3 = 0 \text{ and } phi_T_4 \geq 0 \text{ and}
#phi_S != 0 and phi_T != 0
phi_S_0 + phi_S_1 + phi_S_2 + phi_S_3 >= 1 and
phi_S_0 + phi_T_1 + phi_T_2 + phi_T_3 >= 1
};

all_level_1 := keep latency_0, latency_1, latency_2, phi_S_0, phi_S_1, phi_S_2, phi_S_3, phi_S_4, phi_T_0, phi_T_1, phi_T_2, phi_T_3, phi_T_4

in phi_correct * phi_bounded * phi_filter_level_1;

lexmin all_level_1;

#
# Second hyperplane: avoid the null solution + linear independence with the first hyperplane
#
phi_filter_level_2 := [] -> {[phi_S_0, phi_S_1, phi_S_2, phi_S_3, phi_S_4, phi_T_0, phi_T_1, phi_T_2, phi_T_3, phi_T_4]:

#phi_S: positive coefficients + no parameters + lin. ind (coef(i) > 0)
phi_S_0 >= 0 and
phi_S_1 > 0 and
phi_S_2 = 0 and
phi_S_3 = 0 and
phi_S_4 >= 0 and

#phi_T: positive coefficients + no parameters + lin. ind (coef(i) > 0)
phi_T_0 >= 0 and
phi_T_1 > 0 and
phi_T_2 = 0 and
phi_T_3 = 0 and
phi_T_4 >= 0 and

#phi_S != 0 and phi_T != 0
phi_S_0 + phi_S_1 + phi_S_2 + phi_S_3 >= 1 and
phi_S_0 + phi_T_1 + phi_T_2 + phi_T_3 >= 1
};

all_level_2 := keep latency_0, latency_1, latency_2, phi_S_0, phi_S_1, phi_S_2, phi_S_3, phi_S_4, phi_T_0, phi_T_1, phi_T_2, phi_T_3, phi_T_4

in phi_correct * phi_bounded * phi_filter_level_2;

lexmin all_level_2;