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Proof Theory of Riesz Modal Logic

Théorie de la preuve de la Logique Modale de Riesz

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Abstract

The focus of this thesis is to design a hypersequent calculus called **HMR** for Riesz modal logic, or equivalently, for the equational theory of modal Riesz spaces. It is part of a line of research aiming to provide a structural proof system for well–known probabilistic logics like the probabilistic μ–calculus or probabilistic Computational Tree Logic.

Riesz modal logic is a real–valued modal logic, i.e., a modal logic whose terms are interpreted as real numbers instead of Booleans. If extended with fixed–point defined operators, Riesz modal logic is expressive enough to encode most of the usual probabilistic logics like the probabilistic Computational Tree Logic mentioned above. Moreover, an equational axiomatisation has been provided for Riesz modal logic: there is a set of axioms such that two terms of Riesz modal logic are equivalent, i.e., they have the same interpretations in all models, if and only if they can be proved equal using this set of axioms and the rules of equational reasoning.

The goal of providing a hypersequent calculus for Riesz modal logic is to have a better–behaved proof system. The main limitation of equational reasoning is that its rules are not all analytical: it may be necessary to "guess" formulas during the construction of a derivation, which require some human ingenuity. Therefore we want to provide a proof system that is both sound and complete with regards to the axiomatisation of Riesz modal logic, while having only analytical rules, making the process of building derivations much simpler.

To do so, we build upon another existing hypersequent calculus introduced for the Abelian logic. Riesz modal logic can be seen as the Abelian logic extended with a scalar multiplication (a term can be multiplied by a real scalar $r \in \mathbb{R}$) and a modal operator $\Diamond$. Therefore, **HMR** is built by extending the hypersequent calculus for the Abelian logic with new rules to deal with those new operations, and by using weighted hypersequents instead of regular hypersequents, i.e., using pairs of real numbers and Riesz modal logic terms as elementary blocks instead of just Riesz modal logic terms. We show that this new hypersequent calculus is both sound and complete with regards to the axiomatisation of Riesz modal logic. We also prove that it satisfies the CAN elimination theorem, or equivalently the CUT elimination theorem, effectively removing the only non–analytical rule.

Lastly, we use this new hypersequent calculus to prove new and interesting results. Among them, we show that the equivalence problem of Riesz modal logic is decidable (i.e., there is an algorithm to decide whether or not two Riesz modal logic terms are equivalent). We also solve a problem that was previously left open regarding the axiomatisation of Riesz modal logic.

Résumé

L'objectif de cette thèse est de concevoir un calcul d'hypersequents appelé **HMR** pour la Logique Modale de Riesz, ou de manière équivalente, pour la théorie équationnelle des espaces de Riesz modaux. Elle s'inscrit dans une ligne de recherche visant à fournir un système de preuve structurelle pour des logiques probabilistes bien connues comme le μ–calcul probabiliste ou la logique probabiliste du temps arborescent (probabilistic Computational Tree Logic or pCTL).

La logique modale de Riesz est une logique modale à valeurs réelles, c'est-à-dire une logique modale dont les termes sont interprétés comme des nombres réels au lieu de Booléens. Si elle est étendue avec des opérateurs de point fixes, la logique modale de Riesz est suffisamment expressive pour coder la plupart des logiques probabilistes habituelles comme la logique probabiliste du temps arborescent mentionnée ci-dessus. De plus, une axiomatisation équationnelle a été fournie pour la logique modale de Riesz : il existe un ensemble d'axiomes tels que deux termes de la logique modale de Riesz sont équivalents, c'est-à-dire qu'ils ont les mêmes interprétations dans tous les modèles, si et seulement s'ils peuvent être prouvés égaux en utilisant cet ensemble d'axiomes et les règles du raisonnement équationnel.

Le but de fournir un calcul d'hypersequents pour la logique modale de Riesz est d'avoir un système de preuve ayant certaines propriétés intéressantes. La principale limite du raisonnement équationnel est que ses règles ne sont pas toutes analytiques : il peut être nécessaire de « deviner » des formules lors de la construction d'une dérivation, ce qui demande une certaine ingéniosité humaine. Par conséquent, nous voulons fournir un système de preuve à la fois correct et complet en ce qui concerne l'axiomatisation de la logique modale de Riesz, tout en n'ayant que des règles analytiques, ce qui rend le processus de construction des dérivations beaucoup plus simple.

Pour ce faire, nous nous appuyons sur un autre calcul d'hypersequents existant introduit pour la logique Abélienne. La logique modale de Riesz peut être vue comme la logique Abélienne étendue avec une multiplication par un nombre réel (un terme peut être multiplié par un nombre réel $r \in \mathbb{R}$) et un opérateur modal $\Diamond$. Par conséquent, **HMR** est construit en étendant le calcul d'hypersequents pour la logique Abélienne avec de nouvelles règles pour traiter ces nouvelles opérations, et en utilisant des hypersequents pondérés au lieu des hypersequents habituels, c'est-à-dire en utilisant des paires de nombres réels et des termes de la logique modale de Riesz comme blocs élémentaires au lieu de simples termes de la logique modale de Riesz. Nous montrons que ce nouveau calcul d'hypersequents est à la fois correct et complet en ce qui concerne l'axiomatisation de la logique modale de Riesz. Nous prouvons également qu'il satisfait le théorème d'élimination de la règle CAN, ou de manière équivalente le théorème d'élimination de la règle CUT, supprimant la seule règle non analytique.

Enfin, nous utilisons ce nouveau calcul d'hypersequents pour prouver des résultats nouveaux et intéressants. Parmi eux, nous montrons que le problème d'équivalence de la logique modale de Riesz est décidable (c'est-à-dire qu'il existe un algorithme pour décider si oui ou non deux termes de la logique modale de Riesz sont équivalents). Nous résolvons également un problème précédemment laissé ouvert concernant l'axiomatisation de la logique modale de Riesz.

Contents

Introduction

In the last decades, many applications for mathematical logic have been found, notably regarding the formal verification of hardware and software systems: logics are used to express and verify properties of programs. By using tools for the verification of properties of complex systems based on logical methods, it is possible to find and correct bugs which could otherwise be very costly or even be dangerous to human lives. As a concrete example, in [LFZG13] the authors used logical methods to verify some correctness properties of the software regulating air–traffic.

To verify those properties, they first have to be expressed formally. One approach is to express them using a chosen *logic*. For instance, the well–known modal μ–calculus [Koz83] can be used to express interesting properties, like the fact that a program is deadlock–free, i.e., that it will not be stuck in some state (see also [BdRV02, Sti01, CF08] for an introduction to modal logics).

It then becomes useful to be able to reason on the formulas that can be expressed in the logic. It is, for instance, often desirable to be able to check whether or not two formulas ϕ and ψ are equivalent, i.e., that a program satisfies ϕ if and only if it satisfies ψ. One of the uses is, for instance, that if it is known that a program satisfies ϕ, and that ϕ is equivalent to ψ, then we automatically get that the program satisfies ψ, avoiding the (possibly expensive) cost of having to check the second property.

For some logics, it is possible to automatically check whether or not two formulas are equivalent, like for modal μ–calculus (see, e.g., [BK08]). However it is not always possible, and even when it is possible, the process can be quite costly. It is thus often useful to design alternative and complementary approaches to establish relations on formulas. Those approaches can be human–aided, and help the user construct a formal proof of a desired statement.

A classic human–aided approach is equational reasoning, where one can construct formal proofs of equalities between formulas using a set of axioms and the well–known rules of equational reasoning. An example is to use the axioms of Boolean algebras to construct proofs that two formulas of Boolean logic are equivalent.

However, human ingenuity can be necessary in such systems. In equational reasoning, because of the transitivity rule

$$\frac{\mathcal{A} \vdash A = C \quad \mathcal{A} \vdash C = B}{\mathcal{A} \vdash A = B} \text{ trans}$$

the user may have to guess the formula C while constructing the proof of $A = B$, which can require quite a lot of ingenuity. Such a rule where it is necessary to "guess" a new formula is called a non–analytical rule. Even some basic equalities such as $x \vee x = x$ may heavily rely on the user to provide those intermediary steps, e.g., the proof $x \vee x = (x \vee x) \wedge \top = (x \vee x) \wedge (x \wedge \neg x) = x \vee (x \wedge \neg x) = x \vee \bot x = x$ requires four intermediary formulas. This makes the process of deriving such proofs difficult to automatize. Such a rule where we have to guess a new formula is called a non–analytic rule.

Structural proof theory is a field of mathematics (see [Bus98]) that focuses on developing proof systems that require less human ingenuity to build proofs. For instance, regarding Boolean logic, Gentzen's sequent calculus [Gen34] is a structural proof system with good properties, most notably the CUT–elimination theorem that effectively removes the only non–analytical rule of the system.

When coming with a new structural proof system **S** for a logic **L**, there are three properties that are very desirable:

- **Soundness:** every property that can be proved in **S** can be proved in **L**,
- **Completeness:** every property that can be proved in **L** can be proved in **S**, and
- **Elimination of the non–analytical rules:** the system **S** is complete even without any non–analytical rule (in the case of Gentzen's sequent calculus, this is the CUT–elimination theorem).

Those three properties are often enough to ensure that the structural proof system **S** behaves well–enough. Structural proof systems have been developed even for very expressive logics such as the modal μ–calculus among others [Wal95, Stu07, DHL06, Dou17].

Despite being well studied and well understood, ordinary modal logics like modal μ–calculus are not adequate for reasoning on all kinds of programs. One of the limitations concerns probabilistic programs, i.e., programs that use instructions to generate random numbers (e.g., `rand()` in C).

To handle such programs, it is often natural and useful to express properties about probabilities. Thus, the interest in designing probabilistic logics since the early 1980's. Those can express properties regarding probabilities such as "the program will not face any deadlock with probability strictly greater than $\frac{1}{5}$".

Quite a few probabilistic logics have been designed and studied, in particular probabilistic Computation Tree Logic (pCTL) [LS82, HS86, HJ94, BK08]. pCTL is one of the most used probabilistic logic since it has a simple definition but is already quite expressive. For instance, the deadlock–free property above can be stated in pCTL as $\neg(\mathbb{P}_{\geq \frac{4}{5}}(\text{F terminal}))$.

However, the progress in designing a well–behaved structural proof system for probabilistic logics has, this far, been quite underwhelming. There is currently, to the best of our knowledge, no well–behaved structural proof system for the logic pCTL, or any other expressive probabilistic logic (see, e.g., [DFHM16] for the logic pCTL, [BGZB09, Hsu17] for pRHL and [Koz85] pPDL). We can separate the design of a well–behaved structural proof system into two distinct steps:

1. first design a sound and complete proof system for deriving the equivalence of formulas – this is a called the axiomatisation problem,
2. then build upon this proof system to design a better behaved structural proof system, more specifically, a proof system without any non–analytic rule.

In the case of Boolean logic, the first step could be to find the set of axioms used in equational reasoning, while the second step would be to build Gentzen's sequent calculus and prove the CUT–elimination theorem.

The axiomatisation problem for pCTL has been left open for 35 years (see [LS82] or [HS86]): all the proof systems for pCTL are currently incomplete.

One of the approaches to design a good proof system for probabilistic logics, which was democratised by the seminal work of Kozen on probabilistic PDL [Koz85], is to take some distance from the usual Boolean semantics. Instead of interpreting formulas as either true or false (like in pCTL), pPDL formulas are interpreted as real numbers. Such logics are called real–valued logics. Further insights on the usefulness of real–valued logics for probabilistic programs can be found in [Koz81]. Importantly, a sound and complete axiomatisation was found for pPDL [Koz85]. However, pPDL is quite limited, and can not be easily directly extended to other expressive logics like pCTL. This comes from the fact that the language of pPDL is both used as a programming language and as the logic used to reason on those programs, and the programs built using pPDL are quite limited. For instance, pPDL satisfies the finite model property where one of the key points of pCTL is that the finite model property fails for pCTL.

Even though all the methods used by Kozen to design pPDL can not be directly used to develop a well–behaved proof system for pCTL, it gives some new insights into the problems faced in designing this system. For instance, the use of a real valued logic instead of a Boolean one is quite promising. Indeed, recent works [Mio12, MS13, Mio18] have shown that pCTL can be encoded using a simple real–valued modal logic extended with (co)inductively defined operators like Łukasiewicz μ–calculus.

Therefore, one natural direction of research is the following.

1. Find a simple real–valued modal logic having nice properties.
2. Design a good structural proof system for this logic. **This thesis focuses on designing such a system for Riesz modal logic.**
3. Extend the logic with the (co)inductive defined operators necessary to obtain the expressiveness of pCTL.
4. Build upon the proof system designed at step 2 to obtain a well–behaved proof system for the logic obtained at step 3, thus having a good proof system for pCTL.

As mentioned above, good real–valued modal logics have already been designed. Among those, this thesis will focus on Riesz modal logic, introduced by Furber, Mardare and Mio in [MFM17, FMM20]. Riesz modal logic can interpret other basic real–valued logics, including Łukasiewicz modal logic. Moreover, a sound and complete axiomatisation for Riesz modal logic has been found in [MFM17]. Moreover, a structural proof system called **GA** for Abelian logic, which can be seen as a fragment of Riesz modal logic, was designed by Gabbay, Metcalfe and Olivetti in [MOG05]. Thus Riesz modal logic is a good candidate for building a proof system for the second step. Note that there are other good candidates to study for designing a well–behaved structural proof system. E.g., Diaconescu, Metcalfe and Schnüriger recently introduced another extension to Abelian logic with modal operators called K(A) in [DMS18].

To provide a sound and complete axiomatisation for Riesz modal logic, Furber, Mardare and Mio introduce the notion of modal Riesz spaces [MFM17]. A modal Riesz space is a Riesz space, i.e., a vector space equipped with a lattice order compatible with the vector operations, extended with a modal operator $\Diamond$ satisfying the following axioms:

- Linearity: for all terms t_1, t_2 and real scalar r, $\Diamond(t_1 + rt_2) = \Diamond(t_1) + r\Diamond(t_2)$,
- Positivity: if $0 \leq t$ then $0 \leq \Diamond(t)$,
- 1–decreasing : $\Diamond(1) \leq 1$.

It was shown that two formulas for Riesz modal logic are equivalent, when interpreted on Markov chains, if and only if the two corresponding terms are equal in all modal Riesz spaces [MFM17, FMM20].

An interesting class of proof systems to investigate for Riesz modal logic, and modal logics in general, is the class of hypersequent calculi. Hypersequent calculus is a generalisation of Gentzen's sequent calculus introduced by A. Avron [Avr87], and independently by G. Pottinger [Pot83]. In hypersequent calculus, instead of having only one sequent, the system is used to derive a *multiset* of sequents. This modification significantly increases the expressiveness of Gentzen's calculus as it allows additional transfers of information between the sequents.

Contributions of this thesis: The focus of this thesis is to design a hypersequent calculus called **HMR** for Riesz modal logic, or equivalently, for the equational theory of modal Riesz spaces. To do so, we will build on the hypersequent calculus **GA** introduced in [MOG05] by Gabbay, Metcalfe and Olivetti.

Since **GA** is a hypersequent calculus for Abelian lattice–ordered groups, i.e., Abelian groups equipped with a lattice order, there are two operations that needs to be added to the system: the scalar multiplication and the modal operator $\Diamond$.

This will be done incrementally: I separately introduce a hypersequent calculus for Riesz spaces, i.e., Abelian lattice–ordered groups with scalar multiplication, called **HR** and a hypersequent calculus for modal Abelian lattice–ordered groups called **MGA** before introducing **HMR**. This iterative process is useful to better appreciate the difficulties and insights concerning those two additional operations.

Organisation of this thesis. In Chapter 1, we provide the required technical background. We give the basic definitions of the different algebras considered in this thesis, i.e., Abelian lattice–ordered groups, Riesz spaces and modal Riesz spaces. We also introduce the Riesz modal logic and provide the definition of the hypersequent calculus **GA** of [MOG05].

In Chapter 2, we start building the iterative hypersequent calculi. We first introduce the hypersequent calculus **GA**$\|$, which is equivalent to **GA** but has a structure better suited to the systems introduced after. We then introduce **HR** and **MGA** before merging them into **HMR**.

In Chapter 3, we give some applications of the system **HMR**. Notably, we prove that the equational theory of modal Riesz spaces is decidable, and answer a problem left open in [FMM20] by showing that free modal Riesz spaces are Archimedean.

Chapter 1

Technical background

1.1 Lattice–ordered Abelian groups

Lattice–ordered groups constitute a mature field of research, dating back to fundamental papers by Birkhoff, Nakano and Lorenzen among others and are closely related to (modal) Riesz spaces as we will see. Moreover, a hypersequent calculus was introduced by Metcalfe, Gabbay and Olivetti for the theory of lattice–ordered Abelian groups, and this hypersequent calculus will serve as the basis for the work presented in this thesis. Thus, we will first introduce the definitions of lattice–ordered Abelian groups and the results used in our work. We refer to [AF89, KM94] for a comprehensive reference to the subject.

A lattice–ordered Abelian group, or Abelian l–group, is an algebraic structure $(R, 0, +, -, \sqcup, \sqcap)$ such that $(R, 0, +, -)$ is an Abelian group, $(R, \sqcup, \sqcap)$ is a lattice and the induced order ($a \leq b \Leftrightarrow a \sqcap b = a$) is compatible with addition in the sense that for all $a, b, c \in R$, if $a \leq b$ then $a+c \leq b+c$. Formally we have:

Definition 1.1.1 (Abelian l–group)**.** The *language* $\mathcal{L}_{\mathbf{A}}$ of Abelian l–groups is given by the signature $\{0, +, -, \sqcup, \sqcap\}$ where 0 is a constant, $+$, $\sqcup$ and $\sqcap$ are binary functions and $-$ is a unary function. A *lattice–ordered Abelian group*, or *Abelian l–group*, is a $\mathcal{L}_{\mathbf{A}}$-algebra, i.e., a set equipped with interpretations for the operations, satisfying the set $\mathcal{A}_{\text{l–groups}}$ of equational axioms of Figure 1.1. We use the standard abbreviation of $x \leq y$ for $x \sqcap y = x$.

1. Axioms of Abelian groups: $x+(y+z) = (x+y)+z$, $x+y = y+x$, $x+0 = x$, $x-x = 0$,
2. Lattice axioms: (associativity) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (commutativity) $z \sqcup y = y \sqcup z$, $z \sqcap y = y \sqcap z$, (absorption) $z \sqcup (z \sqcap y) = z$, $z \sqcap (z \sqcup y) = z$.
3. Compatibility axiom: $(x \sqcap y) + z \leq y + z$.

Figure 1.1: Set $\mathcal{A}_{\text{l–group}}$ of equational axioms of Abelian l–groups.

Remark 1. Note how the compatibility axiom has been equivalently formalised in Figure 1.1 as an inequality and not as an implication by using $(x \sqcap y)$ and y as two general terms automatically satisfying the hypothesis $(x \sqcap y) \leq y$. Moreover the inequality

$$(x \sqcap y) + z \leq y + z$$

can be rewritten as the equation

$$((x \sqcap y) + z) \sqcap (y + z) = (x \sqcap y) + z$$

using the lattice operations ($x \leq y \Leftrightarrow x \sqcap y = x$). Hence, since Abelian l–groups are axiomatised by a set of equations, the family of Abelian l–groups is a variety in the sense of universal algebra.

Example 1. The integers $\mathbb{Z}$ together with their standard linear order ($\leq$), expressed by taking $k_1 \sqcap k_2 = \min(k_1, k_2)$ and $k_1 \sqcup k_2 = \max(k_1, k_2)$, is an Abelian l–group. This is a fundamental example also due to the following fact (see, e.g., [Wei63] for a proof): for any two terms A, B, the equality $A = B$ holds in all Abelian l–groups if and only if $A = B$ holds in the Abelian l–group $(\mathbb{Z}, \leq)$.

Example 2. For a given set X, the set $\mathbb{Z}^X$ of functions $f : X \to \mathbb{Z}$ is a Abelian l–group when all operations are defined pointwise: $(-f)(x) = -(f(x))$, $(f + g)(x) = f(x) + g(x)$, $(f \sqcup g)(x) = f(x) \sqcup g(x)$, $(f \sqcap g)(x) = f(x) \sqcap g(x)$. Thus, for instance, the space of n-dimensional vectors $\mathbb{Z}^n$ is an Abelian l–group whose lattice order is not linear.

Remark 2. We use the capital letters A, B, C to range over terms built from a set of variables ranged over by x, y, z. We write $A[B/x]$ for the term, defined as expected, obtained by substituting all occurrences of the variable x in the term A with the term B.

As observed in Remark 1, the family of Abelian l–groups is a variety of algebras. This means, by Birkhoff completeness theorem, that two terms A and B are equivalent in all Abelian l–groups if and only if the identity $A = B$ can be derived using the familiar deductive rules of equational logic, written as $\mathcal{A}_{\text{l–groups}} \vdash A = B$ (see Theorem 21 of [Wec92] or the seminal work of Birkhoff [Bir35]).

Definition 1.1.2 (Deductive Rules of Equational Logic)**.** Rules for deriving identities between terms from a set $\mathcal{A}$ of equational axioms:

$$\frac{(A = B) \in \mathcal{A}}{\mathcal{A} \vdash A = B}\ \text{Ax} \qquad \frac{}{\mathcal{A} \vdash A = A}\ \text{refl} \qquad \frac{\mathcal{A} \vdash B = A}{\mathcal{A} \vdash A = B}\ \text{sym} \qquad \frac{\mathcal{A} \vdash A = B}{\mathcal{A} \vdash C[A] = C[B]}\ \text{ctxt}$$

$$\frac{\mathcal{A} \vdash A = B \quad \mathcal{A} \vdash B = C}{\mathcal{A} \vdash A = C}\ \text{trans} \qquad \frac{\mathcal{A} \vdash A = B}{\mathcal{A} \vdash A[C/x] = B[C/x]}\ \text{subst}$$

where A, B, C are terms of the algebraic signature under consideration built from a countable collection of variables and $C[\cdot]$ is a context.

In what follows we denote with $\mathcal{A}_{\text{l–groups}} \vdash A \leq B$ the judgement $\mathcal{A}_{\text{l–groups}} \vdash A = A \sqcap B$.

The following elementary facts (see, e.g., [KM94] for proofs) imply that, in the theory of Abelian l–groups, a proof system for deriving equalities can be equivalently seen as a proof system for deriving equalities with 0 or inequalities.

Lemma 1. *The following assertions hold:*

- $\mathcal{A}_{\text{l–groups}} \vdash A = B \;\Leftrightarrow\; \mathcal{A}_{\text{l–groups}} \vdash A - B = 0$,
- $\mathcal{A}_{\text{l–groups}} \vdash A = 0 \;\Leftrightarrow\; \big(\mathcal{A}_{\text{l–groups}} \vdash 0 \leq A$ and $\mathcal{A}_{\text{l–groups}} \vdash 0 \leq -A\big)$.

Definition 1.1.3. A term A is in *negation normal form* (NNF) if the operator $(-)$ is only applied to variables.

For example, the term $(-x) \sqcap (-y)$ is in NNF, while the term $-(x \sqcup y)$ is not.

Lemma 2. *Every term A can be rewritten to an equivalent term in NNF.*

Proof. Negation can be pushed towards the variables by the following rewritings: $-(-A) = A$, $-(A+B) = (-A)+(-B)$, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$ (see Lemma 5 (4) below). □

Negation can be defined on terms in NNF as follows.

Definition 1.1.4. Given a term A in NNF, the term $\overline{A}$ is defined inductively as follows: $\overline{x} = -x$, $\overline{-x} = x$, $\overline{A + B} = \overline{A} + \overline{B}$, $\overline{A \sqcup B} = \overline{A} \sqcap \overline{B}$, $\overline{A \sqcap B} = \overline{A} \sqcup \overline{B}$.

The following are basic facts regarding negation of NNF terms.

Lemma 3. *For any term A in NNF, the term $\overline{A}$ is also in NNF and it holds that $\mathcal{A}_{\text{l–groups}} \vdash \overline{A} = -A$.*

Proof. We prove the result by straightforward induction on A. See Lemma 5 (4) below for the $\sqcup$ and $\sqcap$ cases. □

Lemma 4. *For any terms A, B in NNF, it holds that $\overline{A}[B/x] = \overline{A[B/x]}$.*

It is convenient to define a notion of complexity on the terms to reason by induction on the size of the terms later on.

Definition 1.1.5 (Complexity)**.** We define the complexity $c(A)$ of a term in NNF A by induction as follows:

- $c(x) = 0$
- $c(\overline{x}) = 0$
- $c(0) = 1$
- $c(A + B) = 1 + c(A) + c(B)$
- $c(A \sqcup B) = 1 + c(A) + c(B)$
- $c(A \sqcap B) = 1 + c(A) + c(B)$

We now list some useful facts that will be used throughout the paper. The following are useful derived operators frequently used in the theory of Abelian l–groups:

Symbol	Terminology	Definition
A^+	The positive part	$A \sqcup 0$
A^-	The negative part	$(-A) \sqcup 0$
$\lvert A \rvert$	The absolute value	$A^+ + A^-$

Lemma 5. *The following equations hold:*

- *(1) For all A, $2A^- = (2A)^-$*
- *(2) For all A, B, $2(A \sqcup B)^- \leq (A + B)^-$*
- *(3) For all A, B, $(A + B)^- \leq A^- + B^-$*
- *(4) For all A, B, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$.*
- *(5) For all A, B, C, $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$ and $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$.*
- *(6) For all A, B, C, $A + (B \sqcup C) = (A + B) \sqcup (A + C)$ and $A + (B \sqcap C) = (A + B) \sqcap (A + C)$.*
- *(7) For all A, $0 \leq A \sqcup (-A)$.*

Most notably, observe that Abelian l–groups are distributive lattices (Lemma 5 (5)), that sum distributes over lattice operations (Lemma 5 (6)) and that the least upper bound of any element with its negation is always positive (Lemma 5 (7)).

Proof. As mentioned in Example 1, the Abelian l–group $\mathbb{Z}$ is complete for the equational theory of Abelian l–groups. This means that a universally quantified equation $\mathcal{A}_{\text{l–groups}} \vdash A = B$ holds in all Abelian l–groups if and only if it holds in the Abelian l–group $(\mathbb{Z}, \leq)$. It is then straightforward to check the validity of all equations in $\mathbb{Z}$. □

Lemma 6. *For all* A, B*,* $A \sqcup B \geq 0$ *if and only if* $A^- \sqcap B^- = 0$*.*

Proof. For all A, B we have:

$$\begin{array}{rcl} 0 \sqcap (A \sqcup B) & = & (A \sqcap 0) \sqcup (B \sqcap 0) \\ & = & -(((-A) \sqcup (-0)) \sqcap ((-B) \sqcup (-0))) \\ & = & -(A^- \sqcap B^-) \end{array}$$

Hence $0 \sqcap (A \sqcup B) = 0$ if and only if $-(A^- \sqcap B^-) = 0$ if and only $(A^- \sqcap B^-) = 0$. The proof is complete recalling that $0 \leq A \sqcup B$ means, by definition, that $0 = 0 \sqcap (A \sqcup B)$. □

1.2 Riesz Spaces

The theory of Riesz spaces, also known as vector lattices, is a branch of mathematics at the intersection of algebra and functional analysis, introduced by F. Riesz, L. Kantorovich and H. Freudenthal among others. It merges the notions of lattice order and that of real vector spaces. The former is pervasive in logic and the latter is at the heart of probability theory (e.g., convex combinations, linearity of the expected value operator, etc.). Kozen was the first to observe in a series of seminal works (see, e.g., [Koz81, Koz85]) that, for the above reasons, the theory of Riesz spaces provides a convenient mathematical setting for the study and design of probabilistic logics. Mio, Furber and Mardare extend the theory of Riesz spaces with a modal operator to obtain the theory of modal Riesz spaces, which is the theory we are interested in for the different hypersequent calculi introduced later on.

A Riesz space is an algebraic structure $(R, 0, +, (r)_{r\in\mathbb{R}}, \sqcup, \sqcap)$ such that $(R, 0, +, (r)_{r\in\mathbb{R}})$ is a vector space over the reals, $(R, \sqcup, \sqcap)$ is a lattice and the induced order $(a \leq b \Leftrightarrow a \sqcap b = a)$ is compatible with addition and with the scalar multiplication, in the sense that: (i) for all $a, b, c \in R$, if $a \leq b$ then $a + c \leq b + c$, and (ii) if $a \geq b$ and $r \in \mathbb{R}_{\geq 0}$ is a non–negative real, then $ra \geq rb$.

Following Definition 1.1.1, a Riesz space can be seen as an Abelian l–group where the language is extended with scalar multiplications (r) for all $r \in \mathbb{R}$ and the axioms regarding the scalar multiplications are added to the axioms of Abelian l–groups.

This section contains the basic definitions and results related to Riesz spaces. We refer to [LZ71, JR77] for a more thorough introduction to the topic. Formally we have:

Definition 1.2.1 (Riesz Space)**.** The *language* $\mathcal{L}_R$ of Riesz spaces is given by the (uncountable) signature $\{0, +, (r)_{r\in\mathbb{R}}, \sqcup, \sqcap\}$ where 0 is a constant, $+$, $\sqcup$ and $\sqcap$ are binary functions and r is a unary function, for all $r \in \mathbb{R}$. A *Riesz space* is a $\mathcal{L}_R$-algebra satisfying the set $\mathcal{A}_{\text{Riesz}}$ of equational axioms of Figure 1.2. We use the standard abbreviations of $-x$ for $(-1)x$ and $x \leq y$ for $x \sqcap y = x$.

Remark 3. Similarly to Abelian l–groups (see Remark 1, Riesz spaces are axiomatised by a set of equations, and so the family of Riesz spaces is a variety in the sense of universal algebra.

1. Axioms of real vector spaces:
   - Additive group: $x + (y + z) = (x + y) + z$, $x + y = y + x$, $x + 0 = x$, $x - x = 0$,
   - Axioms of scalar multiplication: $r_1(r_2x) = (r_1 \cdot r_2)x$, $1x = x$, $r(x + y) = (rx) + (ry)$, $(r_1 + r_2)x = (r_1x) + (r_2x)$,
2. Lattice axioms: (associativity) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (commutativity) $z \sqcup y = y \sqcup z$, $z \sqcap y = y \sqcap z$, (absorption) $z \sqcup (z \sqcap y) = z$, $z \sqcap (z \sqcup y) = z$.
3. Compatibility axioms:
   - $(x \sqcap y) + z \leq y + z$,
   - $r(x \sqcap y) \leq ry$, for all scalars $r \geq 0$.

Figure 1.2: Set $\mathcal{A}_{\text{Riesz}}$ of equational axioms of Riesz spaces.

Example 3. The real numbers $\mathbb{R}$ together with their standard linear order ($\leq$), expressed by taking $r_1 \sqcap r_2 = \min(r_1, r_2)$ and $r_1 \sqcup r_2 = \max(r_1, r_2)$, is a Riesz space. It has the same importance as $\mathbb{Z}$ for Abelian l–groups in the sense that the real numbers $\mathbb{R}$ is complete for the quasi–equational theory of Riesz spaces (see, e.g., [LvA07] for a proof). Thus a quasi–equation, i.e., an implication of the form

$$\bigwedge_{i=1}^{n} A_i = B_i \Rightarrow A = B$$

holds in all Riesz spaces if and only if it holds in $\mathbb{R}$.

Example 4. For a given set X, the set $\mathbb{R}^X$ of functions $f : X \to \mathbb{R}$ is a Riesz space when all operations are defined pointwise: $(rf)(x) = r(f(x))$, $(f + g)(x) = f(x) + g(x)$, $(f \sqcup g)(x) = f(x) \sqcup g(x)$, $(f \sqcap g)(x) = f(x) \sqcap g(x)$. Thus, for instance, the space of n-dimensional vectors $\mathbb{R}^n$ is a Riesz space whose lattice order is not linear.

The following elementary facts (see, e.g., [LZ71, §2.12] for proofs) imply that, in the theory of Riesz spaces, a proof system for deriving equalities can be equivalently seen as a proof system for deriving equalities with 0 or inequalities.

Lemma 7. *The following assertions hold:*

- $\mathcal{A}_{\text{Riesz}} \vdash A = B \;\Leftrightarrow\; \mathcal{A}_{\text{Riesz}} \vdash A - B = 0$,
- $\mathcal{A}_{\text{Riesz}} \vdash A = B \;\Leftrightarrow\; \big(\mathcal{A}_{\text{Riesz}} \vdash A \leq B \text{ and } \mathcal{A}_{\text{Riesz}} \vdash B \leq A\big)$.

Remark 4. From now on, in the rest of this paper, it will be convenient to take the derived negation operation $(-A) = (-1)A$ as part of the signature and restrict all scalars r to be strictly positive ($r > 0$). The scalar $0 \in \mathbb{R}$ can be removed by rewriting $(0)A$ as 0.

Definition 1.2.2. A term A is in *negation normal form* (NNF) if the operator $(-)$ is only applied to variables.

For example, the term $(-x) \sqcap (-y)$ is in NNF, while the term $-(x \sqcup y)$ is not.

Lemma 8. *Every term A can be rewritten to an equivalent term in NNF.*

Proof. Negation can be pushed towards the variables by the following rewritings: $-(-A) = A$, $-(rA) = r(-A)$, $-(A+B) = (-A)+(-B)$, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$ (see Lemma 10(4) below). $\square$

Negation can be defined on terms in NNF as follows.

Definition 1.2.3. Given a term A in NNF, the term $\overline{A}$ is defined as follows: $\overline{x} = -x$, $\overline{-x} = x$, $\overline{rA} = r\overline{A}$, $\overline{A + B} = \overline{A} + \overline{B}$, $\overline{A \sqcup B} = \overline{A} \sqcap \overline{B}$, $\overline{A \sqcap B} = \overline{A} \sqcup \overline{B}$.

The following are basic facts regarding negation of NNF terms.

Lemma 9. *For any term A in NNF, the term $\overline{A}$ is also in NNF and it holds that $\mathcal{A}_{\text{Riesz}} \vdash \overline{A} = -A$.*

Proof. We prove the result by straightforward induction on A. See Lemma 10(4) below for the $\sqcup$ and $\sqcap$ cases. $\square$

Proposition 1. *For any terms A, B in NNF, it holds that $\overline{A}[B/x] = \overline{A[B/x]}$.*

We now define the complexity of a Riesz term.

Definition 1.2.4 (Complexity)**.** We define the complexity $c(A)$ of a term in NNF A by induction as follows:

- $c(x) = 0$
- $c(\overline{x}) = 0$
- $c(0) = 1$
- $c(rA) = 1 + c(A)$
- $c(A + B) = 1 + c(A) + c(B)$
- $c(A \sqcup B) = 1 + c(A) + c(B)$
- $c(A \sqcap B) = 1 + c(A) + c(B)$

We now list some useful facts that will be used throughout the paper.

Lemma 10. *The following equations hold:*

- *(1) For all A and $r > 0$, $r(A^-) = (rA)^-$.*
- *(2) For all A, B, $2(A \sqcup B)^- \leq (A + B)^-$*
- *(3) For all A, B, $(A + B)^- \leq A^- + B^-$.*
- *(4) For all $r > 0$, $0 \leq A$ if and only if $0 \leq rA$.*
- *(5) For all A, B, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$.*
- *(6) For all A, B, C, $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$ and $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$.*
- *(7) For all A, B, C, $A + (B \sqcup C) = (A + B) \sqcup (A + C)$ and $A + (B \sqcap C) = (A + B) \sqcap (A + C)$.*
- *(8) For all A, $0 \leq A \sqcup (-A)$.*

Most notably, observe that Riesz spaces are distributive lattices (Lemma 10(6)), that sum distributes over lattice operations (Lemma 10(7)) and that the least upper bound of any element with its negation is always positive (Lemma 10(8)).

Proof. As mentioned in Example 3, the Riesz space $\mathbb{R}$ is complete for the quasi–equational theory of Riesz spaces. This means that a universally quantified Horn clause $\bigwedge_{i\in I} \mathcal{A}_{\text{Riesz}} \vdash A_i = B_i \Rightarrow \mathcal{A}_{\text{Riesz}} \vdash A = B$ holds in all Riesz spaces if and only if it holds in the Riesz space $(\mathbb{R}, \leq)$. It is then straightforward to check the validity of all equations and equational implications in $\mathbb{R}$. □

Lemma 11. *For all A, B, $A \sqcup B \geq 0$ if and only if $A^- \sqcap B^- = 0$.*

Proof. For all A, B we have:

$$\begin{aligned} 0 \sqcap (A \sqcup B) &= (A \sqcap 0) \sqcup (B \sqcap 0) \\ &= -(((-A) \sqcup (-0)) \sqcap ((-B) \sqcup (-0))) \\ &= -(A^- \sqcap B^-) \end{aligned}$$

Hence $0 \sqcap (A \sqcup B) = 0$ if and only if $-(A^- \sqcap B^-) = 0$ if and only $(A^- \sqcap B^-) = 0$. The proof is complete recalling that $0 \leq A \sqcup B$ means, by definition, that $0 = 0 \sqcap (A \sqcup B)$. □

The Archimedean property plays a central role in the representation theory of Riesz spaces. Indeed the Archimedean spaces are precisely those that can be shown to be isomorphic to spaces of real-valued functions on certain topological spaces (see, e.g., [LZ71, Sec. 46]) Furthermore, special representation theorems are available if the Riesz space possesses a weak-unit [LZ71, Sec. 50]) or even a strong unit ([LZ71, Sec. 45]).

Definition 1.2.5 (Archimedean Riesz spaces)**.** A Riesz space R is said to be *Archimedean* if it satisfies the following property

$$\forall a, b \in R, (\forall n \in \mathbb{N}, na \leq b) \Rightarrow a \leq 0$$

Example 5. The Riesz space $\mathbb{R}$ is Archimedean.

Example 6. The Riesz space $\mathbb{R}^2$ with the lexicographic order, i.e.,

$$(a, b) \leq (c, d) \Leftrightarrow \begin{cases} a < c \\ a = c \text{ and } b \leq d \end{cases}$$

is not Archimedean. Indeed, $n(0, 1) = (0, n) \leq (1, 0)$ for all $n \in \mathbb{N}$ but $(0, 1) \not\leq (0, 0)$.

Definition 1.2.6 (Weak unit)**.** Let R be a Riesz space. A positive element $u \in R$ is said to be a *weak unit* if for all positive element $x \in R$, if $x \sqcap u = 0$ then $x = 0$.

Definition 1.2.7 (Strong unit)**.** Let R be a Riesz space. A positive element $u \in R$ is said to be a *strong unit* if for all $x \in R$, there exists $n \in \mathbb{N}$ such that $|x| \leq nu$.

1.3 Modal Riesz Spaces

This section contains the basic definitions and results related to modal Riesz spaces, as introduced in [MFM17, FMM20].

The language of modal Riesz spaces extends that of Riesz spaces with two symbols: a constant 1 and an unary operator $\Diamond$.

Definition 1.3.1 (Modal Riesz Space)**.** The *language* $\mathcal{L}_R^{\Diamond}$ of modal Riesz spaces is $\mathcal{L}_R \cup \{1, \Diamond\}$ where $\mathcal{L}_R$ is the language of Riesz spaces as specified in Definition 1.2.1. A modal Riesz space is a $\mathcal{L}_R^{\Diamond}$-algebra satisfying the set $\mathcal{A}_{\text{Riesz}}^{\Diamond}$ of axioms of Figure 1.2.

Axioms of Riesz spaces	see Figure 1.2
+	
Positivity of 1:	$0 \leq 1$
Linearity of $\Diamond$:	$\Diamond(r_1 A + r_2 B) = r_1 \Diamond(A) + r_2 \Diamond(B)$
Positivity of $\Diamond$:	$\Diamond(0 \sqcup A) \geq 0$
1-decreasing property of $\Diamond$:	$\Diamond(1) \leq 1$

Figure 1.3: Set $\mathcal{A}^{\Diamond}_{\text{Riesz}}$ of equational axioms of modal Riesz spaces.

Remark 5. Note that the positivity axiom is equivalent to the following implication, which is easier to read.

$$A \geq 0 \Rightarrow \Diamond A \geq 0$$

However, as in Remark 1, we formalise the axiom as an inequality to ensure that modal Riesz spaces are fully axiomatised by a set of equations, and thus that the family of modal Riesz spaces is a variety.

Example 7. Every Riesz space R can be made into a modal Riesz space by interpreting 1 with any positive element and by interpreting $\Diamond$ as the identity function ($\Diamond(x) = x$) or the constant 0 function $\Diamond(x) = 0$.

Example 8. The Riesz space $(\mathbb{R}, \leq)$ of linearly ordered real numbers becomes a modal Riesz space by interpreting 1 with the number 1, and $\Diamond$ by any linear (due to the linearity axiom) function $x \mapsto rx$ for a scalar $r \in \mathbb{R}$ such that $r \geq 0$ (due to the positivity axiom) and $r \leq 1$ (due to the 1-decreasing axiom).

Example 9. Generalising the previous example, the Riesz space $\mathbb{R}^n$ (with operations defined pointwise, see Example 4) becomes a modal Riesz space by interpreting 1 with the constant 1 vector and $\Diamond$ by a linear (due to the linearity axiom) map $M : \mathbb{R}^n \to \mathbb{R}^n$, thus representable as a square matrix,

$$1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \Diamond = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1} & r_{n,2} & \cdots & r_{n,n} \end{pmatrix}$$

such that all entries $r_{i,j}$ are non–strictly positive (due to the positivity axiom) and where all the rows sum up to a value ≤ 1, i.e., for all $1 \leq i \leq n$ it holds that $\sum_{j=1}^{k} r_{i,j} \leq 1$ (due to the 1–decreasing axiom). Such matrices are known as sub–stochastic matrices. Each sub–stochastic matrix M can be regarded as a probabilistic transition system (also referred to as Markov chain, see Section 1.5) whose set S of states is $S = \{s_1, \dots, s_n\}$ and whose transition function $\tau_M : S \to \mathcal{D}^{\leq 1}(S)$, defined as:

$$\tau_M(s_i)(s_j) = r_{i,j}$$

assigns to each state $s_i \in S$ a sub–probability distribution $\tau_M(s_i) \in \mathcal{D}^{\leq 1}(S)$ specifying the probability of reaching s_j from s_i, for any $s_i, s_j \in S$.

Example 10. Consider the equality $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$. Does it hold in all modal Riesz spaces? In other words, does $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash \Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$? The answer is negative. Take as example the modal Riesz space $\mathbb{R}^2$ with:

$$1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Diamond = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{pmatrix}$$

and let $a = (1, 0)$ and $b = (0, 1)$. One verifies that $\Diamond(a \sqcup b) = (1, 0)$ while $\Diamond(a) \sqcup \Diamond(b) = (\frac{2}{3}, 0)$. This example shows that unlike the theory of Riesz spaces (cf. Example 3), the theory of modal Riesz spaces cannot be generated by a linear model, i.e. a model where either $a \leq b$ or $b \leq a$ for all a and b. Indeed, in any linear model, the equality $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$ clearly holds while it does not hold in the example above.

Remark 6. The choice of using the $\Diamond$ symbol for the unary operation of modal Riesz spaces might suggest the existence of a distinct De Morgan dual operator $\Box x = -\Diamond - x$. This is not the case since, due to linearity, $\Box x = \Diamond x$, i.e., $\Diamond$ is self dual. While using a different symbol such as $(\circ)$ might have been a better choice, we decided to stick to $\Diamond$ for backwards compatibility with previous works on modal Riesz spaces [MFM17, FMM20, LM19]. Another source of potential ambiguity lies in the "modal" adjective itself. Of course other axioms for $\Diamond$ can be conceived (e.g., $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$ instead of our $\Diamond(x+y) = \Diamond(x) + \Diamond(y)$, see, e.g., [DMS18]). Therefore different notions of modal Riesz spaces can be investigated, just like many types of classical modal logic exist (K, S4, S5, etc). Once again, our choice of terminology is motivated by backwards compatibility with previous works.

We now expand the definitions and properties related to terms in negation normal form to modal Riesz spaces.

Definition 1.3.2. A term A is in *negation normal form* (NNF) if the operator $(-)$ is only applied to variables and the constant 1.

Lemma 12. *Every term A can be rewritten to an equivalent term in NNF.*

Proof. Negation can be pushed towards the variables by the following rewritings: $-\Diamond(A) = \Diamond(-A)$ (see Lemma 8 for the other operators). $\square$

Negation can be defined on terms in NNF as follows.

Definition 1.3.3. Given a term A in NNF, we expand the operator $\overline{A}$ as follows: $\overline{\Diamond A} = \Diamond \overline{A}$, $\overline{1} = -1$, $\overline{-1} = 1$.

The following are basic facts regarding negation of NNF terms.

Lemma 13. *For any term A in NNF, the term $\overline{A}$ is also in NNF and it holds that $\mathcal{A}_{\text{Riesz}} \vdash \overline{A} = -A$.*

Lemma 14. *For any terms A, B in NNF, it holds that $\overline{A}[B/x] = \overline{A[B/x]}$.*

To define the complexity of a modal Riesz term, special care has to be taken for the $\Diamond$ operator and the 1 constant. Indeed, as we will see in Sections 2.3 and 2.4, it is useful for the induction proofs to consider the complexity of a $\Diamond$ formula and the 1 constant to be 0. Therefore we define the complexity of a modal Riesz term as follows.

Definition 1.3.4 (Complexity)**.** We define the complexity $c^{\Diamond}(A)$ of a term A in NNF by induction as follows:

- $c^{\Diamond}(x) = 0$
- $c^{\Diamond}(1) = 0$
- $c^{\Diamond}(\overline{x}) = 0$
- $c^{\Diamond}(\overline{1}) = 0$

- $c^{\Diamond}(\Diamond A) = 0$
- $c^{\Diamond}(0) = 1$
- $c^{\Diamond}(rA) = 1 + c(A)$
- $c^{\Diamond}(A + B) = 1 + c(A) + c(B)$
- $c^{\Diamond}(A \sqcup B) = 1 + c(A) + c(B)$
- $c^{\Diamond}(A \sqcap B) = 1 + c(A) + c(B)$

We now list the usual facts that we require. First, note that modal Riesz spaces are an extension of Riesz spaces, i.e., any equation that holds in all Riesz spaces also holds in all modal Riesz spaces.

Lemma 15. *Let A and B be Riesz terms, i.e., not containing any $\Diamond$ nor 1. If $\mathcal{A}_{\text{Riesz}} \vdash A = B$, then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$.*

Proof. We prove this result by a straightforward induction on the derivation of $\mathcal{A}_{\text{Riesz}} \vdash A = B$. The induction is trivial since every axiom of Riesz spaces is also an axiom of modal Riesz spaces. □

Moreover, the following elementary facts still hold in the theory of modal Riesz spaces.

Lemma 16. *The following assertions hold:*

- $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B \;\Leftrightarrow\; \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A - B = 0$,
- $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B \;\Leftrightarrow\; \big(\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq B \text{ and } \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash B \leq A\big)$.

Proof. The first assertion comes from the following equalities:

$$A = A + 0 = A + (B - B) = (A - B) + B$$

Thus if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A - B = B - B = 0$ and if $A - B = 0$ then $(A - B) + B = 0 + B = 0$.

For the second assertion, recall that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq B$ is a notation for $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \sqcap B = A$. Moreover, since $\mathcal{A}_{\text{Riesz}} \vdash x \sqcap x = x$ holds, by Lemma 15, we have $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash x \sqcap x = x$ and by using the substitution rule, $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \sqcap A = A$ and $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash B \sqcap B = B$. Therefore if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$, then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \sqcap B = A \sqcap A = A$ and $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash B \sqcap A = B \sqcap B = B$.

For the other direction, if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq B$ and $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash B \leq A$, then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = A \sqcap B = B \sqcap A = B$. □

Lemma 17. *The following equations hold:*

- *(1) For all A and $r > 0$, $r(A^-) = (rA)^-$.*
- *(2) For all A, B, $2(A \sqcup B)^- \leq (A + B)^-$*
- *(3) For all A, B, $(A + B)^- \leq A^- + B^-$.*
- *(4) For all $r > 0$, $0 \leq A$ if and only if $0 \leq rA$.*
- *(5) For all A, B, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$.*
- *(6) For all A, B, C, $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$ and $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$.*
- *(7) For all A, B, C, $A + (B \sqcup C) = (A + B) \sqcup (A + C)$ and $A + (B \sqcap C) = (A + B) \sqcap (A + C)$.*
- *(8) For all A, $0 \leq A \sqcup (-A)$.*

Most notably, observe that Riesz spaces are distributive lattices (Lemma 10(6)), that sum distributes over lattice operations (Lemma 10(7)) and that the least upper bound of any element with its negation is always positive (Lemma 10(8)).

Proof. Since $\mathbb{R}$ is not complete for the (quasi)equational theory of modal Riesz spaces (see Example 10), we can not use the same proof as in Lemma 10.

However, apart from the fact (4), every fact is an equality, and thus still holds in modal Riesz spaces according to Lemma 15. For instance, to show (1), according to Lemma 10, we know that $\mathcal{A}_{\text{Riesz}} \vdash r(x^-) = (rx)^-$ for a variable x and for all $r > 0$ and thus $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash r(x^-) = (rx)^-$. Therefore, for all A and $r > 0$ we can derive $r(A^-) = (rA)^-$ with the following derivation:

$$\frac{\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash r(x^-) = (rx)^-}{\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash r(A^-) = (rA)^-}\ \text{subst}$$

The last fact (4) is a direct consequence of the compatibility axiom $r(x \sqcap y) \leq ry$ with $x = 0$ and $y = A$. Indeed, if $0 \leq A$ then $0 \sqcap A = 0$, and thus $0 = r(0 \sqcap A) \leq rA$. Therefore, if $0 \leq A$ then $0 \leq rA$ for all $r \geq 0$. For the other direction, if $0 \leq rA$ then we just showed that $0 \leq \frac{1}{r} rA = A$. ☐

Lemma 18. *For all A, B, $A \sqcup B \geq 0$ if and only if $A^- \sqcap B^- = 0$.*

Proof. For all A, B we have:

$$\begin{aligned} 0 \sqcap (A \sqcup B) &= (A \sqcap 0) \sqcup (B \sqcap 0) \\ &= -(((-A) \sqcup (-0)) \sqcap ((-B) \sqcup (-0))) \\ &= -(A^- \sqcap B^-) \end{aligned}$$

Hence $0 \sqcap (A \sqcup B) = 0$ if and only if $-(A^- \sqcap B^-) = 0$ if and only $(A^- \sqcap B^-) = 0$. The proof is complete recalling that $0 \leq A \sqcup B$ means, by definition, that $0 = 0 \sqcap (A \sqcup B)$. ☐

1.4 Free algebras

Being definable purely by equations, the class of modal Riesz spaces is a variety in the sense of universal algebras. Therefore the category of modal Riesz spaces has free objects. Moreover, the study of those free objects can be of interest. For instance, the open problem of [FMM20] concerns the initial object of the category of modal Riesz spaces – which is a free object.

In this section, we consider a language $\mathcal{L}$ containing a least one constant and a variety of algebras $\mathbb{A}$ over $\mathcal{L}$ (e.g., the variety of modal Riesz spaces).

Definition 1.4.1 (Free objects)**.** For every set X, there is an unique *free algebra* $\mathbb{F}_{\mathbb{A}}(X) \in \mathbb{A}$ over the set of generators X satisfying the following universal property: there is an injective map $i : X \to \mathbb{F}_{\mathbb{A}}(X)$ such that for any given algebra $A \in \mathbb{A}$ and map $k : X \to A$, there is an unique $\mathbb{A}$–homomorphism $f : \mathbb{F}_{\mathbb{A}}(X) \to A$ satisfying $f \circ i = k$.

The free algebra $\mathbb{F}_{\mathbb{A}}(\emptyset)$ is the initial object.

Remark 7. Note that since there is an injective map i from X to $\mathbb{F}_{\mathbb{A}}(X)$, by identifying $x \in X$ and $i(x)$, we can always consider that $X \subseteq \mathbb{F}_{\mathbb{A}}(X)$.

Definition 1.4.2 (Term algebras)**.** Let $\mathcal{V}$ be a set. The *term algebra* over the variables $\mathcal{V}$ is the set of terms built using the signature $\mathcal{L}$ and the variables $\mathcal{V}$ quotiented by the axioms of $\mathbb{A}$.

Example 11. In the case of modal Riesz spaces, the term algebra over the variables $\mathcal{V}$ is the set of modal Riesz space terms on $\mathcal{V}$, i.e., the set defined by the grammar

$$A, B := x \in \mathcal{V} \mid 0 \mid 1 \mid A + B \mid rA \mid A \sqcap B \mid A \sqcup B \mid \Diamond A$$

quotiented by the axioms of modal Riesz spaces.

Lemma 19 (Thm 12 of [Wec92]). *The term algebra over the variables $\mathcal{V}$ is isomorphic to the free algebra $\mathbb{F}_{\mathbb{A}}(\mathcal{V})$.*

Corollary 1. *Let A, B two $\mathbb{A}$–terms over the set of variables $\mathcal{V}$, i.e., the variables appearing in A and B are all in $\mathcal{V}$. Then*

$$\mathcal{A}_{\mathbb{A}} \vdash A = B \text{ if and only if } A = B \text{ in } \mathbb{F}_{\mathbb{A}}(\mathcal{V})$$

1.5 Riesz modal logic

In this section, we will introduce the Riesz modal logic. Riesz modal logic was introduced by Furber, Mardare and Mio in [FMM20] as a real–valued probabilistic logic for Markov chains. As mentioned earlier, modal Riesz spaces were introduced as the algebraic semantic for Riesz modal logic.

Definition 1.5.1 (Sub–probability). Given a set X, we denote by $\mathcal{D}^{\leq 1}(X)$

$$\mathcal{D}^{\leq 1}(X) = \{d : X \to [0,1] \mid \sum_x d(x) \leq 1\}$$

the set of sub–probability distributions on X.

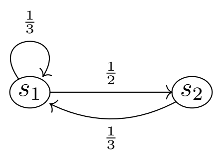
Given a distribution $d \in \mathcal{D}^{\leq 1}(X)$ and a subset $A \subseteq X$, we note by $d(A)$ the cumulative probability of A:

$$d(A) = \sum_{x \in A} d(x)$$

Definition 1.5.2 (Markov chain). A *Markov chain* is a pair (X, τ) where X is the (possibly infinite) set of states and $\tau : X \to \mathcal{D}^{\leq 1}(X)$ is the transition function which maps each state to a sub–probability distribution over states.

The intended interpretation is that, at a state $x \in X$, the computation stops with probability $1 - d(X)$, where $d = \tau(x)$, and continues with probability $d(X)$ following the sub–probability distribution d.

Example 12. Consider the Markov chain having state space $S = \{s_1, s_2\}$ and transition function τ defined by: $\tau(s_1) = (s_1 \mapsto \frac{1}{3}, s_2 \mapsto \frac{1}{2})$ and $\tau(s_2) = (s_1 \mapsto \frac{1}{3}, s_2 \mapsto 0)$:



From the state s_1 the computation progresses to s_1 itself with probability $\frac{1}{3}$, to s_2 with probability $\frac{1}{2}$ and it halts with probability $\frac{1}{6}$ (i.e., with the remaining probability $1 - (\frac{1}{2} + \frac{1}{3})$). From the state s_2 the computation progresses to s_1 with probability $\frac{1}{3}$ and it halts with probability $\frac{2}{3}$.

Definition 1.5.3 (Terms)**.** The terms of Riesz modal logic are defined using the following grammar:

$$A, B ::= 0 \mid 1 \mid A \sqcap B \mid A \sqcup B \mid rA \mid A + B \mid \Diamond A$$

Remark 8. Note that all Riesz modal logic terms are also modal Riesz space terms. Conversely, all *closed* modal Riesz space terms (i.e., modal Riesz space terms without any variable) are also Riesz modal logic terms.

Given any Markov chain (S, τ_M), each Riesz modal logic term A is interpreted as a function $[\![A]\!] = S \to \mathbb{R}$.

Definition 1.5.4 (Transition semantics)**.** For a Markov chain $C = (S, \tau_M)$, the interpretation $[\![A]\!]_C$ of a Riesz modal logic term A is inductively defined as:

$$[\![0]\!]_C(s_i) = 0 \qquad [\![1]\!]_C(s_i) = 1$$

$$[\![rA]\!]_C(s_i) = r \cdot \big([\![A]\!]_C(s_i)\big) \qquad [\![A+B]\!]_C(s_i) = [\![A]\!]_C(s_i) + [\![B]\!]_C(s_i)$$

$$[\![A \sqcup B]\!]_C(s_i) = \max\{[\![A]\!]_C(s_i), [\![B]\!]_C(s_i)\}$$

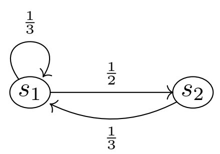
$$[\![A \sqcap B]\!]_C(s_i) = \min\{[\![A]\!]_C(s_i), [\![B]\!]_C(s_i)\}$$

$$[\![\Diamond A]\!]_C(s_i) = \sum_{s_j \in S} \big(\tau_M(s_i)(s_j) \cdot [\![A]\!]_C(s_j)\big)$$

for all $s_i \in S$.

Therefore, the semantics $[\![A]\!]_C$ of a term A can be understood as a (real–valued) quantitative property of states and $[\![\Diamond A]\!]_C$ denotes the expected value of $[\![A]\!]_C$ after a transition step.

Example 13. Consider for example the Markov chain of the previous example:



and the Riesz modal logic terms $\Diamond 1$ and $\Diamond(\Diamond 1)$. They are interpreted as the two functions on $S = \{s_1, s_2\}$ illustrated as vectors below:

$$[\![\Diamond 1]\!] = \begin{pmatrix} \frac{5}{6} \\ \frac{1}{3} \end{pmatrix} \qquad \text{and} \qquad [\![\Diamond(\Diamond 1)]\!] = \begin{pmatrix} \frac{4}{9} \\ \frac{5}{18} \end{pmatrix}$$

The term $\Diamond 1$ assigns to each state $s_i \in \{s_1, s_2\}$ the probability of making a computational step from s_i to any other state (and thus not halting). Similarly, the term $\Diamond\Diamond 1$ is the vector assigning to each state s_i the probability of making two consecutive computational steps from s_i.

More generally, $\Diamond^n 1$ assigns to each state s_i the probability of making n computational step from s_i.

Importantly, in [FMM20] the transition semantic is proved to be sound and complete with respect to the equational theory of modal Riesz spaces.

Lemma 20 ([FMM20][Thm 8.1])**.** *Two closed modal Riesz terms A and B are provably equal from the axioms of modal Riesz spaces if and only if $[\![A]\!]_C = [\![B]\!]_C$ when interpreted in all possible Markov chains*[1] *C.*

Moreover, sophisticated real–valued properties of Markov chains can be expressed using Riesz modal logic [MFM17, FMM20]. For instance, Riesz modal logic is powerful enough to characterise bisimulation (see Corollary 8.3 of [FMM20]).

Riesz modal logic with fixed–points

The remainder of this section is an informal discussion to give some insights on how to extend the Riesz modal logic with fixed–points to obtain a more expressive logic. Note, however, that this extension has not been studied during this thesis, and thus reading this section is not required to understand this work.

In its current state, Riesz modal logic terms can only express properties concerning a bounded number of steps (e.g., $\lozenge^n 1$ concerns only the next n steps). Therefore, some interesting properties can not be expressed using Riesz modal logic. For instance, the property "the Markov chain will eventually halt" can not be expressed since the number of steps before a Markov chain halts can not, in general, be bounded. Note that this is also the case for other classical modal logic on Kripke frames.

To express such properties, fixed–points or (co)inductively defined operators are necessary. For instance, we will investigate the property "the Markov chain will never halt". Let consider a Markov chain $C = (S, \tau)$. With some abuse of notations, we denote by $[\![P_{NT}]\!]_C$ the function that maps a state s to the probability that the Markov chain does not halt starting from s. The Markov chain does not halt starting from s_i if it does one step, and then does not halt either. Thus by the law of total probability

$$[\![P_{NT}]\!]_C(s_i) = \sum_{j=1}^{n} \big(\tau(s_i)(s_j) \cdot [\![P_{NT}]\!]_C(s_j)\big)$$

or, again, with some abuse of notations,

$$[\![P_{NT}]\!]_C(s_i) = [\![\lozenge P_{NT}]\!]_C(s_i)$$

Thus we would want to define the term P_{NT} as the fixed–point of the function $X \mapsto \lozenge X$. By taking inspirations from the μ–calculus, we define the term P_{NT} as

$$P_{NT} = \nu X.\lozenge X \sqcup 0 \sqcap 1$$

Remark 9. Note the presence of $\sqcup 0 \sqcap 1$ in the definition of P_{NT}. This is done to ensure that the interpretation of P_{NT} is in $[0,1]^S$ and not $\mathbb{R}^S$.

We can proceed in a similar way for the property "the Markov chain will eventually halt" mentioned above. Indeed, for a Markov chain $C = (S, \tau)$, we can denote by $[\![P_T]\!]_C$ the function that maps a state s to the probability that the Markov chain will eventually halt starting from s. C will eventually halt starting from a state s_i if it halts at this step (with probability $[\![1-\lozenge 1]\!]_C(s_i)$) or it does one more step and then eventually halts (with probability $[\![\lozenge P_T]\!]_C(s_i)$). Thus by the law of total probability

$$[\![P_T]\!]_C(s_i) = [\![(1-\lozenge 1) + \lozenge P_T]\!]_C(s_i)$$

[1]In fact, a generalised semantics is required for completeness, where Markov chains have state spaces endowed with a compact Hausdorff topology. The examples above have finite state spaces, and are thus trivially compact Hausdorff.

and we define the term P_T as

$$P_T = \mu X.((1 - \Diamond 1) + \Diamond X \sqcup 0 \sqcap 1)$$

Informally, we can extend the terms of Riesz modal logic with the fixed–points à la μ–calculus

$$\mu X.\phi(X) \mid \ \nu X.\phi(X)$$

where ϕ is defined using the grammar

$$\phi, \psi := X \mid \ 0 \mid \ 1 \mid \ \phi \sqcap \psi \mid \ \phi \sqcup \psi \mid \ r\phi \mid \ \phi + \psi \mid \ \Diamond \phi \mid \ \mu X.\phi(X) \mid \ \nu X.\phi(X)$$

The interpretation of a term ϕ with k variables can be seen as an operator that takes k functions and returns a function. We define

$$[\![\phi(X_1, ..., X_k)]\!]_C : (\mathbb{R}^S)^k \to \mathbb{R}^S$$

inductively as

$$[\![0]\!]_C(\vec{f}) = s \mapsto 0 \qquad [\![1]\!]_C(\vec{f}) = s \mapsto 1 \qquad [\![X_i]\!]_C(\vec{f}) = f_i$$

$$[\![r\phi]\!]_C(\vec{f}) = r \cdot \left([\![\phi]\!]_C(\vec{f})\right) \qquad [\![\phi_1 + \phi_2]\!]_C(\vec{f}) = [\![\phi_1]\!]_C(\vec{f}) + [\![\phi_2]\!]_C(\vec{f})$$

$$[\![\phi_1 \sqcup \phi_2]\!]_C(\vec{f}) = \max\{[\![\phi_1]\!]_C(\vec{f}), [\![\phi_2]\!]_C(\vec{f})\} \quad [\![\phi_1 \sqcap \phi_2]\!]_C(\vec{f}) = \min\{[\![\phi_1]\!]_C(\vec{f}), [\![\phi_2]\!]_C(\vec{f})\}$$

$$[\![\Diamond \phi]\!]_C(\vec{f}) = \sum_{j=1}^{n} \left(\tau_M(s_i)(s_j) \cdot [\![\phi]\!]_C(\vec{f})\right)$$

$$[\![\mu X.\phi(X)]\!]_C(\vec{f}) = \mathrm{lfp}(g \mapsto [\![\phi]\!]_C(\vec{f}, g)) \quad [\![\nu X.\phi(X)]\!]_C(\vec{f}) = \mathrm{gfp}(g \mapsto [\![\phi]\!]_C(\vec{f}, g))$$

where $\mathrm{lfp}(F)$ is the least fixed–point of the function F and $\mathrm{gfp}(F)$ is the greatest fixed–point of the function F. We then extend the semantics of Riesz modal logic terms with

$$[\![\mu X.\phi(X)]\!]_C = \mathrm{lfp}([\![\phi]\!]_c) \quad [\![\nu X.\phi(X)]\!]_C = \mathrm{gfp}([\![\phi]\!]_C)$$

Remark 10. To ensure that the fixed–points always exist, some considerations must be taken when defining the grammar of ϕ (e.g., restricting the variable X to only appear in positive occurrences). Since we are interested in probabilistic properties, we will only consider fixed–point expressions to monotone functions from $[0,1]^S$ to $[0,1]^S$ in the discussion below.

Thus we define

$$\begin{aligned} P_{NT} &= \nu X.\Diamond X \sqcup 0 \sqcap 1 \\ P_T &= \mu X.((1 - \Diamond 1) + \Diamond X) \sqcup 0 \sqcap 1 \end{aligned}$$

and we will show that $[\![P_{NT}]\!]_C$ is the function that maps a state s to the probability that the Markov chain does not halt starting from s, and $[\![P_T]\!]_C$ is the function that maps a state s to the probability that the Markov chain will eventually halt starting from s.

Lemma 21. *In modal Riesz logic extended with fixed–point, for all Markov chain C, $[\![P_{NT}]\!]_C$ is the function that maps a state s to the probability that the Markov chain does not halt starting from s.*

Proof. Let $C = (S,\tau)$ be a Markov chain. We have shown that the formula $\Diamond^n 1$ computes the probability to do at least n steps. Thus

$$\lim_{n\to+\infty} [\![\Diamond^n 1]\!]_C$$

maps a state s to the probability that the Markov chain does an infinite number of steps starting from s and therefore never halts. Thus we want to show that $[\![P_{NT}]\!]_C = \lim_{n\to+\infty} [\![\Diamond^n 1]\!]_C$.

First note that the limit of the sequence $[\![\Diamond^n 1]\!]_C$ does exist for every Markov chain. Indeed, we define the operator $\Diamond_C : \mathbb{R}^S \to \mathbb{R}^S$ by

$$\Diamond_C(f)(s_i) = \sum_{s_j\in S} \tau_M(s_i)(s_j)\dot{f}(s_j)$$

Then $[\![\Diamond^n 1]\!]_C = \Diamond^n_C(\underline{1})$ where $\underline{1}$ is the constant function equal to 1. By definition of $\Diamond_C$, we have

$$\begin{array}{lll} \Diamond_C(\underline{1}) & \leq & \underline{1} \\ \Diamond_C(f) & \leq & \Diamond_C(g) \quad \text{whenever } f \leq g \end{array}$$

Thus the sequence $(\Diamond^n_C(\underline{1}))_n$ is decreasing and bounded below by the function constant equal to 0, and therefore has a limit.

We will now show that $\lim_{n\to+\infty} [\![\Diamond^n 1]\!]_C$ is the greatest fixed–point of the operator $[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C$.

First, we show that $\lim_{n\to+\infty} \Diamond^n_C(\underline{1}) = [\![\Diamond X \sqcup 0 \sqcap 1]\!]_C(\lim_{n\to+\infty} \Diamond^n_C(\underline{1}))$. Indeed for all state $s_i \in S$, we have

$$[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C(\lim_{n\to+\infty} \Diamond^n_C(\underline{1})) = \Diamond_C(\lim_{n\to+\infty} \Diamond^n_C(\underline{1}))$$

since $\lim_{n\to+\infty} \Diamond^n_C(\underline{1}) \in [0,1]^S$. Also,

$$\begin{array}{lll} \Diamond_C(\lim_{n\to+\infty} \Diamond^n_C(\underline{1}))(s_i) & = & \sum_{s_j\in S} \tau_M(s_i)(s_j) \lim_{n\to+\infty} \Diamond^n_C(\underline{1})(s_j) \\ & = & \lim_{n\to+\infty} \sum_{s_j\in S} \tau_M(s_i)(s_j)\Diamond^n_C(\underline{1})(s_j) \\ & = & \lim_{n\to+\infty} \Diamond^{n+1}_C(\underline{1})(s_i) \\ & = & \lim_{n\to+\infty} \Diamond^n_C(\underline{1})(s_i) \end{array}$$

Therefore $[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C(\lim_{n\to+\infty} \Diamond^n_C(\underline{1})) = \lim_{n\to+\infty} \Diamond^n_C(\underline{1})$.

Then, we will show that $\lim_{n\to+\infty} \Diamond^n_C(\underline{1})$ is the greatest fixed–point. Let f be a fixed–point of $[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C$. Then $f = [\![\Diamond X \sqcup 0 \sqcap 1]\!]_C(f) \in [0,1]^S$ and thus $f \leq \underline{1}$. Since $f \in [0,1]^S$ then $[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C(f) = \Diamond_C(f)$. Thus, we can show by a straightforward induction that for all n,

$$f \leq \Diamond^n_C(\underline{1})$$

Indeed, if $f \leq \Diamond^n_C(\underline{1})$ for some n then $f = \Diamond_C(f) \leq \Diamond^{n+1}_C(f)$. Since $f \leq \Diamond^n_C(\underline{1})$ for all n, we have

$$f \leq \lim_{n\to+\infty} \Diamond^n_C(\underline{1})$$

Therefore $\lim_{n\to+\infty} \Diamond^n_C(\underline{1})$ is indeed the greatest–fixed point of $[\![\Diamond X \sqcup 0 \sqcap 1]\!]_C$.

Thus we have

$$\begin{aligned}[\![P_{NT}]\!]_C &= \text{gfp}([\![\Diamond X \sqcup 0 \sqcap 1]\!]_C) \\ &= \lim_{n\to+\infty} \Diamond_C^n(\underline{1})\end{aligned}$$

□

Lemma 22. *In modal Riesz logic extended with fixed–point, for all Markov chain C, $[\![P_T]\!]_C$ is the function that maps a state s to the probability that the Markov chain will eventually halt starting from s.*

Proof. If we restrict the set of function to $[0,1]^S$, we have shown that $[\![P_{NT}]\!]_C = \text{gfp}(\Diamond_C)$. Moreover $1 - \text{gfp}(\phi) = \text{lfp}(f \mapsto 1 - \phi(1-f))$ for all $\phi : [0,1]^S \to [0,1]^S$ (see Proposition 1.2.25 of [NA01][§1.2] for a proof). Thus

$$\begin{aligned}1 - [\![P_{NT}]\!]_C &= \text{lfp}([\![1 - \Diamond(1 - X)]\!]_C) \\ &= \text{lfp}([\![1 - \Diamond 1 + \Diamond X]\!]_C) \\ &= [\![P_T]\!]_C\end{aligned}$$

We can then conclude since $[\![P_{NT}]\!]_C$ is the function that maps a state s to the probability that the Markov chain will never halt starting from s. □

Most of the interesting properties of Markov chains can be expressed using Riesz modal logic and fixed–point operators. Indeed, the full logic pCTL can be encoded in Riesz modal logic with fixed–point operators (see [Mio12, MS13, Mio18]). One of the first steps to do so is to simulate a Boolean logic in Riesz modal logic. To do so, we can define threshold operators $\mathbb{T}_{*p}$ for $* \in \{\leq, <, \geq, >\}$ and $p \in [0,1]$ such that

$$[\![\mathbb{T}_{*p}(A)]\!]_C(s_i) = \begin{cases} 1 & \text{if } [\![A]\!]_C(s_i) * p \\ 0 & \text{otherwise} \end{cases}$$

and thus $[\![\mathbb{T}_{*p}(A)]\!]_C$ is always a *Boolean* function (i.e., $\{0,1\}$–valued). Such threshold operators can be defined using fixed–point (see [Mio18] for details).

1.6 Hypersequent calculus GA

Hypersequent calculus is a generalisation of Gentzen's sequent calculus introduced by A. Avron [Avr87], and independently by G. Pottinger [Pot83]. In hypersequent calculus, instead of having only one sequent, the proof system is used to derive a *multiset* of sequents. This modification significantly increases the expressiveness of Gentzen's calculus as it allows additional transfers of information between the sequents.

Hypersequent calculus was introduced to provide a cut–free formalisation of many nonclassical logics including modal, relevant, multi–valued and fuzzy logics. In particular, replacing ordinary sequents with hypersequents made possible obtaining different CUT–free systems for S5 ([Avr87, Pot83]).

In [MOG05], G. Metcalfe, N. Olivetti and D. M. Gabbay introduced a hypersequent calculus for the theory of Abelian l–groups called **GA**. The system **GA** is the basis on which we will build the other hypersequent calculi introduced in this thesis. We will give the formal definition of this system.

Definition 1.6.1. A *term* is a formal expression A where A is an Abelian l–group term.

We use the greek letters $\Gamma, \Delta, \Theta, \Sigma$ to range over possibly empty finite multisets of terms. We often write these multisets as lists but they should always be understood as being taken modulo reordering of their elements. As usual, we write Γ, Δ for the concatenation of Γ and Δ.

Definition 1.6.2. A *sequent* is a formal expression of the form $\Gamma \vdash \Delta$.

If $\Gamma = \emptyset$ and $\Delta = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 1.6.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \dots | \vdash \Gamma_n$.

We use the letter G, H to range over hypersequents.

We now describe how sequents and hypersequents can be interpreted by Abelian l–group terms. This means that **GA**, like all the systems introduced in this thesis, is a *structural proof system*, i.e., by manipulating sequents and hypersequents it in fact deals with terms of a certain specific form.

Definition 1.6.4 (Interpretation)**.** We interpret sequents $\Gamma \vdash \Delta$ and hypersequents G as the Abelian l–group terms $(\!|\Gamma \vdash \Delta|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Sequents	$A_1, \dots, A_n \vdash B_1, \dots, B_m$	$(B_1 + \dots + B_m) - (A_1 + \dots + A_n)$
Hypersequents	$\Gamma_1 \vdash \Delta_1 | \dots | \Gamma_n \vdash \Delta_n$	$(\!|\Gamma_1 \vdash \Delta_1|\!) \sqcup \dots \sqcup (\!|\Gamma_1 \vdash \Delta_n|\!)$

Hence a sequent is simply interpreted as sum ($\sum$) and a hypersequent is interpreted as a join of sums ($\bigsqcup \sum$).

Example 14. The interpretation of the hypersequent:

$$x - y \vdash x, (y \sqcap z) \mid \vdash ((-x) \sqcap y)$$

is the Abelian l–group term:

$$\big(x + (y \sqcap z) - (x - y)\big) \sqcup \big(((-x) \sqcap y)\big).$$

The hypersequent calculus **GA**, is a deductive system for deriving hypersequents whose interpretation is positive, i.e., the hypersequents G such that $\mathcal{A}_{\text{l-groups}} \vdash 0 \leq (\!|G|\!)$. The rules of **GA** are presented in Figure 1.4.

In what follows we say that an hypersequent G has a *CAN–free derivation* (resp., *M–free*, *etc.*) if it has a derivation that never uses the CAN can (resp., M rule, *etc.*).

Definition 1.6.5 (Elimination theorem)**.** For a rule R, the R *elimination theorem* states that if a hypersequent has a derivation, then it has a R–free derivation.

Remark 11. Note that the following CUT rule

$$\frac{G \mid \Gamma_1, A \vdash \Delta_1 \quad G \mid \Gamma_2 \vdash \Delta_2, A}{G \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ CUT}$$

is equivalent (i.e., mutually derivability) to the CAN rule in the **GA** hypersequent calculus:

Axioms:

$$\frac{}{\vdash}\ \Delta\text{–ax} \qquad \frac{}{A \vdash A}\ \text{ID–ax}$$

Structural rules:

$$\frac{G}{G|\Gamma \vdash \Delta}\ \text{Weakening (W)} \qquad \frac{G|\Gamma \vdash \Delta|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta}\ \text{Contraction (C)}$$

$$\frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2}\ \text{Split (S)} \qquad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{Mix (M)}$$

Logical rules:

$$\frac{G|\Gamma \vdash \Delta}{G|\Gamma, 0 \vdash \Delta}\ 0_L \qquad \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta, 0}\ 0_R$$

$$\frac{G|\Gamma, A, B \vdash \Delta}{G|\Gamma, A + B \vdash \Delta}\ +_L \qquad \frac{G|\Gamma \vdash \Delta, A, B}{G|\Gamma \vdash \Delta, A + B}\ +_R$$

$$\frac{G|\Gamma \vdash \Delta, A}{G|\Gamma, -A \vdash \Delta}\ -_L \qquad \frac{G|\Gamma, A \vdash \Delta}{G|\Gamma \vdash \Delta, -A}\ -_R$$

$$\frac{G|\Gamma, A \vdash \Delta \quad G|\Gamma, B \vdash \Delta}{G|\Gamma, A \sqcup B \vdash \Delta}\ \sqcup_L \qquad \frac{G|\Gamma \vdash \Delta, A|\Gamma \vdash \Delta, B}{G|\Gamma \vdash \Delta, A \sqcup B}\ \sqcup_R$$

$$\frac{G|\Gamma, A \vdash \Delta|\Gamma, B \vdash \Delta}{G|\Gamma, A \sqcap B \vdash \Delta}\ \sqcap_L \qquad \frac{G|\Gamma \vdash \Delta, A \quad G|\Gamma \vdash \Delta, B}{G|\Gamma \vdash \Delta, A \sqcap B}\ \sqcap_R$$

CAN rule:

$$\frac{G \mid \Gamma, A \vdash \Delta, A}{G \mid \Gamma \vdash \Delta}\ \text{CAN}$$

Figure 1.4: Inference rules of the hypersequent system GA of [MOG05].

$$\cfrac{\cfrac{G \mid \ \Gamma_1, A \vdash \Delta_1 \quad G \mid \ \Gamma_2 \vdash \Delta_2, A}{G \mid \ \Gamma_1, \Gamma_2, A \vdash \Delta_1, \Delta_2, A}\ \text{M}}{G \mid \ \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \text{CAN}$$

Figure 1.5: Derivability of the CUT rule.

$$\cfrac{\cfrac{\cfrac{G \mid \ \Gamma, A \vdash \Delta, A}{G \mid \ \Gamma, A, -A \vdash \Delta}\ -_L}{G \mid \ \Gamma, (A - A) \vdash \Delta}\ +_L \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\ \text{ID-ax}}{\vdash A, -A}\ -_R}{\vdash (A - A)}\ +_R}{G \mid \vdash (A - A)}\ \text{W}^*}{G \mid \ \Gamma \vdash \Delta}\ \text{CUT}$$

Figure 1.6: Derivability of the CAN rule.

Thus, instead of the traditional CUT elimination theorem, the main result of interest concerning the system **GA** is the CAN elimination theorem.

The main results concerning the system **GA** are proved in [MOG05, MOG09], namely the soundness and completeness of the calculus with regard to the equational theory of Abelian l–groups, as well as the CAN–elimination theorem. Those results can be summarised as

For all hypersequent G, $\mathcal{A}_{\text{l–groups}} \vdash 0 \leq (\!|G|\!)$
if and only if
G has a CAN–free derivation.

Notice that the system **GA** is used to derive inequality instead of equality. However, Lemma 1 shows that a system for deriving inequalities can be used to derive equalities. Thus the system **GA** is expressive enough to reason on the equational theory of Abelian l–groups.

Chapter 2

Hypersequent calculi

In this chapter, we build upon the hypersequent calculus **GA** introduced in [MOG05, MOG09] to achieve a system for modal Riesz spaces presented in Section 1.2. The system **GA** was built to reason on Abelian l–groups. Thus we will add rules for the scalar multiplication as well as for the $\Diamond$ operator and the 1 constant to the system **GA** to build a system for modal Riesz spaces.

Before adding the new operations, we will implement a few changes to the system **GA** for it to be more convenient when we will add the $\Diamond$ operator. Indeed, the proof of the CAN elimination for **GA** of [MOG09] is not easily adapted to deal with the $\Diamond$ operator. To recall, they first show how to remove one instance of the CAN rule. Thus, they first prove that if $G \mid \Gamma, A \vdash \Delta, A$ has a CAN–free derivation, then so does $G \mid \Gamma \vdash \Delta$. To do so, they proceed in two steps:

- they first show the *atomic* CAN elimination, i.e., if

  $$G \mid \Gamma, x \vdash \Delta, x$$

  is derivable, then so is

  $$G \mid \Gamma \vdash \Delta$$

- and then they use the CAN–free invertibility of the logical rules, i.e., the fact that if the conclusion of a logical rule has a CAN–free derivation, then so do the premises, to reduce the problem to the atomic CAN elimination.

For instance, to show that if $G \mid \Gamma, x+y \vdash \Delta, x+y$ is CAN–free derivable, then so is $G \mid \Gamma \vdash \Delta$, they first show that $G \mid \Gamma, x, y \vdash \Delta, x, y$ is CAN–free derivable using the invertibility of the $+$ rule and they conclude with the atomic CAN elimination.

However, the invertibility of the $\Diamond$ rule can not be used in the second step. As we will see, the $\Diamond$ rule

$$\frac{\Gamma \vdash \Delta, 1^n}{\Diamond\Gamma \vdash \Diamond\Delta, 1^n}\ \Diamond$$

has very strong constraints for its shape. One of those constraints is that the $\Diamond$ rule can be used only if the hypersequent has only one sequent. The other is that the terms appearing in the sequent must be either $\Diamond$ terms or 1. Thus, we can not use the invertibility of the $\Diamond$ rule to reduce the complexity of the CAN term. For instance, even if $x \vdash y \mid \Gamma, \Diamond x \vdash \Delta, \Diamond x$ has a CAN–free derivation, we can not use the invertibility of the $\Diamond$ rule to show that $x \vdash y \mid \Gamma, x \vdash \Delta, x$ has a CAN–free derivation since $x \vdash y \mid \Gamma, \Diamond x \vdash \Delta, \Diamond x$ is not a valid instance of the conclusion of the $\Diamond$ rule.

To address this limitation, it is possible to deal with the case of CAN terms being $\Diamond$ terms in a different way, by induction on the structure of the derivation (in the style of the classic inductive proof techniques for eliminating CUT applications in sequent calculi, see, e.g., [Bus98]). In this inductive proof, however, there is a critically difficult case when the derivation ends with an M rule, since the M rule can break the symmetry of the CAN rule. For instance, we do not know how to deal with the following instance of the M rule:

$$\frac{G \mid \ \Gamma_1, \Diamond A \vdash \Delta_1 \quad G \mid \ \Gamma_2 \vdash \Delta_2, \Diamond A}{G \mid \ \Gamma_1, \Gamma_2, \Diamond A \vdash \Delta_1, \Delta_2, \Diamond A} \text{ M}$$

since we can not use the induction hypothesis on the two premises (because $\Diamond A$ does not appear on the two sides of the $\vdash$).

Therefore, it would be useful to prove a M elimination theorem to get rid of this difficult case. However, one can easily notice that the system **GA** does not satisfy the M elimination theorem, since the hypersequent $x, y \vdash x, y$ is derivable as follows

$$\frac{\dfrac{}{x \vdash x} \text{ ID–ax} \quad \dfrac{}{y \vdash y} \text{ ID–ax}}{x, y \vdash x, y} \text{ M}$$

but does not have a M–free derivation. We implement three changes to the system **GA** for it to satisfy the M elimination theorem.

The first one is to go from two–sided hypersequent calculus to one–sided hypersequent calculus, thus removing half of the logical rules. Since most of our proofs are done by induction on the derivations, removing half of the logical rules effectively removes quite a few cases in the proofs. Even though this change is not necessary for the M elimination theorem, it makes most of the proofs shorter and more simple, so this change is quite convenient.

More importantly, we replace the ID axiom with the following ID rule

$$\frac{G \mid \ \Gamma \vdash \Delta}{G \mid \ \Gamma, x \vdash \Delta, x} \text{ ID}$$

which makes, e.g., the hypersequent $x, y \vdash x, y$ mentioned above M–free derivable.

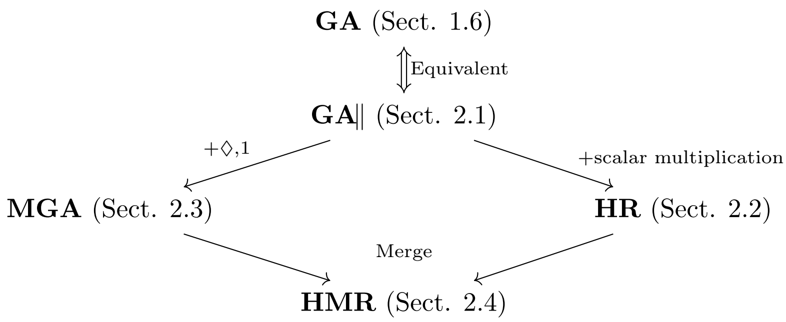
Lastly, we allow the logical rule to act on several instances of a term in a sequent. For instance, we will now allow the following rule

$$\frac{G \mid \ \Gamma, A, B, A, B \vdash \Delta}{G \mid \ \Gamma, A + B, A + B \vdash \Delta} +$$

where we apply the $+$ rule to two instances of the formula $A + B$. This is also a crucial change as we do not know how to prove the M elimination theorem without this flexibility in the logical rules.

Because of the novelty and the significant usefulness of the last change, where we allow to use a rule *in parallel* on several instances of a term, we call the system where we implement these three changes on top of **GA** as **GA**$\|$. Then we add the rules for the new operators to the system **GA**$\|$. Adding the scalar multiplication requires changing the structure of the hypersequents we manipulate - we no longer have regular sequents like in **GA** but *weighted* sequents, i.e. sequents where formulas have weights. It is thus convenient to add scalar multiplication separately from the 1 constant and the $\Diamond$ operator. Thus we introduce two systems, the system **HR** where we add the scalar multiplication the the system **GA**$\|$ and the system **MGA** where we add both the $\Diamond$ operator and the 1 constant.

We then merge the two systems into the system **HMR** that implements both the scalar multiplication as well as the $\Diamond$ operator and the 1 constant.



We will build each hypersequent calculus in the same way so the following sections will have the same structure. Thus for $X \in \{1, 2, 3, 4\}$, Section 2.X has the following structure

- We start by recalling the basic definitions concerning the hypersequent calculus we are considering in the introduction of Section 2.X
- In Section 2.X.1, we state some technical lemmas which are necessary for the main results.
- And then we prove the main results mentioned above concerning the system, namely
  - in Section 2.X.2, we show the soundness of the system,
  - in Section 2.X.3, we show the completeness,
  - in Section 2.X.4, we show the CAN–free invertibility of the logical rules,
  - in Section 2.X.5, we show the M elimination theorem,
  - in Section 2.X.6, we show the CAN elimination theorem,
  - and finally in Section 2.X.7, we show a result we call the *algebraic property*.

2.1 Hypersequent calculus GA||

In this section, we introduce the hypersequent calculus **GA**||. As mentioned earlier, **GA**|| is equivalent to the system **GA** but is closer to the systems we will introduce later on and thus this section will allow the readers to get used to the new features of our systems, such as the rules acting on several instances of a formula. Those new features are important for proving some of our results, for instance we do not know yet how to prove the results of Section 2.1.5 if the rules can only act on one formula (see Remark 18).

We start with a sequence of syntactical definitions and notational conventions necessary to present the rules of the system. We use the letters A, B, C to range over Abelian l–group terms in negation normal form (NNF, see Definition 1.1.3) built from a countable set of variables x, y, z and negated variables $\overline{x}$, $\overline{y}$, $\overline{z}$.

Definition 2.1.1. A *term* is a formal expression A where A is an Abelian l–group term in NNF.

We use the greek letters $\Gamma, \Delta, \Theta, \Sigma$ to range over possibly empty finite multisets of terms. We often write these multisets as lists but they should always be understood as being taken modulo reordering of their elements. As usual, we write Γ, Δ for the concatenation of Γ and Δ. We use the notation A^n for the multiset $A, ..., A$ consisting of n copies of A, and the notation Γ^n for the multiset $\Gamma, ..., \Gamma$ consisting of the concatenation of n copies of Γ.

Definition 2.1.2. A *sequent* is a formal expression of the form $\vdash \Gamma$.

If $\Gamma = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 2.1.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \dots | \vdash \Gamma_n$.

We use the letter G, H to range over hypersequents. Note that, under these notational conventions, the expression $\vdash \Gamma$ could either denote the sequent $\vdash \Gamma$ itself or the hypersequent $[\vdash \Gamma]$ containing only one sequent. The context will always determine which of these two interpretations is intended.

We introduce a notion of "simple" hypersequent which arises as the basic case in some of the proofs since one of the inductive schemes used to reason on the system **GA**$\|$ is to reduce the complexity of the formulas appearing in the hypersequents until only atoms remain.

Definition 2.1.4. A hypersequent is said *atomic* if it only contains atoms, i.e., formulas of the form x or $\overline{x}$.

We now describe how sequents and hypersequents can be interpreted by Abelian l–group terms. This means that **GA**$\|$, like all the systems introduced in this thesis, is a *structural proof system*, i.e., by manipulating sequents and hypersequents it in fact deals with terms of a certain specific form.

Definition 2.1.5 (Interpretation)**.** We interpret sequents $\vdash \Gamma$ and hypersequents G as the Abelian l–group terms $(\!|\vdash \Gamma|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Sequents	$\vdash A_1, \dots, A_n$	$A_1 + \dots + A_n$
Hypersequents	$\vdash \Gamma_1 | \dots | \vdash \Gamma_n$	$(\!|\vdash \Gamma_1|\!) \sqcup \dots \sqcup (\!|\vdash \Gamma_n|\!)$

Hence a sequent is simply interpreted as sum ($\sum$) and a hypersequent is interpreted as a join of sums ($\bigsqcup \sum$).

Example 15. The interpretation of the hypersequent:

$$\vdash x, (y \sqcap z) \;|\; \vdash (\overline{x} \sqcap y)$$

is the l-group term:

$$\big(x + (y \sqcap z)\big) \sqcup \big((\overline{x} \sqcap y)\big).$$

The hypersequent calculus **GA**$\|$, as for the system **GA**, is a deductive system for deriving hypersequents whose interpretation is positive, i.e., the hypersequents G such that $\mathcal{A}_{\text{l-groups}} \vdash 0 \leq (\!|G|\!)$. The rules of **GA**$\|$ are presented in Figure 2.1 and are very similar to the rules of the system **GA** of [MOG05, MOG09] (see Figure 1.4) where the main difference is that the rules act on several instances of a term in a sequent. We write $\rhd_{\mathbf{GA}\|} G$ if the hypersequent G is derivable in the system **GA**$\|$.

The axiom INIT allows for the derivation of ($\vdash$), the hypersequent containing only the empty sequent, thus it corresponds to the positivity of the constant 0. The C rule (contraction) allows treating hypersequents as (always non–empty) sets of sequents. The M (mix) and S (split) rules are as in the system GA of [MOG05, MOG09] (see Section 1.6). As mentioned earlier, we instead adopted the rule ID, in place of the axiom ID-ax of **GA**. While the two are equivalent (i.e., mutually derivable) in presence of the other rules, the formulation of ID as a rule is necessary in the statement of the M elimination theorem later on. Finally, note that the logical rules are

Axiom:

$$\frac{}{\vdash}\ \text{INIT}$$

Structural rules:

$$\frac{G}{G \mid \vdash \Gamma}\ \text{W} \qquad \frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ \text{C}$$

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S} \qquad \frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{M}$$

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, x^n, \overline{x}^n}\ \text{ID}$$

Logical rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, 0^n}\ 0 \qquad \frac{G \mid \vdash \Gamma, A^n, B^n}{G \mid \vdash \Gamma, (A+B)^n}\ +$$

$$\frac{G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcup B)^n}\ \sqcup \qquad \frac{G \mid \vdash \Gamma, A^n \quad G \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcap B)^n}\ \sqcap$$

CAN rule:

$$\frac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma}\ \text{CAN}$$

Figure 2.1: Inference rules of **GA**$\|$.

all presented using the syntactic sugaring A^n described above. For example, one valid instance of the rule (+) is the following:

$$\frac{\vdash \Gamma, y, x, y, x}{\vdash \Gamma, (y+x), (y+x)}\ +$$

This effectively allows us to apply the rule to several terms in the sequent at the same time. This feature adds some flexibility in the process of derivation construction and is necessary for some proofs, such as the M elimination theorem.

Note 1. We often have to use the same rule multiple times when building a derivation. For convenience, we may write the rule only once with the number of times the rule is used as exponent, as follows:

$$\frac{G}{G \mid \vdash \Gamma \mid \vdash \Delta}\ W^2$$

If the number of times a rule is used is not known, we use a wildcard as exponent, as in the following example where the weakening rule is used to remove all sequents appearing in G:

$$\frac{\vdash \Gamma}{G \mid \vdash \Gamma}\ W^*$$

Remark 12. On the one hand, we could have introduced appropriate exchange (i.e., reordering) rules and defined sequents and hypersequents as lists, rather than multisets. In the opposite direction, we could have defined hypersequents as (non–empty) sets and dispose of the rules (C). Our choice is motivated by a balance between readability and fine control over the derivation steps in the proofs.

Remark 13. Note that the following CUT rule

$$\dfrac{G \mid \vdash \Gamma_1, A^n \quad G \mid \vdash \Gamma_2, \overline{A}^n}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CUT}$$

is, as in **GA**, equivalent to the CAN rule in the **GA**‖ hypersequent calculus:

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, A^n \quad G \mid \vdash \Gamma_2, \overline{A}^n}{G \mid \vdash \Gamma_1, \Gamma_2, A^n, \overline{A}^n}\ \text{M}}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CAN}$$

Figure 2.2: Derivability of the CUT rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma, (A + \overline{A})^n}\ + \quad \dfrac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma, (\overline{A} + A)^n}\ +}{G \mid \vdash \Gamma, \Gamma}\ \text{CUT}}{G \mid \vdash \Gamma \mid \vdash \Gamma}\ \text{S}}{G \mid \vdash \Gamma}\ \text{C}$$

Figure 2.3: Derivability of the CAN rule.

Our choice (following [MOG09, MOG05]) of presenting the system **GA**‖ using the CAN rule, rather than the equivalent CUT rule, is just motivated by elegance and technical convenience.

Example 16. Example of derivation of the hypersequent $\vdash \big((x + \overline{y}) \sqcup (y + \overline{x})\big)$ which consists of only one sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash y, \overline{y}}\ \text{ID}}{\vdash x, y, \overline{x}, \overline{y}}\ \text{ID}}{\vdash x, \overline{y} \mid \vdash y, \overline{x}}\ \text{S}}{\vdash x, \overline{y} \mid \vdash (y + \overline{x})}\ +}{\vdash (x + \overline{y}) \mid \vdash (y + \overline{x})}\ +}{\vdash \big((x + \overline{y}) \sqcup (y + \overline{x})\big)}\ \sqcup$$

We now define an appropriate notion of complexity of a hypersequent. One natural way to reason on hypersequents is to use the logical rules to reduce the "complexity" of the hypersequent and to do so, we have to define the complexity of a hypersequent in such a way that it decreases when applying the logical rules. Unfortunately, we can not simply use the number of operators for the complexity of a hypersequent since the application of the $\sqcup$ rule

$$\dfrac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.A \sqcup B}\ \sqcup$$

may increase the number of operators in the hypersequent because of the duplicated Γ. Therefore we use the following definition for the complexity of a hypersequent.

Definition 2.1.6. We define the complexity of a sequent $\vdash \Gamma$, noted $c(\vdash \Gamma)$, as the sum of the operators used in the terms of Γ,i.e., if $\Gamma = A_1, ..., A_n$, $c(\vdash \Gamma) = \sum_{i=1}^{n} c(A_i)$ (see Definition 1.1.5 for the definition of $c(A_i)$).

The complexity of a hypersequent G, noted $c(G)$, is then defined as the pair $c(G) = (a, b)$ where

- a is the maximum complexity of a sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $a = \max\limits_{i \in [1..n]} c(\vdash \Gamma_i)$, and
- b is the number of sequents in G having a complexity of a, i.e., $b = \#\{\vdash \Gamma_i \mid c(\vdash \Gamma_i) = a\}$.

We say that a sequent $\vdash \Gamma$ of G is *maximal* if $c(\vdash \Gamma) = a$.

The complexities of hypersequents is then ordered using the lexicographic order, meaning that

$$(a, b) < (c, d) \text{ if and only if } \begin{cases} a < c & \text{or} \\ a = c \text{ and } b < d \end{cases}$$

Remark 14. The premises of a logical rule acting on a *maximal sequent* have a strictly lower complexity than the conclusion of the logical rule with regard to the lexicographic order.

For instance, for the $\sqcup$ rule

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.A \sqcup B} \sqcup$$

with $c(G \mid \vdash \Gamma, \vec{r}.A \sqcup B) = (a, b)$ and $c(\vdash \Gamma, \vec{r}.A \sqcup B) = a$. Then $c(\vdash \Gamma, \vec{r}.A) < a$ and $c(\vdash \Gamma, \vec{r}.B) < a$ and so we have two possibilities:

- either $b > 1$ and $c(G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B) = (a, b-1)$,
- or $b = 1$ and $c(G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B) = (a', b')$ for some a' and b' such that $a' < a$.

In both cases, the complexity of the hypersequent has decreased.

Remark 15. Since the lexicographic order is well-founded, together with Remark 14, we can ensure that the process of applying the logical rules to maximal sequents until we reach an atomic hypersequent always finishes.

2.1.1 Preliminary lemmas

Before embarking on the proofs of the main theorems, we prove in this section a few useful routine lemmas that will be used often.

Our first lemma states that the following variant of the ID rule (see Figure 2.1) where general terms A are considered rather than just variables, is admissible in the proof system **GA**$\|$.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, A^n, \overline{A}^n} \text{ ID}$$

Lemma 23. *For all terms A and natural numbers n*

$$\textit{if } \rhd_{\mathbf{GA}\|} G \mid \vdash \Gamma \textit{ then } \rhd_{\mathbf{GA}\|} G \mid \vdash \Gamma, A^n, \overline{A}^n$$

Proof. We prove the result by induction on A.

- If A is a variable, we simply use the ID rule.

- If $A = 0$, we use the 0 rule.
- If $A = B + C$, we use the $+$ rule twice (for $A + B$ and $\overline{A} + \overline{B}$) and conclude with the induction hypothesis.
- For the case $A = B \sqcap C$ or $A = B \sqcup C$, we first use the $\sqcap$ rule and then the $\sqcup$ rule on each premise and the W rule on each premise to remove the sequents with both B and C in them. We can then conclude with the induction hypothesis.

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, B^n, \overline{B}^n}{G \mid \vdash \Gamma, B^n, \overline{B}^n \mid \vdash \Gamma, B^n, \overline{C}^n}\,\text{W}}{G \mid \vdash \Gamma, B^n, (\overline{B} \sqcup \overline{C})^n}\,\sqcup \qquad \dfrac{\dfrac{G \mid \vdash \Gamma, C^n, \overline{C}^n}{G \mid \vdash \Gamma, C^n, \overline{B}^n \mid \vdash \Gamma, C^n, \overline{C}^n}\,\text{W}}{G \mid \vdash \Gamma, C^n, (\overline{B} \sqcup \overline{C})^n}\,\sqcup}{G \mid \vdash \Gamma, (B \sqcap C)^n, (\overline{B} \sqcup \overline{C})^n}\,\sqcap$$

□

The next result states that derivability in the **GA**$\|$ system is preserved by substitution of terms for variables.

Lemma 24. *For all hypersequents G and terms A, if $\triangleright_{\mathbf{GA}\|} G$ then $\triangleright_{\mathbf{GA}\|} G[A/x]$.*

Proof. We prove the result by induction on the derivation of G. Most cases are quite straightforward, we simply use the induction hypothesis on the premises and then use the same rule. For instance, if the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma, B^n, C^n}{G \mid \vdash \Gamma, (B + C)^n}\, +$$

by induction hypothesis $\triangleright_{\mathbf{GA}\|} G[A/x] \mid \vdash \Gamma[A/x], B[A/x]^n, C[A/x]^n$ so

$$\dfrac{G[A/x] \mid \vdash \Gamma[A/x], B[A/x]^n, C[A/x]^n}{G[A/x] \mid \vdash \Gamma[A/x], (B + C)[A/x]^n}\, +$$

The only tricky case is when the ID rule is used on the variable x, where we conclude using Lemma 23. □

The next lemma states that the logical rules are invertible using the CAN rule, meaning that if the conclusion is derivable, then the premises are also derivable. Unlike a stronger result we will prove later in Section 2.1.4 where we prove the CAN–free version of this lemma, the derivations of the premises may contain CAN rules and thus this result is not sufficient to imply the CAN elimination theorem.

Lemma 25. *All logical rules are invertible.*

Proof. We simply use the CAN rule to introduce the operators. We will show one example. The $\sqcap$ rule: we assume that $G \mid \vdash \Gamma, (A \sqcap B)^n$ is derivable. The derivation of $G \mid \vdash \Gamma, A^n$ is then:

$$\dfrac{\dfrac{G \mid \vdash \Gamma, (A \sqcap B)^n \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash A^n, \overline{A}^n}\,\text{Lemma 23}}{\vdash A^n, \overline{A}^n \mid \vdash A^n, \overline{B}^n}\,\text{W}}{\vdash A^n, (\overline{A} \sqcup \overline{B})^n}\,\sqcup}{G \mid \vdash A^n, (\overline{A} \sqcup \overline{B})^n}\,\text{W}^*}{G \mid \vdash \Gamma, A^n, (A \sqcap B)^n, (\overline{A} \sqcup \overline{B})^n}\,\text{M}}{G \mid \vdash \Gamma, A^n}\,\text{CAN}$$

The derivation of $G \mid \vdash \Gamma, B^n$ is similar. □

The next lemmas state that CAN–free derivability in the **GA**$\|$ system is preserved by multiplication by a natural number.

Lemma 26. *Let $n > 0$ be a natural number and G a hypersequent. If* $\triangleright_{\boldsymbol{GA}\|\setminus\{CAN\}} G \mid \vdash \Gamma^n$ *then* $\triangleright_{\boldsymbol{GA}\|\setminus\{CAN\}} G \mid \vdash \Gamma$.

Proof. We simply use the C and S rules :

$$\dfrac{\dfrac{G \mid \vdash \Gamma^n}{G \mid \vdash \Gamma \mid \ ... \mid \vdash \Gamma}\ \mathrm{S}^{n-1}}{G \mid \vdash \Gamma}\ \mathrm{C}^{n-1}$$

□

Lemma 27. *Let $n > 0$ be a natural number and G a hypersequent. If* $\triangleright_{\boldsymbol{GA}\|\setminus\{CAN\}} G \mid \vdash \Gamma$ *then* $\triangleright_{\boldsymbol{GA}\|\setminus\{CAN\}} G \mid \vdash \Gamma^n$.

Proof. We simply use the M rule $n - 1$ times. □

Equivalence between GA and GA$\|$

We want to emphasise that the results we prove in this section are already known and have been proved by Metcalfe, Gabbay and Olivetti in [MOG05]. The Section 2.1 must be understood as an introduction to the features of our systems (parallel rules and M elimination theorem), and not as new results.

Indeed, we will show that the system **GA** and **GA**$\|$ are equivalent in the sense that there is a one to one translation of a **GA** derivation to a **GA**$\|$ derivation, and vice–versa. Moreover, those translations preserve the CAN rule in the sense that a CAN–free derivation is sent to a CAN–free derivation. Thus all the results in this section (except the M elimination theorem) are direct corollaries of those translations.

Lemma 28. *Let G be a hypersequent. If* $\triangleright_{\boldsymbol{GA}} G$ *then* $\triangleright_{\boldsymbol{GA}\|} G$.

Proof. We prove this result by induction on the derivation of $\triangleright_{\mathbf{GA}} G$. Note that, apart from the ID axiom, every single inference rules of the system **GA** is also a valid instance of a rule of the system **GA**$\|$. Thus we only show how to translate the ID axiom.

If the derivation of $\triangleright_{\mathbf{GA}} G$ finishes with

$$\dfrac{}{\vdash A, \overline{A}}\ \text{ID–ax}$$

then the derivation of the hypersequent in **GA**$\|$ is

$$\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash A, \overline{A}}\ \text{Lemma 23}$$

□

Lemma 29. *Let G be a hypersequent. If* $\triangleright_{\boldsymbol{GA}\|} G$ *then* $\triangleright_{\boldsymbol{GA}} G$.

Proof. We prove this result by induction on the derivation of $\triangleright_{\mathbf{GA}\|} G$. Note that the INIT, S, C,W,M rules are also valid rules of the system **GA**. Thus we only show how to translate the other rules.

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, x^n, \overline{x}^n}\ \text{ID}$$

then by induction hypothesis, $\triangleright_{\mathbf{GA}} G \mid \vdash \Gamma$ and thus the derivation of the hypersequent in **GA** is

$$\frac{G \mid \vdash \Gamma \quad \dfrac{\dfrac{\dfrac{}{\vdash x, \overline{x}}\ \text{ID–ax}}{\vdash x^n, \overline{x}^n}\ \text{M}^{n-1}}{G \mid \vdash x^n, \overline{x}^n}\ \text{W}^*}{G \mid \vdash \Gamma, x^n, \overline{x}^n}\ \text{M}$$

- The $+$, 0 and CAN are similar, so we will only the case of the $+$ rule. If the derivation finishes with

$$\frac{G \mid \vdash \Gamma, A^n, B^n}{G \mid \vdash \Gamma, (A + B)^n}\ +$$

then by induction hypothesis $\triangleright_{\mathbf{GA}} G \mid \vdash Gamma, A^n, B^n$ and thus the derivation of the hypersequent in **GA** is

$$\frac{G \mid \vdash \Gamma, A^n, B^n}{G \mid \vdash \Gamma, (A + B)^n}\ +^n$$

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcup B)^n}\ \sqcup$$

then by induction hypothesis $\triangleright_{\mathbf{GA}} G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n$. We first show the case where $n = 2$ to help the reader understand the idea of the derivation before doing the general case. Therefore we show how to derive $G \mid \vdash \Gamma, (A \sqcup B)^2$ in **GA**:

$$\frac{\dfrac{G \mid \vdash, \Gamma, A, A \mid \vdash \Gamma, B, B}{G \mid \vdash \Gamma, A, A \mid \vdash \Gamma, A, B \mid \vdash \Gamma, B, A \mid \vdash \Gamma, B, B}\ \text{W}^2}{G \mid \vdash \Gamma, (A \sqcup B)^2}\ \sqcup^3$$

The general derivation is

$$\frac{\dfrac{G \mid \vdash, \Gamma, A^n \mid \vdash \Gamma, B^n}{G \mid \ \left[[\vdash \Gamma, A^i, B^j]^{\binom{n}{i}} \right]_{i+j=n}}\ \text{W}^{2^n-2}}{G \mid \vdash \Gamma, (A \sqcup B)^n}\ \sqcup^{2^n-1}$$

where $[\vdash \Gamma, A^i, B^j]^{\binom{n}{i}}$ means that the sequent $\vdash \Gamma, A^i, B^j$ appears $\binom{n}{i}$ times in the hypersequent.

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma, A^n \quad G \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcap B)^n}\ \sqcup$$

then by induction hypothesis $\triangleright_{\mathbf{GA}} G \mid \vdash \Gamma, A^n$ and $\triangleright_{\mathbf{GA}} G \mid \vdash \Gamma, B^n$. We first show the case where $n = 2$ to help the reader understand the idea of the derivation before doing the general case. Therefore we show how to derive $G \mid \vdash \Gamma, (A \sqcap B)^2$ in **GA**:

$$\dfrac{\dfrac{G \mid \vdash \Gamma, A, B}{G \mid \vdash \Gamma, A, A \sqcap B}\ \sqcap \qquad \dfrac{G \mid \vdash \Gamma, B, A \quad G \mid \vdash \Gamma, B, B}{G \mid \vdash \Gamma, B, A \sqcap B}\ \sqcap}{G \mid \vdash \Gamma, (A \sqcap B)^2}\ \sqcap$$

thus we need to derive the hypersequent $G \mid \vdash \Gamma, A, B$ which we can do with the following derivation

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, A, A \quad G \mid \vdash \Gamma, B, B}{G \mid \vdash (\Gamma, A, B)^2}\ \mathrm{M}}{G \mid \vdash \Gamma, A, B \mid \vdash \Gamma, A, B}\ \mathrm{S}}{G \mid \vdash \Gamma, A, B}\ \mathrm{C}$$

More generally, to derive $G \mid \vdash \Gamma, (A \sqcap B)^n$ in the system **GA**, we need to derive $G \mid \vdash \Gamma, A^i, B^j$ for all $i, j \in \mathbb{N}$ such that $i + j = n$. We can derive those hypersequents with the following derivation:

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, A^n}{G \mid \ (\vdash \Gamma, A^n)^i}\ \text{Lemma 27} \qquad \dfrac{G \mid \vdash \Gamma, B^n}{G \mid \ (\vdash \Gamma, B^n)^j}\ \text{Lemma 27}}{G \mid \ (\vdash \Gamma, A^i, B^j)^n}\ \mathrm{M}}{G \mid \vdash \Gamma, A^i, B^j}\ \text{Lemma 26}$$

□

Note that in the translations shown above, a CAN–free derivation is indeed sent to a CAN–free derivation since the translation of the other rules does not add any new CAN rule. Thus since those results have been proved for **GA**, we obtain automatically that

- **GA**|| is sound and complete (Theorems 32 and 39 of [MOG05]),
- the logical rules of **GA**|| are CAN–free invertible (Proposition 36 of [MOG05]),
- the CAN elimination theorem holds for **GA**|| (corollary of the previous results since the completeness is proved without the CAN rule, see the note at the end of Section 4.1 of [MOG05]), and
- what we will refer to as the "algebraic property" holds for **GA**|| (Proposition 37 o f[MOG05]).

However the proofs of these results in [MOG05] rely heavily on the fact that $\mathbb{Q}$ is complete for the equational theory of Abelian l–groups (in the same sense that $\mathbb{Z}$ is complete, see Remark 1). We do not have such a universal model for the equational theory of modal Riesz spaces, and thus we can not easily adapt the proofs of [MOG05] for the other hypersequent calculi we will introduce.

On the other hand, the proofs we present in this section are purely syntactic. This has two main advantages: this gives us concrete procedure to, for instance, eliminate the CAN rule or translate a derivation $\mathcal{A}_{\text{l–groups}} \vdash (\!|G|\!)$ in equational theory to a derivation of $\triangleright_{\mathbf{GA}||} G$. Moreover, it provides a solid foundation on which to build to extend our system.

Remark 16. Note however that Metcalfe, Gabbay and Olivetti provided a syntactical proof of the CAN elimination theorem for a class of hypersequent calculi similar to **GA** in [MOG09], proof on which we based the proof of the CAN elimination theorem provided in this thesis. Yet, this class of hypersequent calculi does not have a rule for the $\Diamond$ operator, nor have a rule for the scalar multiplication, which we add to **GA**.

2.1.2 Soundness

We need to prove that if there exists a **GA**$\|$ derivation of a hypersequent G then $(\!|G|\!) \geq 0$ is derivable in equational logic (written $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$). This is done in a straightforward way by showing that each deduction rule of the system **GA**$\|$ is sound. The desired result then follows immediately by induction on the derivation of G.

Theorem 2.1.1 (Soundness of **GA**$\|$)**.** *For all hypersequent G, if $\triangleright_{\mathbf{GA}\|} G$ then $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$.*

Proof. By induction on the derivation of G.

- For the rule

$$\frac{}{\vdash}\ \text{INIT}$$

The semantics of the hypersequent consisting only of the empty sequent is $(\!|\vdash|\!) = 0$ and therefore $(\!|\vdash|\!) \geq 0$, as desired.

- For the rule

$$\frac{G}{G \mid \vdash \Gamma}\ \text{W}$$

the hypothesis is $(\!|G|\!) \geq 0$ so

$$\begin{array}{rcl} (\!|G \mid \vdash \Gamma|\!) & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma|\!) \\ & \geq & (\!|G|\!) \\ & \geq & 0 \end{array}$$

- For the $\text{C}, \text{ID}, +, 0$ and CAN rules, it is immediate to observe that the interpretation of the only premise and the interpretation of its conclusion are equal, therefore the result is trivial.

- For the rule

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S}$$

the hypothesis is $(\!|G \mid \vdash \Gamma_1, \Gamma_2|\!) \geq 0$ so according to Lemma 6, $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- = 0$. Our goal is to prove that $(\!|G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2|\!) \geq 0$. Again, using Lemma 6, we equivalently need to prove that

$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- = 0.$$

The above expression is of the form $A^- \sqcap B^-$, and since $A^- = (-A) \sqcup 0 \geq 0$ always holds for every A, it is clear that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \geq 0$. It remains therefore to show that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \leq 0$. This is done as follows (where $2.A$ is a notation for $A + A$):

$$
\begin{array}{rcll}
(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- & \leq & (\!|G|\!)^- \sqcap 2.(\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- & \text{since } (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \geq 0 \\
& \leq & (\!|G|\!)^- \sqcap (2.((\!|\vdash \Gamma_1|\!) \sqcup (\!|\vdash \Gamma_2|\!)))^- & \text{Lemma 5[1]} \\
& \leq & (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!) + (\!|\vdash \Gamma_2|\!))^- & \text{Lemma 5[2]} \\
& = & (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1, \Gamma_2|\!))^- & \\
& = & 0 &
\end{array}
$$

- For the rule

$$\frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \mathrm{M}$$

the hypothesis is

$$(\!|G \mid \vdash \Gamma_1|\!) \geq 0$$

$$(\!|G \mid \vdash \Gamma_2|\!) \geq 0$$

so according to Lemma 6,

$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1|\!)^- = 0$$

$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_2|\!)^- = 0$$

Following the same reasoning of the previous case (S rule) our goal is to show that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- \leq 0$. This is done as follows:

$$
\begin{array}{rcll}
(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- & = & (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!) + (\!|\vdash \Gamma_2|\!))^- & \\
& \leq & (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!)^- + (\!|\vdash \Gamma_2|\!)^-) & \text{Lemma 5[3]} \\
& \leq & (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1|\!)^- + (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_2|\!)^- & \text{distributivity of } \sqcap \text{ over } +
\end{array}
$$

- For the rule

$$\frac{G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcup B)^n}\ \sqcup$$

the hypothesis is $(\!|G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n|\!) \geq 0$. So :

$$
\begin{array}{rcll}
(\!|G \mid \vdash \Gamma, (A \sqcup B)^n|\!) & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, (A \sqcup B)^n|\!) & \\
& = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, A^n|\!) \sqcup (\!|\vdash \Gamma, B^n|\!) & \text{distributivity of } \sqcup \text{ over } + \\
& \geq & 0 &
\end{array}
$$

- For the rule

$$\frac{G \mid \vdash \Gamma, A^n \quad G \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, (A \sqcap B)^n}\ \sqcap$$

the hypothesis is

$$(\!|G \mid \vdash \Gamma, A^n|\!) \geq 0$$

$$(\!|G \mid \vdash \Gamma, B^n|\!) \geq 0$$

So

$$
\begin{array}{rcll}
(\!|G \mid \vdash \Gamma, (A \sqcap B)^n|\!) & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, (A \sqcap B)^n|\!) & \\
& = & (\!|G|\!) \sqcup ((\!|\vdash \Gamma, A^n|\!) \sqcap (\!|\Gamma, B^n|\!)) & \text{distributivity of } \sqcap \text{ over } + \\
& = & ((\!|G|\!) \sqcup (\!|\vdash \Gamma, A^n|\!)) \sqcap ((\!|G|\!) \sqcup (\!|\vdash \Gamma, B^n|\!)) & \text{distributivity of } \sqcup \text{ over } \sqcap \\
& \geq & 0 &
\end{array}
$$

$\square$

2.1.3 Completeness

In order to prove the completeness of the system **GA**$\|$, i.e. that if $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ then $\triangleright_{\mathbf{GA}\|} G$, we first prove an equivalent result (Lemma 30 below) stating that if $\mathcal{A}_{\text{l-groups}} \vdash A = B$ then the hypersequents $\vdash A, \overline{B}$ and $\vdash B, \overline{A}$ are both derivable. The advantage of this formulation is that it allows for a simpler proof by induction.

From Lemma 30 one indeed obtains Theorem 2.1.2 as a corollary.

Theorem 2.1.2 (Completeness of **GA**$\|$)**.** *For all hypersequent G, if $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ then $\triangleright_{\boldsymbol{GA}\|} G$.*

Proof. Recall that $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ is a shorthand for $\mathcal{A}_{\text{l-groups}} \vdash 0 = (\!|G|\!) \sqcap 0$. Hence, from the hypothesis $\mathcal{A}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ we can deduce, by using Lemma 30, that $\triangleright_{\mathbf{GA}\|} \vdash (0 \sqcap (\!|G|\!)), 0$.

From this we can show that $\triangleright_{\mathbf{GA}\|} G$ by invoking Lemma 25. Indeed, if G is $\vdash \Gamma_1 \mid \; ... \mid \vdash \Gamma_n$ then $(\!|G|\!) = (\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)$ and

1. by using the invertibility of the 0 rule, $\vdash (0 \sqcap ((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)))$ is derivable,
2. by using the invertibility of the $\sqcap$ rule, $\vdash ((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!))$ is derivable,
3. by using the invertibility of the $\sqcup$ rule $n-1$ times, $\vdash (\!|\vdash \Gamma_1|\!) \mid \; ... \mid \vdash (\!|\vdash \Gamma_n|\!)$ is derivable,
4. and finally, by using the invertibility of the $+$ rule, $\vdash \Gamma_1 \mid \; ... \mid \vdash \Gamma_n$ is derivable.

□

Lemma 30. *If $\mathcal{A}_{\text{l-groups}} \vdash A = B$ then $\vdash A, \overline{B}$ and $\vdash B, \overline{A}$ are derivable.*

Proof. We prove this result by induction on the derivation in equational logic (see Definition 1.1.2) of $\mathcal{A}_{\text{l-groups}} \vdash A = B$.

- If the derivation finishes with

$$\dfrac{}{\mathcal{A}_{\text{l-groups}} \vdash A = A}\ \text{refl}$$

  we can conclude with Lemma 23.

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{l-groups}} \vdash B = A}{\mathcal{A}_{\text{l-groups}} \vdash A = B}\ \text{sym}$$

  then the induction hypothesis allows us to conclude.

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{l-groups}} \vdash A = C \quad \mathcal{A}_{\text{l-groups}} \vdash C = B}{\mathcal{A}_{\text{l-groups}} \vdash A = B}\ \text{trans}$$

  then the induction hypothesis is

$$\triangleright_{\mathbf{GA}\|} \vdash A, \overline{C}$$
$$\triangleright_{\mathbf{GA}\|} \vdash C, \overline{A}$$
$$\triangleright_{\mathbf{GA}\|} \vdash C, \overline{B}$$
$$\triangleright_{\mathbf{GA}\|} \vdash B, \overline{C}$$

We will show that $\triangleright_{\mathbf{GA}\|} \vdash A, \overline{B}$, the other one is similar.

$$\dfrac{\dfrac{\vdash A, \overline{C} \quad \vdash C, \overline{B}}{\vdash A, \overline{B}, C, \overline{C}}\ \mathrm{M}}{\vdash A, \overline{B}}\ \mathrm{CAN}$$

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{l-groups}} \vdash A = B}{\mathcal{A}_{\text{l-groups}} \vdash A[C/x] = B[C/x]}\ \text{subst}$$

we conclude using the induction hypothesis and Lemma 24.

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{l-groups}} \vdash A = B}{\mathcal{A}_{\text{l-groups}} \vdash C[A] = C[B]}\ \text{ctxt}$$

we prove the result by induction on C. For instance, if $C = C_1 + C_2$, then the induction hypothesis is

$$\triangleright_{\mathbf{GA}\|} \vdash C_1[A], \overline{C_1[B]}$$
$$\triangleright_{\mathbf{GA}\|} \vdash C_1[B], \overline{C_1[A]}$$
$$\triangleright_{\mathbf{GA}\|} \vdash C_2[A], \overline{C_2[B]}$$
$$\triangleright_{\mathbf{GA}\|} \vdash C_2[B], \overline{C_2[A]}$$

We then have the following derivation for $\vdash (C_1[A] + C_2[A]), (\overline{C_1[B]} + \overline{C_2[B]})$ (the other one is similar):

$$\dfrac{\dfrac{\vdash C_1[A], \overline{C_1[B]} \quad \vdash C_2[A], \overline{C_2[B]}}{\vdash C_1[A], C_2[A], \overline{C_1[B]}, \overline{C_2[B]})}\ \mathrm{M}}{\vdash (C_1[A] + C_2[A]), (\overline{C_1[B]} + \overline{C_2[B]})}\ +^*$$

- It now remains to consider the cases when the derivation finishes with one of the axioms of Figure 1.1. Since all cases are quite straightforward, we only show one.

  - If the derivation finishes with

$$\dfrac{}{\mathcal{A}_{\text{l-groups}} \vdash x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z}\ \text{ax}$$

then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash x, \overline{x}}\ \mathrm{ID}}{\vdash (x \sqcup (y \sqcup z)), \overline{x}}\ \sqcup^3 - \mathrm{W}^2 \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash y, \overline{y}}\ \mathrm{ID}}{\vdash (x \sqcup (y \sqcup z)), \overline{y}}\ \sqcup^3 - \mathrm{W}^2 \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash z, \overline{z}}\ \mathrm{ID}}{\vdash (x \sqcup (y \sqcup z)), \overline{z}}\ \sqcup^3 - \mathrm{W}^2}{\vdash (x \sqcup (y \sqcup z)), ((\overline{x} \sqcap \overline{y}) \sqcap \overline{z})}\ \sqcap^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash x, \overline{x}}\ \mathrm{ID}}{\vdash ((x \sqcup y) \sqcup z), \overline{x}}\ \sqcup^2 - \mathrm{W}^2 \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash y, \overline{y}}\ \mathrm{ID}}{\vdash ((x \sqcup y) \sqcup z), \overline{y}}\ \sqcup^2 - \mathrm{W}^2 \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash z, \overline{z}}\ \mathrm{ID}}{\vdash ((x \sqcup y) \sqcup z), \overline{z}}\ \sqcup^2 - \mathrm{W}^2}{\vdash ((x \sqcup y) \sqcup z), (\overline{x} \sqcap (\overline{y} \sqcap \overline{z}))}\ \sqcap^2$$

□

2.1.4 CAN–free invertibility

In this section, we prove that all logical rules of the system **GA**$\|$ are CAN–free invertible, meaning that if there is CAN–free derivation of the conclusion, then there are CAN–free derivations for the premises. This result will be necessary for the CAN–elimination theorem, and thus it is important that we do not add any CAN rule in the proofs of invertibility. For this reason, the CAN–free invertibility result is stronger than Lemma 25 of Section 2.1.1.

It is technically convenient, in order to carry out the inductive argument, to prove a slightly stronger result, expressed as the invertibility of more general logical rules that can act on the same term on *different* sequents of the hypersequent, at the same time. The generalised rules are the following:

Logical rules:

$$\frac{[\vdash \Gamma_i]_{i=1}^n}{[\vdash \Gamma_i, 0^{n_i}]_{i=1}^n}\ 0 \qquad \frac{[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^n}\ +$$

$$\frac{[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=1}^n}\ \sqcup \qquad \frac{[\vdash \Gamma_i, A^{n_i}]_{i=1}^n \quad [\vdash \Gamma_i, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^n}\ \sqcap$$

Figure 2.4: Generalised logical rules

Indeed, if we try to prove the invertibility of the regular rules, e.g., the + rule, it is difficult to deal with the C or $\sqcup$ rule. For instance, if the derivation of $G \mid\ \vdash \Gamma_1, (A+B)^{n_1}$ finishes with

$$\frac{G \mid\ \vdash \Gamma_1, (A+B)^{n_1} \mid\ \vdash \Gamma_1, (A+B)^{n_1}}{G \mid\ \vdash \Gamma_1, (A+B)^{n_1}}\ \text{C}$$

we need to be able to use the induction hypothesis on both sequents at the same time. Thus the need to prove the invertibility of the generalised rules. This situation where we want to prove a result concerning only one sequent by induction on the derivation will appear quite a few more times (e.g., Lemma 38 for the M elimination or Lemma 41 for the CAN elimination). Each time, the situation is resolved by proving a more general result concerning the whole hypersquent.

Remark 17. It would have been possible to define **GA**$\|$ directly using the generalised rules. However, we feel that the notation would have then been too heavy, and so we decided against this.

We conceptually divide the logical rules in three categories:

Type 1 The rule with only one premise but that adds one sequent to the hypersequent: the $\sqcup$ rule.

Type 2 The rules with only one premise and that do not change the number of sequents: the $0, +$ rules.

Type 3 The rule with two premises: the $\sqcap$ rule.

Because of the similarities of the rules in each of these categories, we just prove the CAN–free invertibility of one rule in each category by means of a sequence of lemmas.

Lemma 31 (Type 1)**.** *If* $[\vdash \Gamma_i, (A \sqcup B)_i^n]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, A_i^n \mid \vdash \Gamma_i, B^{n_i}]_{i=1}^n$ *has a CAN–free derivation.*

Proof. By induction on the derivation of $[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=1}^n$. Most cases are easy except the cases for when the derivation ends with a M rule, a $\sqcap$ rule or a $\sqcup$ rule so we will only show those cases.

- If the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma_1, (A \sqcup B)^{n_1} \quad G \mid \vdash \Gamma_2, (A \sqcup B)^{n_2}}{G \mid \vdash \Gamma_1, \Gamma_2, (A \sqcup B)^{n_1}, (A \sqcup B)^{n_2}}\ \mathrm{M}$$

with $G = [\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=3}^n$ and $G' = [\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=3}^n$ then by induction hypothesis on the CAN–free derivations of the premises we have that

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_1, B^{n_1}$$

and

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_2, B^{n_2}$$

are derivable by CAN–free derivations. We want to prove that both

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_2, B^{n_2}$$

and

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_1, B^{n_1}$$

are CAN–free derivable, as this will allow us to conclude by application of the M rule as follows:

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_2, B^{n_2} \quad G \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_2, B^{n_2}}{G \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_1, \Gamma_2, B^{n_1}, B^{n_2}}\ \mathrm{M} \quad \dfrac{G \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_1, B^{n_1} \quad G \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_2, B^{n_2}}{G \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_1, \Gamma_2, B^{n_1}, B^{n_2}}\ \mathrm{M}}{G \mid \vdash \Gamma_1, \Gamma_2, A^{n_1}, A^{n_2} \mid \vdash \Gamma_1, \Gamma_2, B^{n_1}, B^{n_2}}\ \mathrm{M}$$

If $n_1 = 0$ or $n_2 = 0$, those two hypersequents are derivable using the C rule then the W rule.
Otherwise, by using the W rule, Lemma 27 and the M rule, we have

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_2, B^{n_2} \mid \vdash \Gamma_1^{n_2}, \Gamma_2^{n_1}, A^{n_1 n_2}, B^{n_1 n_2}$$

and

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, A^{n_2} \mid \vdash \Gamma_1, B^{n_1} \mid \vdash \Gamma_1^{n_2}, \Gamma_2^{n_1}, A^{n_1 n_2}, B^{n_1 n_2}$$

We can then conclude using the S rule, Lemma 26 and the C rule.

- If the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma_1, (A \sqcup B)^{n_1}, C^k \quad G \mid \vdash \Gamma_1, (A \sqcup B)^{n_1}, D^k}{G \mid \vdash \Gamma_1, (A \sqcup B)^{n_1}, (C \sqcap D)^k}\ \sqcap$$

with $G = [\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=2}^n$ and $G' = [\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^n$, then by induction hypothesis on the CAN–free derivations of the premises we have that

$$\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, C^k \mid \vdash \Gamma_1, B^{n_1}, C^k$$

and

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, D^k \mid \vdash \Gamma_1, B^{n_1}, D^k$$

so by using the W rule and the M rule, we can derive

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, C^k \mid \vdash \Gamma_1, B^{n_1}, D^k \mid \vdash \Gamma_1, \Gamma_1, A^{n_1}, B^{n_1}, C^k, D^k$$

and

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, D^k \mid \vdash \Gamma_1, B^{n_1}, C^k \mid \vdash \Gamma_1, \Gamma_1, A^{n_1}, B^{n_1}, C^k, D^k$$

and then with the S rule and the C rule

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, C^k \mid \vdash \Gamma_1, B^{n_1}, D^k$$

and

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, D^k \mid \vdash \Gamma_1, B^{n_1}, C^k$$

We can then conclude with the $\sqcap$ rule.

- if the derivation finishes with an application on the $\sqcup$ rule acting on the formula $A \sqcup B$, we need to carefully analyse where the $A \sqcup B$ formulas appear. There are three cases:
  - the formulas $A \sqcup B$ active in the rule, but which are not under consideration in the lemma, i.e., there are not the instances of $A \sqcup B$ we want to reduce, which are $(A \sqcup B)^{m_1}$ below,
  - the formulas $A \sqcup B$ which are both active in the rule, and under consideration in the lemma, which are $(A \sqcup B)^{m_2}$ below, and
  - the formulas $A \sqcup B$ which are not active in the rule but under consideration in the lemma, which are $(A \sqcup B)^{m_3}$ below.

  Thus, the derivation finishes with

$$\frac{G \mid \vdash \Gamma_1, A^{m_1+m_2}, (A \sqcup B)^{m_3} \mid \vdash \Gamma_1, B^{m_1+m_2}, (A \sqcup B)^{m_3}}{G \mid \vdash \Gamma_1, (A \sqcup B)^{m_1+m_2}, (A \sqcup B)^{m_3}} \sqcup$$

  with $G = [\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=2}^n$, $G' = [\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^n$ and $n_1 = m_2 + m_3$ and we want to derive

$$G' \mid \vdash \Gamma_1, (A \sqcup B)^{m_1}, A^{m_2+m_3} \mid \vdash \Gamma_1, (A \sqcup B)^{m_1}, B^{m_2+m_3}$$

  We note $\Gamma_{\alpha_1,\alpha_2,\alpha_3} = \Gamma_1, \alpha_1^{m_1}, \alpha_2^{m_2}, \alpha_3^{m_3}$ for $\alpha_i \in \{A, B\}$. For instance, $\Gamma_{A,B,B} = \Gamma_1, A^{m_1}, B^{m_2}, B^{m_3}$.
  Notice that $\Gamma^{m_2}_{\alpha_1,\alpha_2,\alpha_2}, \Gamma^{m_3}_{\alpha_1,\alpha_3,\alpha_3} = \Gamma^{m_2+m_3}_{\alpha_1,\alpha_2,\alpha_3}$.
  The induction hypothesis is

$$\triangleright_{\mathbf{GA}\Vert\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,A} \mid \vdash \Gamma_{B,B,B}$$

Then the derivation is

$$
\begin{array}{cl}
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,A} & \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma^{m_2+m_3}_{B,B,A} & \text{Lemma 27} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma^{m_2+m_3}_{B,B,A} & \text{W} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma^{m_2}_{B,B,B} \mid \vdash \Gamma^{m_3}_{B,A,A} & \text{S} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{B,A,A} & (\text{Lemma 26})^2 \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} & \text{C}^2 \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma^{m_2+m_3}_{A,A,B} & \text{Lemma 27} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma^{m_2+m_3}_{A,A,B} & \text{W} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma^{m_2}_{A,A,A} \mid \vdash \Gamma^{m_3}_{A,B,B} & \text{S} \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{A,B,B} & (\text{Lemma 26})^2 \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} & \text{C}^2 \\
\hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_1, (A \sqcup B)^{m_1}, B^{m_2+m_3} & \sqcup \\
\hline
G' \mid \vdash \Gamma_1, (A \sqcup B)^{m_1}, A^{m_2+m_3} \mid \vdash \Gamma_1, (A \sqcup B)^{m_1}, B^{m_2+m_3} & \sqcup
\end{array}
$$

□

Lemma 32 (Type 2)**.** *If* $[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=1}^n$ *has a CAN–free derivation.*

Proof. Straightforward induction on the derivation of $[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^n$. For instance if the derivation finishes with

$$\frac{G \mid \vdash \Gamma_1, (A+B)^{n_1} \quad G \mid \vdash \Gamma_2, (A+B)^{n_2}}{G \mid \vdash \Gamma_1, \Gamma_2, (A+B)^{n_1}, (A+B)^{n_2}} \text{ M}$$

with $G = [\vdash \Gamma_i, (A+B)^{n_i}]_{i=3}^n$ and $G' = [\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=3}^n$, then by induction hypothesis on the CAN–free derivations of the premises we have that

$$\triangleright_{\mathbf{GA}\|\setminus\{\text{CAN}\}} G' \mid \vdash \Gamma_1, A^{n_1}, B^{n_1}$$

and

$$\triangleright_{\mathbf{GA}\|\setminus\{\text{CAN}\}} G' \mid \vdash \Gamma_2, A^{n_2}, B^{n_2}$$

so

$$\frac{G' \mid \vdash \Gamma_1, A^{n_1}, B^{n_1} \quad G' \mid \vdash \Gamma_2, A^{n_2}, B^{n_2}}{G' \mid \vdash \Gamma_1, \Gamma_2, A^{n_1}, A^{n_2}, B^{n_1}, B^{n_2}} \text{ M}$$

□

Lemma 33 (Type 3)**.** *If* $[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, A^{n_i}]_{i=1}^n$ *and* $[\vdash \Gamma_i, B^{n_i}]_{i=1}^n$ *have a CAN–free derivation.*

Proof. A straightforward induction on the derivation of $[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^n$. We will show the only complicated case, i.e., the $\sqcap$ rule acting on $A \sqcap B$.

As previously, we need to distinguish three types of $A \sqcap B$ formulas:

- the formulas $A \sqcap B$ active in the rule, but which are not under consideration in the lemma, i.e., there are not the instances of $A \sqcap B$ we want to reduce, which are $(A \sqcap B)^{m_1}$ below,

- the formulas $A \sqcap B$ which are both active in the rule and under consideration in the lemma, which are $(A \sqcap B)^{m_2}$ below, and
- the formulas $A \sqcap B$ which are not active in the rule but under consideration in the lemma, which are $(A \sqcap B)^{m_3}$ below.

Thus the derivation finishes with

$$\frac{G \mid \vdash \Gamma_1, A^{m_1+m_2}, (A \sqcap B)^{m_3} \quad G \mid \vdash \Gamma_1, B^{m_1+m_2}, (A \sqcap B)^{m_3}}{G \mid \vdash \Gamma_1, (A \sqcap B)^{m_1+m_2}, (A \sqcap B)^{m_3}} \sqcap$$

with $G = [\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=2}^n$ and $n_1 = m_2 + m_3$.

We will show how to derive

$$G' \mid \vdash \Gamma_1, (A \sqcap B)^{m_1}, A^{m_2+m_3}$$

where $G' = [\vdash \Gamma_i, A^{n_i}]_{i=2}^n$, the other case is similar.

By using the induction hypothesis, we have that

$$\triangleright_{\mathbf{GA}\|\setminus\{\text{CAN}\}} G' \mid \vdash \Gamma_1, A^{m_1+m_2}, A^{m_3}$$

$$\triangleright_{\mathbf{GA}\|\setminus\{\text{CAN}\}} G' \mid \vdash \Gamma_1, B^{m_1+m_2}, A^{m_3}$$

We will now derive the hypersequent $G' \mid \vdash \Gamma_1, B^{m_1}, A^{m_2+m_3}$ which will allow us to conclude using the $\sqcap$ rule.

$$\dfrac{\dfrac{G' \mid \vdash \Gamma_1, B^{m_1+m_2}, A^{m_3}}{G' \mid \vdash \Gamma_1^{m_1}, B^{(m_1+m_2)m_1}, A^{m_1 m_3}} \text{Lemma 27} \quad \dfrac{G' \mid \vdash \Gamma_1, A^{m_1+m_2+m_3}}{G' \mid \vdash \Gamma_1^{m_2}, A^{m_2(m_1+m_2+m_3)}} \text{Lemma 27}}{\dfrac{G' \mid \vdash \Gamma_1^{m_1+m_2}, B^{(m_1+m_2)m_1}, A^{(m_2+m_3)(m_1+m_2)}}{G' \mid \vdash \Gamma_1, B^{m_1}, A^{m_2+m_3}} \text{Lemma 26}} \text{M}$$

□

2.1.5 M–elimination

In this section, we will show the M elimination theorem. Recall that the M elimination theorem states

if a hypersequent G is derivable, then it has a M–free derivation.

However, since this result will be used in the proof of the CAN elimination theorem, we have to ensure that the M elimination theorem does not add any instance of the CAN rule. Thus we will show the slightly different result

if a hypersequent G is CAN–free derivable, then it has a CAN–free M–free derivation.

To prove this theorem, we need to show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free derivations d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there exists also a CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$.

The idea behind the proof is to combine d_1 and d_2 step-by-step. First we take the derivation d_1 and we modify it into a CAN–free and M–free prederivation (i.e., an open derivation) of

$$G \mid \;\; G \mid \vdash \Gamma, \Delta$$

where all the leaves in the prederivation are either terminated (by the INIT axiom) or non–terminated and of the form:

$$G \mid \vdash \Delta^n$$

for some $n \in \mathbb{N}$.

For instance if the derivation d_1 of $\vdash y \sqcup \overline{y}$ (with $G = \emptyset$ and $\Gamma = \vdash y \sqcup \overline{y}$) is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ ax}}{\vdash y, \overline{y}}\text{ ID}}{\vdash y \mid \vdash \overline{y}}\text{ S}}{\vdash y \sqcup \overline{y}}\ \sqcup$$

then we will modify it into a prederivation of $\vdash y \sqcup \overline{y}, \Delta$ by using the exact same rules, thus obtaining,

$$\dfrac{\dfrac{\dfrac{\vdash \Delta^2}{\vdash y, \overline{y}, \Delta^2}\text{ ID}}{\vdash y, \Delta \mid \vdash \overline{y}, \Delta}\text{ S}}{\vdash y \sqcup \overline{y}, \Delta}\ \sqcup$$

Then we use the derivation d_2 to construct a CAN–free and M–free derivation of each

$$G \mid \vdash \Delta^n$$

hence completing the prederivation of

$$G \mid \; G \mid \vdash \Gamma, \Delta$$

into a full derivation. From this it is possible to obtain the desired CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$ using several times the C rule:

$$\dfrac{G \mid \; G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{C}^*$$

In what follows, the first step is formalised as Lemma 34 and the second step as Lemma 35.

Lemma 34. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$, let H be a hypersequent and Δ be a sequent. Then there exists a CAN–free M–free prederivation of*

$$G \mid \; H \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are of the form $H \mid \vdash \Delta^n$ for some n.

Proof. This is an instance of the slightly more general statement of Lemma 37 below where:

- $[\vdash \Gamma_i]_{i=1}^{k-1} = G$ and $\Gamma_k = \Gamma$.
- $n_i = 0$ for $1 \leq i < k$ and $n_k = 1$.

□

Lemma 35. *Let d_2 be CAN–free and M–free derivation of $H \mid \vdash \Delta$. Then, for every natural number n, there exists a CAN–free and M–free derivation of*

$$H \mid \vdash \Delta^n$$

Proof. This is an instance of the slightly more general statement of Lemma 38 below where:

- $[\vdash \Delta_i]_{i=1}^{k-1} = H$ and $\Delta_k = \Delta$.
- $n_i = 1$ for $1 \leq i < k$ and $n_k = n$.

□

Before proving Lemmas 37 and 38, we now show how to remove one instance of the M rule and then the M-elimination theorem.

Lemma 36. *If $G \mid \vdash \Gamma$ and $H \mid \vdash \Delta$ have CAN–free M–free derivations, then so does $G \mid\ H \mid \vdash \Gamma, \Delta$.*

Proof. By using Lemma 34, we have a prederivation of $G \mid\ H \mid \vdash \Gamma, \Delta$ where all non–terminated leaves are of the form $H \mid \vdash \Delta^n$ for some n.

To conclude, we have to show that every non-terminated leaf is derivable, which can be done using Lemma 35. □

Theorem 2.1.3 (M elimination)**.** *If G is CAN–free derivable, then G is CAN–free M–free derivable.*

Proof. We prove the result by induction on the derivation of G. The only interesting case if the M rule, i.e., if the derivation finishes with

$$\frac{G \mid \vdash \Gamma \quad G \mid \vdash \Delta}{G \mid \vdash \Gamma, \Delta}\ \mathrm{M}$$

then by induction hypothesis, $G \mid \vdash \Gamma$ and $G \mid \vdash \Delta$ have a CAN–free M–free derivation.

By using Lemma 36, we have a CAN-free M-free derivation of $G \mid\ G \mid \vdash \Gamma, \Delta$. The derivation is then

$$\frac{G \mid\ G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta}\ \mathrm{C}^*$$

□

Lastly, we prove the technical version of Lemmas 34 and 35.

Lemma 37. *Let d_1 be a CAN–free and M–free derivation of $[\vdash \Gamma_i]_{i=1}^{k}$ and let H be a hypersequent and Δ be a sequent. Then for every sequence of natural numbers n_i, there exists a CAN–free M–free prederivation of*

$$H \mid\ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^{k}$$

where all non-terminated leaves are of the form $H \mid\ \Delta^n$ for some n.

Proof. By induction on d_1. We will show the case of the INIT axiom and the S rule, the other cases are similar.

- If the derivation finishes with
$$\frac{}{\vdash}\ \mathrm{INIT}$$

  Let n be a natural number. The prederivation of $H \mid \vdash \Delta^n$ is simply the leaf

$$H \mid \vdash \Delta^n$$

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k}{[\vdash \Gamma_i]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1} \mid \vdash \Gamma_k} \text{ S}$$

Let n_i be a sequence of natural numbers. By induction hypothesis, there exists a CAN–free M–free prederivation of

$$H \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k, \Delta^{n_{k-1}+n_k}$$

where all non-terminated leaves are of the form $H \mid \ \Delta^n$ for some n. We then continue this prederivation with

$$\frac{H \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k, \Delta^{n_{k-1}+n_k}}{H \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Delta^{n_{k-1}} \mid \vdash \Gamma_k, \Delta^{n_k}} \text{ S}$$

□

Lemma 38. *If* $[\vdash \Delta_i]_{i=1}^{k}$ *has a CAN–free M–free derivation then for all* n_i*, there is a CAN–free M–free derivation of* $[\vdash \Delta_i^{n_i}]_{i=1}^{k}$.

Proof. By induction on the derivation of $[\vdash \Delta_i]_{i=1}^{k}$. We show the only nontrivial case:

- If the derivation finishes with

$$\frac{[\vdash \Delta_i]_{i=3}^{k} \mid \vdash \Delta_1, \Delta_2}{[\vdash \Delta_i]_{i=3}^{k} \mid \vdash \Delta_1 \mid \vdash \Delta_2} \text{ S}$$

By induction hypothesis there is CAN–free derivation of

$$[\vdash \Delta_i^{n_i}]_{i=3}^{k} \mid \vdash \Delta_1^{n_1 n_2}, \Delta_2^{n_1 n_2}$$

If $n_1 = 0$ or $n_2 = 0$, we have the empty sequent which is derivable. Otherwise,

$$\dfrac{\dfrac{[\vdash \Delta_i^{n_i}]_{i=3}^{n} \mid \vdash \Delta_1^{n_1 n_2}, \Delta_2^{n_1 n_2}}{[\vdash \Delta_i^{n_i}]_{i=3}^{n} \mid \vdash \Delta_1^{n_1 n_2} \mid \vdash \Delta_2^{n_1 n_2}} \text{ S}}{[\vdash \Delta_i^{n_i}]_{i=3}^{n} \mid \vdash \Delta_1^{n_1} \mid \vdash \Delta_2^{n_2}} \text{ Lemma 26}$$

□

Remark 18. Lemma 38 is the main reason for using parallel rules in **GA**‖. Indeed, it is necessary for the logical rules to be able to act on several instances of a term in the same sequent, more precisely for the cases of the ⊔ and ⊓ rules.

To show this, let's consider the case of the ⊔ rule when the rule can only act on one instance of the formula, i.e., if the derivation finishes with

$$\frac{[\vdash \Delta_i]_{i=2}^{k} \mid \vdash \Delta_1, A \mid \vdash \Delta_2, B}{[\vdash \Delta_i]_{i=2}^{k} \mid \vdash \Delta_1, A \sqcup B} \sqcup$$

We want to show that $[\vdash \Delta_i^{n_i}]_{i=2}^{k} \mid \vdash \Delta_1^{n_1}, (A \sqcup B)^{n_1}$ is derivable, which is direct by using the induction hypothesis when the ⊔ rule can act on the n_1 instances of $A \sqcup B$ but would require to show that the hypersequent

$$[\vdash \Delta_i^{n_i}]_{2=1}^{k} \mid \ \left[\vdash \Delta_1^{n_1}, A^{k_A}, B^{k_B}\right]_{k_A + k_B = n_1}$$

is derivable if the ⊔ acts only on one instance of $A \sqcup B$ at a time. However, we do not know how to prove this without using the M rule.

2.1.6 CAN–elimination

The CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma} \text{ CAN}$$

We prove the CAN elimination Theorem 2.1.4 below by showing that if the hypersequent $G \mid \vdash \Gamma, A^n, \overline{A}^n$ has a M–free CAN–free derivation then the hypersequent $G \mid \vdash \Gamma$ also has a M–free CAN–free derivation.

Our proof proceeds by induction on the complexity of the term A. The base case is given by $A = x$ (or equivalently $A = \overline{x}$) for some variable x. The following lemma proves this base case.

Lemma 39. *If there is a M–free CAN–free derivation of $G \mid \vdash \Gamma, x^n, \overline{x}^n$, then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. The statement follows as a special case of Lemma 41 below, a stronger version of Lemma 39 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, x^n, \overline{x}^n$, where:

- $[\vdash \Gamma_i]_{i=1}^{k-1} = G$ and $\Gamma_k = \Gamma$.
- $n_i = 0$ for $1 \leq i < k$.
- $n_k = n$.

□

For complex terms A, we proceed by using the CAN–free invertibility of the logical rules (see Section 2.1.4) as follows.

Lemma 40. *If there is a M–free CAN–free derivation of $G \mid \vdash \Gamma, A^n, \overline{A}^n$, then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. We proceed by induction on A.

- If $A = x$, we are in the base case of Lemma 39.
- If $A = 0$, we can conclude with the CAN–free invertibility of the 0 rule and the M–elimination theorem.
- If $A = B + C$, since the $+$ rule is CAN–free invertible, $G \mid \vdash \Gamma, B^n, C^n, \overline{B}^n, \overline{C}^n$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, B^n, (\overline{B} \sqcap \overline{C})^n \mid \vdash \Gamma, C^n, (\overline{B} \sqcap \overline{C})^n$ has a M–free CAN–free derivation. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, B^n, \overline{B}^n \mid \vdash \Gamma, C^n, \overline{C}^n$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler terms B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:

$$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma} \; C$$

- If $A = B \sqcap C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, (B \sqcap C)^n, \overline{B}^n \mid \vdash \Gamma, (B \sqcap C)^n, \overline{C}^n$ has a M–free CAN–free derivation. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, B^n, \overline{B}^n \mid \vdash \Gamma, C^n, \overline{C}^n$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler terms B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:

$$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ C$$

□

We can now prove the CAN elimination theorem by a straightforward induction on the derivation of the hypersequent, the CAN rule being dealt with using Lemma 40 above.

Theorem 2.1.4 (CAN elimination)**.** *For all hypersequent G, if $\triangleright_{\boldsymbol{GA}\|} G$ then $\triangleright_{\boldsymbol{GA}\|\backslash\{CAN\}} G$.*

Proof. We proceed by induction on the derivation of G. We only show the case of the CAN rule and one example for the other rules since the other cases are all similar.

- If the derivation finishes with:

$$\frac{G}{G \mid \vdash \Gamma}\ \mathrm{W}$$

  then $\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G$ and by using the W rule, we obtain a CAN–free derivation of $G \mid \vdash \Gamma$.

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma}\ \mathrm{CAN}$$

  then $\triangleright_{\mathbf{GA}\|\backslash\{\mathrm{CAN}\}} G \mid \vdash \Gamma, A^n, \overline{A}^n$ and we can conclude with Lemma 40.

□

We now prove Lemma 41, the stronger version of Lemma 39.

Lemma 41. *If there is a CAN–free and M–free derivation of the hypersequent*

$$[\vdash \Gamma_i, x^{n_i}, \overline{x}^{m_i}]_{i=1}^k$$

then for all n_i', m_i' such that $n_i - m_i = n_i' - m_i'$ for all $i \in [1..k]$, there is a CAN–free, M–free derivation of

$$\left[\vdash \Gamma_i, x^{n_i'}, \overline{x}^{m_i'}\right]_{i=1}^k$$

Proof. By induction on the derivation of $[\vdash \Gamma_i, x^{n_i}, \overline{x}^{m_i}]_{i=1}^k$. Most cases are trivial, we just describe the most interesting one.

- If the derivation finishes with:

$$\frac{[\vdash \Gamma_i, x^{n_i}, \overline{x}^{m_i}]_{i\geq 2} \mid \vdash \Gamma_1, x^{n_c}, \overline{x}^{m_c}}{[\vdash \Gamma_i, x^{n_i}, \overline{x}^{m_i}]_{i\geq 2} \mid \vdash \Gamma_1, x^{n_a+n_b+n_c}, \overline{x}^{m_a+m_b+m_c}}\ \mathrm{ID}$$

  with $n_1 = n_b + n_c$, $m_1 = m_b + m_c$ and $n_a + n_b = m_a + m_b$.

We want to show that

$$\triangleright_{\mathbf{GA}\|\backslash\{\text{CAN}\}} \left[\vdash \Gamma_i, x^{n'_i}, \overline{x}^{m'_i}\right]_{i\geq 2} \mid \vdash \Gamma_1, x^{n_a}, x^{n'_1}, \overline{x}^{m_a}, \overline{x}^{m'_1}$$

We will now prove that $n_c - m_c = n'_1 + n_a - (m'_1 + m_a)$ to be able to conclude with the induction hypothesis.

$$\begin{aligned} n_c - m_c &= (n_1 - n_b) - (m_1 - m_b) \\ &= (n_1 - m_1) + (m_b - n_b) \\ &= (n'_1 - m'_1) + (n_a - m_a) \\ &= (n'_1 + n_a) - (m'_1 + m_a) \end{aligned}$$

so by induction hypothesis, we have

$$\triangleright_{\mathbf{GA}\|\backslash\{\text{CAN}\}} \left[\vdash \Gamma_i, x^{n'_i}, \overline{x}^{m'_i}\right]_{i\geq 2} \mid \vdash \Gamma_1, x^{n_a}, x^{n'_1}, \overline{x}^{m_a}, \overline{x}^{m'_1}$$

which is the result we want.

□

2.1.7 Algebraic property

We will now present a property that reduces the problem of derivability of an atomic hypersequent to solving a system of linear equations, allowing us to have an algebraic characterization of the derivability of atomic hypersequents. With some abuse of notations, this property states that an atomic hypersequent

$$\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m$$

is derivable if and only if there are $k_1, ..., k_m \in \mathbb{N}$ with $k_i \neq 0$ for some i such that

$$k_1\Gamma_1 + ... + k_m\Gamma_m = 0 \tag{2.1}$$

This result is a direct consequence of the elimination Theorems 2.1.3 and 2.1.4 stating that if an atomic hypersequent has a derivation, then it has a CAN–free M–free derivation. Thus, to build this CAN–free M–free derivation, only the S,C, W and ID rules can be used before concluding with the INIT axiom. If we analyse the impact of those rules to the hypersequent, then

- the C rule adds some copies the sequents,
- the W rule removes some copies of the sequents,
- the S rule connects two sequents together, such that we can only add or remove copies of those sequents at the same time, i.e., after the rule

$$\frac{\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m, \Gamma_{m+1}}{\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m \mid \vdash \Gamma_{m+1}}\ \text{S}$$

  then we can add or remove copies of Γ_m only if we also add or remove copies of Γ_{m+1} (and vice versa), and
- the ID rule removes axioms that cancel each other, i.e., removes both n x and n $\overline{x}$ for some n and x.

Since only those rules can be used to derive the hypersequent, we can only multiply the different sequents by natural numbers (the natural number being the number of copies of the corresponding sequent) before cancelling all of the remaining ones with the ID rule. Thus, we obtain the equation (2.1).

Theorem 2.1.5. *For all atomic hypersequents G, built using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$, of the form*

$$\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m$$

where $\Gamma_i = x_1^{n_{i,1}}, ..., x_k^{n_{i,k}}, \overline{x_1}^{n'_{i,1}}, ..., \overline{x_k}^{n'_{i,k}}$, the following are equivalent:

1. *G has a derivation.*
2. *there exist numbers $k_1, ..., k_m \in \mathbb{N}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $k_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

   $$\sum_{i=1}^{m} k_i n_{i,j} = \sum_{i=1}^{m} k_i n'_{i,j}$$

   i.e., the scaled (by the numbers $k_1 \ldots k_m$) sum of the numbers of variable x_j is equal to the scaled sum of the numbers of covariable $\overline{x_j}$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the derivation of G. By using Theorem 2.1.4 and Theorem 2.1.3, we can assume that the derivation is CAN–free and M–free. We show only the C case, the other cases being similar:

- If the derivation finishes with

  $$\frac{\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m \mid \vdash \Gamma_m}{\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m} \text{ C}$$

  by induction hypothesis, there are $k_1, ..., k_m, k_{m+1} \in \mathbb{N}$ such that :

  – there exists $i \in [1..m+1]$ such that $k_i \neq 0$.

  – for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=1}^{m} k_i n_{i,j} + k_{m+1} n_{m,j} = \sum_{i=1}^{m} k_i n'_{i,j} + k_{m+1} n'_{m,j}$.

  Then $k_1, ..., k_{m-1}, k_m + k_{m+1}$ satisfies the property.

The other way ((2) $\Rightarrow$ (1)) is more straightforward. If there exist natural numbers $k_1, ..., k_m \in \mathbb{N}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $k_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} k_i n_{i,j} = \sum_{i=1}^{m} k_i n'_{i,j}$$

then we can use the W rule to remove the sequents corresponding to the numbers $k_i = 0$, and use the C rule $k_i - 1$ times then the S rule $k_i - 1$ times on the ith sequent to multiply it by k_i. If we assume that there is a natural number l such that $k_i = 0$ for all $i > l$ and $k_i \neq 0$ for all $i \leq l$, then the CAN–free derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash {\Gamma_1}^{k_1}, \dots, {\Gamma_l}^{k_l}}\text{ ID}^*}{\vdash {\Gamma_1}^{k_1} \mid \dots \mid \vdash {\Gamma_l}^{k_l}}\text{ S}^*}{\vdash \Gamma_1 \mid \dots \mid \vdash \Gamma_l}\text{ C-S}^*}{\vdash \Gamma_1 \mid \dots \mid \Gamma_m}\text{ W}^*$$

□

Remark 19. Note that the M elimination theorem 2.1.3 is actually not necessary to prove the result above. However, it makes the proof much simpler.

2.2 Hypersequent calculus HR

In this section we add rules for the scalar multiplication to the system **GA**‖ to build the system **HR**. Whereas the system **GA**‖ was used to derive positive *Abelian l-group* terms, the system **HR** can be used to derive positive *Riesz* terms. The main difference between **GA**‖ and **HR** lies in the presence of scalars in hypersequents, namely in the weighted terms defined below, and the additional T rule of the system **HR**.

We start by adapting the definitions and conventions used for the system **GA**‖ in the context of Riesz spaces and **HR**.

Definition 2.2.1. A *weighted term* is a formal expression $r.A$ where $r \in \mathbb{R}_{>0}$ and A is a term in NNF.

Recall that the scalars appearing in these terms in NNF are all strictly positive and are ranged over by the letters $r, s, t \in \mathbb{R}_{>0}$. From now on, the term scalar should always be understood as strictly positive scalar.

Given a weighted term $r.A$ and a scalar s we denote with $s.(r.A)$ the weighted term $(sr).A$. Thus we have defined (strictly positive) scalar multiplication on weighted terms.

We adopt the following notation:

- Given a sequence $\vec{r} = (r_1, \dots r_n)$ of scalars and a term A, we denote with $\vec{r}.A$ the multiset $[r_1.A, \dots, r_n.A]$. When $\vec{r}$ is empty, the multiset $\vec{r}.A$ is also empty.
- Given a multiset $\Gamma = [r_1.A_1, \dots, r_n.A_n]$ and a scalar $s > 0$, we denote with $s.\Gamma$ the multiset $[s.r_1.A_1, \dots, s.r_n.A_n]$.
- Given a sequence $\vec{s} = (s_1, \dots s_n)$ of scalars and a multiset Γ, we denote with $\vec{s}.\Gamma$ the multiset $s_1.\Gamma, \dots, s_n.\Gamma$.
- Given two sequences $\vec{r} = (r_1, \dots r_n)$ and $\vec{s} = (s_1, \dots s_m)$ of scalars, we denote $\vec{r}; \vec{s}$ the concatenation of the two sequences, i.e. the sequence $(r_1, \dots r_n, s_1, \dots s_m)$.
- Given a sequence $\vec{s} = (s_1, \dots s_n)$ of scalars and a scalar r, we denote $(r\vec{s})$ the sequence $(rs_1, \dots rs_n)$.

- Given two sequences $\vec{r} = (r_1, \dots r_n)$ and $\vec{s} = (s_1, \dots s_m)$ of scalars, we denote $\vec{r}\vec{s}$ the sequence $r_1\vec{s}; \dots ; r_n\vec{s}$.
- Given a sequence $\vec{s} = (s_1, \dots s_n)$ of scalars, we denote $\sum \vec{s}$ the sum of all scalars in $\vec{s}$, i.e. the scalar $\sum\limits_{i=1}^{n} s_i$.

Note that the notation $A^n = A, ..., A$ used in the system **GA**$\|$ can be seen as a specific instance of the notation $\vec{r}.A$ where every element of $\vec{r}$ is equal to 1. We can adapt most of the definitions and results of the system **GA**$\|$ to the system **HR** by simply replacing A^n with $\vec{r}.A$ and Γ^n with $\vec{r}.\Gamma$, as can be seen in the rules of Figure 2.5.

Definition 2.2.2. A *sequent* is a formal expression of the form $\vdash \Gamma$.

If $\Gamma = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 2.2.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \dots | \vdash \Gamma_n$.

We use the same notion of "simple" hypersequent as in the system **GA**$\|$, once again it can be seen as the basic case when reasoning by induction on hypersequents.

Definition 2.2.4. A hypersequent is said *atomic* if it only contains atoms, i.e., formulas of the form x or $\overline{x}$.

We now describe how sequents and hypersequents can be interpreted by Riesz terms.

Definition 2.2.5 (Interpretation)**.** We interpret weighted terms $(r.A)$, sequents $\vdash \Gamma$ and hypersequents G as the Riesz terms $(\!|r.A|\!)$, $(\!|\vdash \Gamma|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Weighted terms	$r.A$	rA
Sequents	$\vdash r_1.A_1, \dots, r_n.A_n$	$(\!|r_1.A_1|\!) + \dots + (\!|r_n.A_n|\!)$
Hypersequents	$\vdash \Gamma_1 | \dots | \vdash \Gamma_n$	$(\!|\vdash \Gamma_1|\!) \sqcup \dots \sqcup (\!|\vdash \Gamma_n|\!)$

Hence a weighted term is simply interpreted as the term scalar–multiplied by the weight. A sequent is interpreted as sum ($\sum$) and a hypersequent is interpreted as a join of sums ($\bigsqcup \sum$).

Example 17. The interpretation of the hypersequent:

$$\vdash 1.x, 2.(y \sqcap z) \ | \ \vdash 2.(3\overline{x} \sqcap y)$$

is the Riesz term:

$$\big(1x + 2(y \sqcap z)\big) \sqcup \big(2(3\overline{x} \sqcap y)\big).$$

The hypersequent calculus **HR** is a deductive system for deriving hypersequents whose interpretation is positive, i.e., the hypersequents G such that $\mathcal{A}_{\text{Riesz}} \vdash 0 \leq (\!|G|\!)$. The rules of **HR** are presented in Figure 2.5 and are very similar to the rules of the system **GA**$\|$ (see Figure 2.1) where the main difference is the use of weighted terms in sequents. We write $\triangleright_{\mathbf{HR}} G$ if the hypersequent G is derivable in the system **HR**.

The T rule is novel, and can be seen as a real–valued variant of C (contraction) rule in that the weight of a sequent in the hypersequent can be multiplied by an arbitrary positive real number – while the C rule was mostly used to multiply a sequent by a natural number. It is worth knowing that the T rule is not necessary to prove the completeness of the system **HR**, but is necessary for the CAN elimination theorem, and appears naturally in the proofs of the CAN–free invertibility

Axiom:

$$\frac{}{\vdash}\ \text{INIT}$$

Structural rules:

$$\frac{G}{G \mid \vdash \Gamma}\ \text{W} \qquad \frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ \text{C}$$

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S} \qquad \frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{M}$$

$$\frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma}\ \text{T} \qquad \frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}}\ \text{ID}, \sum r_i = \sum s_i$$

Logical rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.0}\ 0 \qquad \frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A+B)}\ + \qquad \frac{G \mid \ \Gamma \vdash \Gamma, (s\vec{r}).A}{G \mid \ \Gamma \vdash \Gamma, \vec{r}.(sA)}\ \times$$

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)}\ \sqcup \qquad \frac{G \mid \vdash \Gamma, \vec{r}.A \quad G \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)}\ \sqcap$$

CAN rule:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma}\ \text{CAN}, \sum r_i = \sum s_i$$

Figure 2.5: Inference rules of **HR**.

of the logical rules in Section 2.2.4. Finally, note that the logical rules are all presented using the syntactic sugaring $\vec{r}.A$ described above. As in the system **GA**$\|$, this effectively allows us to apply the rule to several terms in the sequent at the same time. We stress again that we can obtain most of the rules of the system **HR** simply by replacing A^n with $\vec{r}.A$ in the rules of the system **GA**$\|$. Note that the following ID rule

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.x, \vec{r}.\overline{x}}\ \text{bad ID}$$

that we would obtain by replacing x^n and $\overline{x}^n$ with $\vec{r}.x$ and $\vec{r}.\overline{x}$ would not allow to prove the completeness of the system **HR**, and more precisely to derive the axiom $(r_1 + r_2)x = r_1 x + r_2 x$ and thus we require the slightly more complex ID rule presented above. The CAN rule was then chosen to keep the symmetry with the ID rule.

Remark 20. Note that the following CUT rule

$$\frac{G \mid \vdash \Gamma_1, \vec{r}.A \quad G \mid \vdash \Gamma_2, \vec{s}.\overline{A}}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CUT}, \sum \vec{r} = \sum \vec{s}$$

is equivalent to the CAN rule in the **HR** hypersequent calculus. Deriving the CAN rule from the CUT rule is done the same way as in **GA**$\|$ but the other way is trickier and actually requires to use three times the CAN rule (instead of one in **GA**$\|$):

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, \vec{r}.A \quad G \mid \vdash \Gamma_2, \vec{s}.\overline{A}}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}.A, \vec{s}.\overline{A}}\ \text{M}}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CAN}, \sum \vec{r} = \sum \vec{s}$$

Figure 2.6: Derivability of the CUT rule.

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 42}}{G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}\ \text{CUT}}{G \mid \vdash \Gamma, \vec{r}.(A + \overline{A})}\ + \quad \dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad \dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 42}}{G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}\ \text{CUT}}{G \mid \vdash \Gamma, \vec{r}.(\overline{A} + A)}\ +}{\dfrac{\dfrac{G \mid \vdash \Gamma, \Gamma}{G \mid \vdash \Gamma \mid \vdash \Gamma}\ \text{S}}{G \mid \vdash \Gamma}\ \text{C}}\ \text{CUT}$$

Figure 2.7: Derivability of the CAN rule.

Example 18. Example of derivation of the hypersequent $\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)$ which consists of only one sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 2.y, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.y, 2.\overline{x}, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.\overline{y} \mid \vdash 2.y, 2.\overline{x}}\ \text{S}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.y, 1.\overline{x}}\ \text{T(multiplication by 2)}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 2.x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.2x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.(2x + 2\overline{y}) \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)}\ \sqcup$$

Definition 2.2.6. We define the complexity of a sequent $\vdash \Gamma$, noted $c(\vdash \Gamma)$, as the sum of the operators used in the terms of Γ,i.e., if $\Gamma = r_1.A_1, ..., r_n.A_n$, $c(\vdash \Gamma) = \sum_{i=1}^{n} c(A_i)$ (see Definition 1.2.4 for the definition of $c(A_i)$).

The complexity of a hypersequent G, noted $c(G)$, is then defined as the pair $c(G) = (a, b)$ where

- a is the maximum complexity of a sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $a = \max\limits_{i \in [1..n]} c(\vdash \Gamma_i)$, and
- b is the number of sequents in G having a complexity of a, i.e., $b = \#\{\vdash \Gamma_i \mid c(\vdash \Gamma_i) = a\}$.

We say that a sequent $\vdash \Gamma$ of G is maximal if $c(\vdash \Gamma) = a$.

2.2.1 Preliminary lemmas

As in Section 2.1, we start by proving a few technical lemmas.

Our first lemma states that the following variant of the ID rule (see Figure 2.5) where general terms A are considered rather than just variables, is admissible in the proof system **HR**.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}} \text{ ID}, \sum \vec{r} = \sum \vec{s}$$

Lemma 42. *For all terms A and vectors $\vec{r}$ and $\vec{s}$*

$$\textit{if } \rhd_{\boldsymbol{HR}} G \mid \vdash \Gamma \textit{ and } \sum \vec{r} = \sum \vec{s} \textit{ then } \rhd_{\boldsymbol{HR}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$$

Proof. We prove the result by induction on A.

- If A is a variable, we simply use the ID rule.
- If $A = 0$, we use the 0 rule.
- If $A = B + C$, we use the + rule twice (for $A + B$ and $\overline{A} + \overline{B}$) and conclude with the induction hypothesis.
- For the case $A = B \sqcap C$ or $A = B \sqcup C$, we first use the $\sqcap$ rule and then the $\sqcup$ rule on each premise and the W rule on each premise to remove the sequents with both B and C in them. We can then conclude with the induction hypothesis.

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B}}{G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{C}} \text{ W}}{G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.(\overline{B} \sqcup \overline{C})} \sqcup \quad \dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}}{G \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}} \text{ W}}{G \mid \vdash \Gamma, \vec{r}.C, \vec{s}.(\overline{B} \sqcup \overline{C})} \sqcup}{G \mid \vdash \Gamma, \vec{r}.(B \sqcap C), \vec{s}.(\overline{B} \sqcup \overline{C})} \sqcap$$

□

The next result states that derivability in the **HR** system is preserved by substitution of terms for variables.

Lemma 43. *For all hypersequents G and terms A, if $\rhd_{\boldsymbol{HR}} G$ then $\rhd_{\boldsymbol{HR}} G[A/x]$.*

Proof. We prove the result by induction on the derivation of G. Most cases are quite straightforward, we simply use the induction hypothesis on the premises and then use the same rule. For instance, if the derivation finishes with

$$\frac{G \mid \vdash \Gamma, \vec{r}.B, \vec{r}.C}{G \mid \vdash \Gamma, \vec{r}.(B + C)} +$$

by induction hypothesis $\rhd_{\mathbf{HR}} G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.B[A/x], \vec{r}.C[A/x]$ so

$$\frac{G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.B[A/x], \vec{r}.C[A/x]}{G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.(B + C)[A/x]} +$$

The only tricky case is when the ID rule is used on the variable x, where we conclude using Lemma 42. □

The next lemma states that the logical rules are invertible using the CAN rule, meaning that if the conclusion is derivable, then the premises are also derivable. Unlike a stronger result we will prove later in Section 2.2.4 where we prove the CAN–free version of this lemma, the derivations of the premises may contain CAN rules and thus this result is not sufficient to imply the CAN elimination theorem.

Lemma 44. *All logical rules are invertible.*

Proof. We simply use the CAN rule to introduce the operators. We will show the two most interesting cases, the other cases are trivial.

- The $\sqcap$ rule: we assume that $G \mid\ \vdash \Gamma, \vec{r}.(A \sqcap B)$ is derivable. The derivation of $G \mid\ \vdash \Gamma, \vec{r}.A$ is then:

$$\dfrac{\dfrac{G \mid\ \vdash \Gamma, \vec{r}.(A \sqcap B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \vec{r}.A, \vec{r}.\overline{A}}\,\text{Lemma 42}}{\vdash \vec{r}.A, \vec{r}.\overline{A} \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B}}\,\text{W}}{\vdash \vec{r}.A, \vec{r}.(\overline{A} \sqcup \overline{B})}\,\sqcup}{G \mid\ \vdash \vec{r}.A, \vec{r}.(\overline{A} \sqcup \overline{B})}\,\text{W}^*}{G \mid\ \vdash \Gamma, \vec{r}.A, \vec{r}.(A \sqcap B), \vec{r}.(\overline{A} \sqcup \overline{B})}\,\text{M}}{G \mid\ \vdash \Gamma, \vec{r}.A}\,\text{CAN}$$

  The derivation of $G \mid\ \vdash \Gamma, \vec{r}.B$ is similar.

- The $\sqcup$ rule: we assume that $G \mid\ \vdash \Gamma, \vec{r}.(A \sqcup B)$ is derivable. The derivation of $G \mid\ \vdash \Gamma, \vec{r}.A \mid$ $\vdash \Gamma, \vec{r}.B$ is then:

$$\dfrac{\dfrac{\dfrac{G \mid\ \vdash \Gamma, \vec{r}.(A \sqcup B)}{G \mid\ \vdash \Gamma, \vec{r}.(A \sqcup B) \mid\ \vdash \Gamma, \vec{r}.B}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \vec{r}.A, \vec{r}.\overline{A}}\,\text{Lemma 42}}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{A} \mid\ \vdash \Gamma, \vec{r}.B}\,\text{W}^* \qquad \dfrac{\Pi}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \Gamma, \vec{r}.B}}{G \mid\ \vdash \vec{r}.A, \vec{r}.(\overline{A} \sqcap \overline{B}) \mid\ \vdash \Gamma, \vec{r}.B}\,\sqcap}{G \mid\ \vdash \Gamma, \vec{r}.A, \vec{r}.(A \sqcup B), \vec{r}.(\overline{A} \sqcap \overline{B}) \mid\ \vdash \Gamma, \vec{r}.B}\,\text{M}}{G \mid\ \vdash \Gamma, \vec{r}.A \mid\ \vdash \Gamma, \vec{r}.B}\,\text{CAN}$$

  where Π is the following derivation:

$$\dfrac{\dfrac{\dfrac{G \mid\ \vdash \Gamma, \vec{r}.(A \sqcup B)}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \Gamma, \vec{r}.(A \sqcup B)}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \vec{r}.B, \vec{r}.\overline{B}}\,\text{Lemma 42}}{\vdash \vec{r}.A, \vec{r}.\overline{B}, \vec{r}.B, \vec{r}.\overline{A}}\,\text{Lemma 42}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \vec{r}.B, \vec{r}.\overline{A}}\,\text{S} \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \vec{r}.B, \vec{r}.\overline{B}}\,\text{Lemma 42}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \vec{r}.B, \vec{r}.\overline{B}}\,\text{W}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \vec{r}.B, \vec{r}.(\overline{A} \sqcap \overline{B})}\,\sqcap}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \vec{r}.B, \vec{r}.(\overline{A} \sqcap \overline{B})}\,\text{W}^*}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \Gamma, \vec{r}.B, \vec{r}.(A \sqcup B), \vec{r}.(\overline{A} \sqcap \overline{B})}\,\text{M}}{G \mid\ \vdash \vec{r}.A, \vec{r}.\overline{B} \mid\ \vdash \Gamma, \vec{r}.B}\,\text{CAN}$$

□

Remark 21. The proof of invertibility does not introduce any new T rule, so if the conclusion of a logical rule has a T–free derivation then the premises also have T–free derivations.

The next lemmas state that CAN–free derivability in the **HR** system is preserved by scalar multiplication.

Lemma 45. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a non-empty vector and G a hypersequent. If $\rhd_{\boldsymbol{HR}\backslash\{CAN\}} G \mid \vdash \vec{r}.\Gamma$ then $\rhd_{\boldsymbol{HR}\backslash\{CAN\}} G \mid \vdash \Gamma$.*

Proof. We simply use the C,T and S rules :

$$\dfrac{\dfrac{\dfrac{G \mid \vdash \vec{r}.\Gamma}{G \mid \vdash r_1.\Gamma \mid \; ... \mid \; r_n.\Gamma}\text{ S}^*}{G \mid \vdash \Gamma \mid \; ... \mid \vdash \Gamma}\text{ T}^*}{G \mid \vdash \Gamma}\text{ C}^*$$

□

Lemma 46. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a vector and G a hypersequent. If $\rhd_{\boldsymbol{HR}\backslash\{CAN\}} G \mid \vdash \Gamma$ then $\rhd_{\boldsymbol{HR}\backslash\{CAN\}} G \mid \vdash \vec{r}.\Gamma$.*

Proof. We reason by induction on the size of $\vec{r}$.

If the size of $\vec{r}$ is 0: Since $\vdash \vec{r}.\Gamma = \vdash$, we simply use the W rule until we can use the INIT rule:

$$\dfrac{\dfrac{}{\vdash}\text{ INIT}}{G \mid \vdash}\text{ W}^*$$

If the size of $\vec{r}$ is 1: we can use the T rule:

$$\dfrac{G \mid \vdash (\frac{1}{r_1} r_1).\Gamma}{G \mid \vdash r_1.\Gamma}\text{ T}$$

Otherwise: Let $(r_1, ..., r_{n+1}) = \vec{r}$. We can invoke the inductive hypothesis and conclude as follows:

$$\dfrac{\dfrac{G \mid \vdash \Gamma}{G \mid \vdash r_1.\Gamma, ..., r_n.\Gamma} \quad \dfrac{G \mid \vdash (\frac{1}{r_{n+1}} r_{n+1}).\Gamma}{G \mid \vdash r_{n+1}.\Gamma}\text{ T}}{G \mid \vdash r_1.\Gamma, ..., r_n.\Gamma, r_{n+1}.\Gamma}\text{ M}$$

□

2.2.2 Soundness

We need to prove that if there exists a **HR** derivation of a hypersequent G then $(\!|G|\!) \geq 0$ is derivable in equational logic (written $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$). This is done in a straightforward way by showing that each deduction rule of the system **HR** is sound. Notice that the soundness of the rules already present in **GA**$\|$ is proved in the exact same way since the scalar multiplication is mostly irrelevant in those rules.

Theorem 2.2.1 (Soundness of **HR**). *For all hypersequent G, if $\rhd_{\boldsymbol{HR}} G$ then $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$.*

Proof. By induction on the derivation of G.

- For the rule
$$\dfrac{}{\vdash}\text{ INIT}$$
The semantics of the hypersequent consisting only of the empty sequent is $(\!|\vdash|\!) = 0$ and therefore $(\!|\vdash|\!) \geq 0$, as desired.

- For the rule

$$\frac{G}{G \mid \vdash \Gamma}\ \mathrm{W}$$

the hypothesis is $(\!|G|\!) \geq 0$ so

$$\begin{aligned} (\!|G \mid \vdash \Gamma|\!) &= (\!|G|\!) \sqcup (\!|\vdash \Gamma|\!) \\ &\geq (\!|G|\!) \\ &\geq 0 \end{aligned}$$

- For the $\mathrm{C}, \mathrm{ID}, +, 0, \times$ and CAN rules, it is immediate to observe that the interpretation of the only premise and the interpretation of its conclusion are equal, therefore the result is trivial.

- For the rule

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \mathrm{S}$$

the hypothesis is $(\!|G \mid \vdash \Gamma_1, \Gamma_2|\!) \geq 0$ so according to Lemma 11, $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- = 0$. Our goal is to prove that $(\!|G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2|\!) \geq 0$. Again, using Lemma 11, we equivalently need to prove that

$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- = 0.$$

The above expression is of the form $A^- \sqcap B^-$, and since $A^- = (-A) \sqcup 0 \geq 0$ always holds for every A, it is clear that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \geq 0$. It remains therefore to show that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \leq 0$. This is done as follows:

$$\begin{aligned} (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- &\leq (\!|G|\!)^- \sqcap 2.(\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- && \text{since } (\!|\vdash \Gamma_1 \mid \vdash \Gamma_2|\!)^- \geq 0 \\ &= (\!|G|\!)^- \sqcap (2.((\!|\vdash \Gamma_1|\!) \sqcup (\!|\vdash \Gamma_2|\!)))^- && \text{Lemma 10[1]} \\ &\leq (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!) + (\!|\vdash \Gamma_2|\!))^- && \text{Lemma 10[2]} \\ &= (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1, \Gamma_2|\!))^- \\ &= 0 \end{aligned}$$

- For the rule

$$\frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \mathrm{M}$$

the hypothesis is

$$(\!|G \mid \vdash \Gamma_1|\!) \geq 0$$
$$(\!|G \mid \vdash \Gamma_2|\!) \geq 0$$

so according to Lemma 11,

$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1|\!)^- = 0$$
$$(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_2|\!)^- = 0$$

Following the same reasoning of the previous case (S rule) our goal is to show that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- \leq 0$. This is done as follows:

$$\begin{aligned} (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1, \Gamma_2|\!)^- &= (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!) + (\!|\vdash \Gamma_2|\!))^- \\ &\leq (\!|G|\!)^- \sqcap ((\!|\vdash \Gamma_1|\!)^- + (\!|\vdash \Gamma_2|\!)^-) && \text{Lemma 10[3]} \\ &\leq (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_1|\!)^- + (\!|G|\!)^- \sqcap (\!|\vdash \Gamma_2|\!)^- && \text{distributivity of } \sqcap \text{ over } + \end{aligned}$$

- For the rule

$$\frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma}\ \mathrm{T}$$

the hypothesis is $(\!|G \mid \vdash r.\Gamma|\!) \geq 0$ so using Lemma 11, we have

$$(\!|G|\!)^- \sqcap r.((\!|\vdash \Gamma|\!))^- = (\!|G|\!)^- \sqcap ((\!|\vdash r.\Gamma|\!))^- = 0$$

By the same reasoning as for the S rule's case, our goal is to show that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^- \leq 0$. To do so, we need to distinguish between two cases: whether or not $r \geq 1$.

If $r \geq 1$, then

$$\begin{array}{rcl} (\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^- & \leq & (\!|G|\!)^- \sqcap r.(\!|\vdash \Gamma|\!)^- \\ & = & 0 \end{array}$$

Otherwise, Lemma 10[4] states that $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^- \leq 0$ if and only if $r.((\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^-) \leq 0$, which is proven as follows:

$$\begin{array}{rcl} r.((\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^-) & = & (r.(\!|G|\!)^-) \sqcap (r.(\!|\vdash \Gamma|\!)^-) \\ & \leq & (\!|G|\!)^- \sqcap (r.(\!|\vdash \Gamma|\!)^-) \\ & = & 0 \end{array}$$

In both cases $(\!|G|\!)^- \sqcap (\!|\vdash \Gamma|\!)^- \leq 0$.

- For the rule

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)}\ \sqcup$$

the hypothesis is $(\!|G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B|\!) \geq 0$. So :

$$\begin{array}{rcll} (\!|G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)|\!) & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.(A \sqcup B)|\!) & \\ & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.A|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.B|\!) & \text{distributivity of } \sqcup \text{ over } + \\ & \geq & 0 & \end{array}$$

- For the rule

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \quad G \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)}\ \sqcap$$

the hypothesis is

$$(\!|G \mid \ \Gamma, \vec{r}.A \vdash \Gamma|\!) \geq 0$$

$$(\!|G \mid \ \Gamma, \vec{r}.B \vdash \Gamma|\!) \geq 0$$

So

$$\begin{array}{rcll} (\!|G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)|\!) & = & (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.(A \sqcap B)|\!) & \\ & = & (\!|G|\!) \sqcup ((\!|\vdash \Gamma, \vec{r}.A|\!) \sqcap (\!|\Gamma, \vec{r}.B|\!)) & \text{distributivity of } \sqcap \text{ over } + \\ & = & ((\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.A|\!)) \sqcap ((\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.B|\!)) & \text{distributivity of } \sqcup \text{ over } \sqcap \\ & \geq & 0 & \end{array}$$

□

2.2.3 Completeness

In order to prove the completeness of the system **HR**, i.e. that if $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ then $\rhd_{\mathbf{HR}} G$, we first prove an equivalent result (Lemma 47 below) stating that if $\mathcal{A}_{\text{Riesz}} \vdash A = B$ then the hypersequents $\vdash r.A, r.\overline{B}$ and $\vdash r.B, r.\overline{A}$ are both derivable for all $r > 0$.

From Lemma 47 one indeed obtains Theorem 2.2.2 as a corollary.

Theorem 2.2.2 (Completeness of **HR**)**.** *For all hypersequent G, if $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ then $\rhd_{\boldsymbol{HR}} G$.*

Proof. Recall that $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ is a shorthand for $\mathcal{A}_{\text{Riesz}} \vdash 0 = (\!|G|\!) \sqcap 0$. Hence, from the hypothesis $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ we can deduce, by using Lemma 47 stated below, that $\rhd_{\mathbf{HR}} \vdash 1.(0 \sqcap (\!|G|\!)), 1.0$.

From this we can show that $\rhd_{\mathbf{HR}} G$ by invoking Lemma 44. Indeed, if G is $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ then $(\!|G|\!) = (\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)$ and

1. by using the invertibility of the 0 rule, $\vdash 1.(0 \sqcap ((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)))$ is derivable,
2. by using the invertibility of the $\sqcap$ rule, $\vdash 1.((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!))$ is derivable,
3. by using the invertibility of the $\sqcup$ rule $n-1$ times, $\vdash 1.(\!|\vdash \Gamma_1|\!) \mid \ ... \mid \vdash 1.(\!|\vdash \Gamma_n|\!)$ is derivable,
4. and finally, by using the invertibility of the $+$ rule and $\times$ rule, $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ is derivable.

□

Remark 22*.* We could simply have used the following weaker formulation for Lemma 47 stated below:

$$\text{If } \mathcal{A}_{\text{Riesz}} \vdash A = B \text{ then } \vdash 1.A, 1.\overline{B} \text{ and } \vdash 1.B, 1.\overline{A} \text{ are provable}$$

However, to prove this version of the lemma, we would have had to use the T rule, and thus we would not be able to show that the T rule is admissible in **HR**.

Lemma 47. *If $\mathcal{A}_{\text{Riesz}} \vdash A = B$ then $\vdash r.A, r.\overline{B}$ and $\vdash r.B, r.\overline{A}$ are derivable for all $r > 0$.*

Proof. We prove this result by induction on the derivation, in equational logic (see Definition 1.1.2) of $\mathcal{A}_{\text{Riesz}} \vdash A = B$.

- If the derivation finishes with
$$\frac{}{\mathcal{A}_{\text{Riesz}} \vdash A = A} \text{ refl}$$
we can conclude with Lemma 42.

- If the derivation finishes with
$$\frac{\mathcal{A}_{\text{Riesz}} \vdash B = A}{\mathcal{A}_{\text{Riesz}} \vdash A = B} \text{ sym}$$
then the induction hypothesis allows us to conclude.

- If the derivation finishes with
$$\frac{\mathcal{A}_{\text{Riesz}} \vdash A = C \quad \mathcal{A}_{\text{Riesz}} \vdash C = B}{\mathcal{A}_{\text{Riesz}} \vdash A = B} \text{ trans}$$
then the induction hypothesis is
$$\rhd_{\mathbf{HR}} \vdash r.A, r.\overline{C}$$

$$\triangleright_{\mathbf{HR}} \vdash r.C, r.\overline{A}$$
$$\triangleright_{\mathbf{HR}} \vdash r'.C, r'.\overline{B}$$
$$\triangleright_{\mathbf{HR}} \vdash r'.B, r'.\overline{C}$$

for all $r, r' > 0$. We will show that $\triangleright_{\mathbf{HR}} \vdash r.A, r.\overline{B}$ for all r, the other one is similar.

$$\dfrac{\dfrac{\vdash r.A, r.\overline{C} \quad \vdash r.C, r.\overline{B}}{\vdash r.A, r.\overline{B}, r.C, r.\overline{C}}\ \text{M}}{\vdash r.A, r.\overline{B}}\ \text{CAN}$$

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{Riesz}} \vdash A = B}{\mathcal{A}_{\text{Riesz}} \vdash A[C/x] = B[C/x]}\ \text{subst}$$

  we conclude using the induction hypothesis and Lemma 43.

- If the derivation finishes with

$$\dfrac{\mathcal{A}_{\text{Riesz}} \vdash A = B}{\mathcal{A}_{\text{Riesz}} \vdash C[A] = C[B]}\ \text{ctxt}$$

  we prove the result by induction on C. For instance, if $C = sC'$ with $s > 0$, then the induction hypothesis is $\triangleright_{\mathbf{HR}} \vdash r.C'[A], r.\overline{C'[B]}$ and $\triangleright_{\mathbf{HR}} \vdash r.C'[B], r.\overline{C'[A]}$ for all $r > 0$ so

$$\dfrac{\vdash rs.C'[A], rs.\overline{C'[B]}}{\vdash r.C[A], r.\overline{C[B]}}\ \times^* \qquad\qquad \dfrac{\vdash rs.C'[B], rs.\overline{C'[A]}}{\vdash r.C[B], r.\overline{C[A]}}\ \times^*$$

- It now remains to consider the cases when the derivation finishes with one of the axioms of Figure 1.2. We only show the nontrivial cases.

  – If the derivation finishes with

$$\dfrac{}{\mathcal{A}_{\text{Riesz}} \vdash (r_1 + r_2)x = r_1x + r_2x}\ \text{ax}$$

  then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash (r_1 + r_2)r.x, r_1r.\overline{x}, r_2r.\overline{x}}\ \text{ID}}{\vdash r.((r_1 + r_2)x), r.r_1\overline{x}, r.r_2\overline{x}}\ \times^*}{\vdash r.((r_1 + r_2)x), r.(r_1\overline{x} + r_2\overline{x})}\ +$$

  and

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash r_1r.x, r_2r.x, (r_1 + r_2)r.\overline{x}}\ \text{ID}}{\vdash r.r_1x, r.r_2x, r.((r_1 + r_2)\overline{x})}\ \times^*}{\vdash r.(r_1x + r_2x), r.((r_1 + r_2)\overline{x})}\ +$$

– If the derivation finishes with

$$\dfrac{}{\mathcal{A}_{\text{Riesz}} \vdash (s(x \sqcap y)) \sqcap sy = s(x \sqcap y)}\ \text{ax}$$

then

$$\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash r.(s(x \sqcap y)), r.s(\overline{x} \sqcup \overline{y})}\ \text{Lemma 42} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash rs.y, rs.\overline{y}}\ \text{ID}}{\vdash rs.y, rs.\overline{x} \mid \vdash rs.y, rs.\overline{y}}\ \text{W}}{\vdash rs.y, rs.(\overline{x} \sqcup \overline{y})}\ \sqcup}{\vdash r.sy, r.s(\overline{x} \sqcup \overline{y})}\ \times^*}{\vdash r.((s(x \sqcap y)) \sqcap sy), r.s(\overline{x} \sqcup \overline{y})}\ \sqcap$$

and

$$\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash r.(s(x \sqcap y)), r.(s(\overline{x} \sqcup \overline{y}))}\ \text{Lemma 42}}{\vdash r.(s(x \sqcap y)), r.((s(\overline{x} \sqcup \overline{y})) \sqcup (s\overline{y}))}\ \sqcup - \text{W}$$

□

Remark 23. By inspecting the proof of Lemma 47 it is possible to verify that the T rule is never used in the construction of $\triangleright_{\mathbf{HR}} G$. This, together with the similar Remark 21 regarding Lemma 44, implies that the T rule is never used in the proof of the completeness Theorem 2.2.2. From this we get the following corollary.

Corollary 2. *The T rule is admissible in the system* ***HR****.*

It turns out, however, that there is no hope of eliminating both the T rule and the CAN rule from the **HR** system.

Lemma 48. *Let r_1 and r_2 be two irrational numbers that are incommensurable (so there is no $q \in \mathbb{Q}$ such that $qr_1 = r_2$). Then the atomic hypersequent G*

$$\vdash r_1.x \mid \vdash r_2.\overline{x}$$

does not have a CAN–free and T–free derivation.

Proof. This is a corollary of the next Lemma 49. The idea is that in the **HR** system without the T rule and the CAN rule, the only way to derive G is by applying the structural rules S, C, W, M and the ID rule. Each of these rules can be seen as adding up the sequents in G or multiplying them up by a positive natural number, very much like the Algebraic Property 2.1.5 of system **GA**‖. Since r_1 and r_2 are incommensurable, it is not possible to construct a derivation. □

Lemma 49. *For all atomic hypersequents G, built using the variables and negated variables $x_1, \overline{x_1}, \dots, x_k, \overline{x_k}$, of the form*

$$\vdash \Gamma_1 \mid \ \dots \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_{i,k}}$, the following are equivalent:

1. *G has a CAN–free and T–free derivation.*
2. *there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $n_i \neq 0$, i.e., the numbers are not all 0's, and*

- *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

$$\sum_{i=1}^{m} n_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} n_i(\sum \vec{s}_{i,j})$$

i.e., the scaled (by the numbers $n_1 \ldots n_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in front of the covariable $\overline{x_j}$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the derivation of G. We show only the M case, the other cases being trivial:

- If the derivation finishes with

$$\frac{\vdash \Gamma_1 \mid ... \mid\vdash \Gamma_m \quad \vdash \Gamma_1 \mid ... \mid\vdash \Gamma'_m}{\vdash \Gamma_1 \mid ... \mid\vdash \Gamma_m, \Gamma'_m}\ \text{M}$$

by induction hypothesis, there are $n_1, ..., n_m \in \mathbb{N}$ and $n'_1, ..., n'_m \in \mathbb{N}$ such that :

  - there exists $i \in [1..m]$ such that $n_i \neq 0$.
  - for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_i n_i. \sum \vec{r}_{i,j} = \sum_i n_i. \sum \vec{s}_{i,j}$.
  - there exists $i \in [1..m]$ such that $n'_i \neq 0$.
  - for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=0}^{m-1} n'_i. \sum \vec{r}_{i,j} + n'_m. \sum \vec{r'}_{m,j} = \sum_{i=0}^{m-1} n'_i. \sum \vec{s}_{i,j} + n'_m. \sum \vec{s'}_{m,j}$.

If $n_m = 0$ then $n_1, ..., n_{m-1}, 0$ satisfies the property.
Otherwise if $n'_m = 0$ then $n'_1, ..., n'_{m-1}, 0$ satisfies the property.
Otherwise, $n_m.n'_1 + n'_m.n_1, n_m.n'_2 + n'_m.n_2, ..., n_m.n'_{m-1} + n'_m.n_{m-1}, n_m.n'_m$ satisfies the property.

The other way ((2) $\Rightarrow$ (1)) is more straightforward. If there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $n_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} n_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} n_i(\sum \vec{s}_{i,j})$$

then we can use the W rule to remove the sequents corresponding to the numbers $n_i = 0$, and use the C rule $n_i - 1$ times then the S rule $n_i - 1$ times on the ith sequent to multiply it by n_i. If we assume that there is a natural number l such that $n_i = 0$ for all $i > l$ and $n_i \neq 0$ for all $i \leq l$, then the CAN–free T–free derivation is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash}\ \text{INIT}}{\vdash {\Gamma_1}^{n_1}, \ldots, {\Gamma_l}^{n_l}}\ \text{ID}^*}{\vdash {\Gamma_1}^{n_1} \mid \ldots \mid \vdash {\Gamma_l}^{n_l}}\ \text{S}^*}{\vdash \Gamma_1 \mid \ldots \mid \vdash \Gamma_l}\ \text{C-S}^*}{\vdash \Gamma_1 \mid \ldots \mid \Gamma_m}\ \text{W}^*$$

where Γ^n stands for $\underbrace{\Gamma, \ldots, \Gamma}_{n}$. □

2.2.4 CAN–free invertibility

In this section, we prove the $0, +, \times, \sqcup$ and $\sqcap$ rules are CAN-free invertible, i.e., that if the conclusion of one of those logical rules has a CAN-free derivation, then so do the premises. As for the system **GA**$\|$, it allows us to reduce the complexity of the formulas in an hypersequent in the proof of the CAN elimination theorem, and thus it is important that we do not add any CAN rule in the proofs of invertibility. For this reason, the CAN–free invertibility result is stronger than Lemma 44 of Section 2.2.1. Note that those results are the ones requiring the T rule. Indeed, if we were able to prove the CAN-free invertibility of all logical rules without using the T rule, all other results can also be proved without using the T rule.

Like in the previous Section 2.1.4, we will prove the CAN-free invertibility of more general rules as those rules are more convenient for the induction process.

Logical rules:

$$\frac{[\vdash \Gamma_i]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.0]_{i=1}^n}\ 0 \qquad \frac{[\vdash \Gamma_i, \vec{r_i}.A, \vec{r_i}.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.(A+B)]_{i=1}^n}\ + \qquad \frac{[\vdash \Gamma_i, (s\vec{r_i}).A]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.(sA)]_{i=1}^n}\ \times$$

$$\frac{[\vdash \Gamma_i, \vec{r_i}.A \mid \vdash \Gamma_i, \vec{r_i}.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.(A \sqcup B)]_{i=1}^n}\ \sqcup \qquad \frac{[\vdash \Gamma_i, \vec{r_i}.A]_{i=1}^n \quad [\vdash \Gamma_i, \vec{r_i}.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=1}^n}\ \sqcap$$

Figure 2.8: Generalised logical rules

Notice again that the rules $0, +, \sqcup, \sqcap$ of Figure 2.8 are exactly the ones of Figure 2.4 with the translation $A^n \leftrightarrow \vec{r}.A$ (with the addition of the $\times$ rule). The proofs of the following lemmas are then very similar to the ones in Section 2.1.4 modulo this translation.

We conceptually divide the logical rules in three categories:

Type 1 The rule with only one premise but that adds one sequent to the hypersequent: the $\sqcup$ rule.

Type 2 The rules with only one premise and that do not change the number of sequents: the $0, +$ rules.

Type 3 The rule with two premises: the $\sqcap$ rule.

Because of the similarities of the rules in each of these categories, we just prove the CAN–free invertibility of one rule in each category by means of a sequence of lemmas.

Lemma 50 (Type 1). *If* $[\vdash \Gamma_i, \vec{r_i}.(A \sqcup B)]_{i=1}^n$ *has a CAN-free derivation then* $[\vdash \Gamma_i, \vec{r_i}.A \mid \vdash \Gamma_i, \vec{r_i}.B]_{i=1}^n$ *has a CAN–free derivation.*

Proof. By induction on the derivation of $[\vdash \Gamma_i, \vec{r_i}.(A \sqcup B)]_{i=1}^n$. We show the two cases that require the T rule.

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma_1, \vec{r_1}.(A \sqcup B) \quad G \mid \vdash \Gamma_2, \vec{r_2}.(A \sqcup B)}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r_1}.(A \sqcup B), \vec{r_2}.(A \sqcup B)}\ \text{M}$$

with $G = [\vdash \Gamma_i, \vec{r_i}.(A \sqcup B)]_{i=3}^n$ and $G' = [\vdash \Gamma_i, \vec{r_i}.A \mid \vdash \Gamma_i, \vec{r_i}.B]_{i=3}^n$ then by induction hypothesis on the CAN–free derivations of the premises we have that

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_1, \vec{r_1}.B$$

and

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_2, \vec{r_2}.B$$

are derivable by CAN–free derivations. We want to prove that both

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_2, \vec{r_2}.B$$

and

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \vec{r_1}.B$$

are CAN–free derivable, as this will allow us to conclude by application of the M rule as follows:

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_2, \vec{r_2}.B \quad G \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_2, \vec{r_2}.B}{G \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r_1}.B, \vec{r_2}.B}\,\mathrm{M} \quad \dfrac{G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \vec{r_1}.B \quad G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_2, \vec{r_2}.B}{G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r_1}.B, \vec{r_2}.B}\,\mathrm{M}}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r_1}.A, \vec{r_2}.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r_1}.B, \vec{r_2}.B}\,\mathrm{M}$$

If $\vec{r_1} = \emptyset$ or $\vec{r_2} = \emptyset$, those two hypersequents are derivable using the C rule then the W rule.
Otherwise, by using the W rule, Lemma 46 and the M rule, we have

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r_1}.A \mid \vdash \Gamma_2, \vec{r_2}.B \mid \vdash \vec{r_2}.\Gamma_1, \vec{r_1}.\Gamma_2, (\vec{r_1}\vec{r_2})A, (\vec{r_1}\vec{r_2})B$$

and

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \vec{r_1}.B \mid \vdash \vec{r_2}.\Gamma_1, \vec{r_1}.\Gamma_2, (\vec{r_1}\vec{r_2})A, (\vec{r_1}\vec{r_2})B$$

We can then conclude using the S rule, Lemma 45 and the C rule.

- if the derivation finishes with an application on the $\sqcup$ rule acting on the formula $A \sqcup B$, we need to carefully analyse where the $A \sqcup B$ formulas appear. There are three cases:
  - the formulas $A \sqcup B$ active in the rule, but which are not under consideration in the lemma, i.e., there are not the instances of $A \sqcup B$ we want to reduce, which are $\vec{a}.(A \sqcup B)$ below,
  - the formulas $A \sqcup B$ which are both active in the rule, and under consideration in the lemma, which are $\vec{b}.(A \sqcup B)$ below, and
  - the formulas $A \sqcup B$ which are not active in the rule but under consideration in the lemma, which are $\vec{c}.(A \sqcup B)$ below.

Thus, the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma_1, \vec{a}; \vec{b}.A, \vec{c}.(A \sqcup B) \mid \vdash \Gamma_1, \vec{a}; \vec{b}.B, \vec{c}.(A \sqcup B)}{G \mid \vdash \Gamma_1, \vec{a}; \vec{b}.(A \sqcup B), \vec{c}.(A \sqcup B)}\,\sqcup$$

with $G = [\vdash \Gamma_i, \vec{r_i}.(A \sqcup B)]_{i=2}^n$, $G' = [\vdash \Gamma_i, \vec{r_i}.A \mid \vdash \Gamma_i, \vec{r_i}.B]_{i=2}^n$ and $\vec{r_1} = \vec{b}; \vec{c}$ and we want to derive

$$G' \mid \vdash \Gamma_1, \vec{a}.(A \sqcup B), \vec{b}; \vec{c}.A \mid \vdash \Gamma_1, \vec{a}.(A \sqcup B), \vec{b}; \vec{c}.B$$

We note $\Gamma_{\alpha_1,\alpha_2,\alpha_3} = \Gamma_1, \vec{a}.\alpha_1, \vec{b}.\alpha_2, \vec{c}.\alpha_3$ for $\alpha_i \in \{A, B\}$. For instance, $\Gamma_{A,B,B} = \Gamma_1, \vec{a}.A, \vec{b}.B, \vec{c}.B$. Notice that $\vec{b}.\Gamma_{\alpha_1,\alpha_2,\alpha_2}, \vec{c}.\Gamma_{\alpha_1,\alpha_3,\alpha_3} = \vec{b};\vec{c}.\Gamma_{\alpha_1,\alpha_2,\alpha_3}$.

The induction hypothesis is

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,A} \mid \vdash \Gamma_{B,B,B}$$

Then the derivation is

$$\begin{array}{cl}
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,A} & \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \vec{b};\vec{c}.\Gamma_{B,B,A} & \text{Lemma 46} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid\ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \vec{b};\vec{c}.\Gamma_{B,B,A} & \text{W} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid\ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \vec{b}.\Gamma_{B,B,B} \mid \vdash \vec{c}.\Gamma_{B,A,A} & \text{S} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid\ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{B,A,A} & (\text{Lemma 45})^2 \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,B} & \text{C}^2 \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid\ \Gamma_{B,A,A} \mid \vdash \Gamma_{B,B,B} \mid \vdash \vec{b};\vec{c}.\Gamma_{A,A,B} & \text{Lemma 46} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \vec{b};\vec{c}.\Gamma_{A,A,B} & \text{W} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \vec{b}.\Gamma_{A,A,A} \mid \vdash \vec{c}.\Gamma_{A,B,B} & \text{S} \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{A,B,B} & (\text{Lemma 45})^2 \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_{A,B,B} \mid \vdash \Gamma_{B,B,B} & \text{C}^2 \\ \hline
G' \mid \vdash \Gamma_{A,A,A} \mid \vdash \Gamma_{B,A,A} \mid \vdash \Gamma_1, \vec{a}.(A \sqcup B), \vec{b};\vec{c}.B & \sqcup \\ \hline
G' \mid \vdash \Gamma_1, \vec{a}.(A \sqcup B), \vec{b};\vec{c}.A \mid \vdash \Gamma_1, \vec{a}.(A \sqcup B), \vec{b};\vec{c}.B & \sqcup
\end{array}$$

□

Lemma 51 (Type 2)**.** *If* $[\vdash \Gamma_i, \vec{r}_i.(A+B)]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=1}^n$ *has a CAN–free derivation.*

Proof. Straightforward induction on the derivation of $[\vdash \Gamma_i, \vec{r}_i.(A + B)]_{i=1}^n$. For instance if the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma_1, \vec{r}_1.(A + B) \quad G \mid \vdash \Gamma_2, \vec{r}_2.(A + B)}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.(A + B), \vec{r}_2.(A + B)}\ \text{M}$$

with $G = [\vdash \Gamma_i, \vec{r}_i.(A + B)]_{i=3}^n$ and $G' = [\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=3}^n$, then by induction hypothesis on the CAN–free derivations of the premises we have that

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B$$

and

$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r}_2.A, \vec{r}_2.B$$

so

$$\dfrac{G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B \quad G' \mid \vdash \Gamma_2, \vec{r}_2.A, \vec{r}_2.B}{G' \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.A, \vec{r}_2.A, \vec{r}_1.B, \vec{r}_2.B}\ \text{M}$$

□

Lemma 52 (Type 3)**.** *If* $[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n$ *and* $[\vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n$ *have a CAN–free derivation.*

Proof. A straightforward induction on the derivation of $[\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=1}^n$. We will show the only case requiring the T rule, i.e., the $\sqcap$ rule acting on $A \sqcap B$.

As previously, we need to distinguish three types of $A \sqcap B$ formulas:

- the formulas $A \sqcap B$ active in the rule, but which are not under consideration in the lemma, i.e., there are not the instances of $A \sqcap B$ we want to reduce, which are $\vec{a}.(A \sqcap B)$ below,
- the formulas $A \sqcap B$ which are both active in the rule and under consideration in the lemma, which are $\vec{b}.(A \sqcap B)$ below, and
- the formulas $A \sqcap B$ which are not active in the rule but under consideration in the lemma, which are $\vec{c}.(A \sqcap B)$ below.

Thus the derivation finishes with

$$\dfrac{G \mid \vdash \Gamma_1, \vec{a}; \vec{b}.A, \vec{c}.(A \sqcap B) \quad G \mid \vdash \Gamma_1, \vec{a}; \vec{b}.B, \vec{c}.(A \sqcap B)}{G \mid \vdash \Gamma_1, \vec{a}; \vec{b}.(A \sqcap B), \vec{c}.(A \sqcap B)} \sqcap$$

with $G = [\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=2}^n$ and $\vec{r_1} = \vec{b}; \vec{c}$.

We will show how to derive

$$G' \mid \vdash \Gamma_1, \vec{a}.(A \sqcap B), \vec{b}; \vec{c}.A$$

where $G' = [\vdash \Gamma_i, \vec{r_i}.A]_{i=2}^n$, the other case is similar.

By using the induction hypothesis, we have that

$$\triangleright_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{a}; \vec{b}.A, \vec{c}.A$$

$$\triangleright_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{a}; \vec{b}.B, \vec{c}.A$$

We will now derive the hypersequent $G' \mid \vdash \Gamma_1, \vec{a}.B, \vec{b}; \vec{c}.A$ which will allow us to conclude using the $\sqcap$ rule.

$$\dfrac{\dfrac{\dfrac{G' \mid \vdash \Gamma_1, \vec{a}; \vec{b}.B, \vec{c}.A}{G' \mid \vdash \vec{a}.\Gamma_1, (\vec{a}; \vec{b})\vec{a}.B, \vec{a}\vec{c}.A} \text{ Lemma 46} \quad \dfrac{G' \mid \vdash \Gamma_1, \vec{a}; \vec{b}; \vec{c}.A}{G' \mid \vdash \vec{b}.\Gamma_1, \vec{b}(\vec{a}; \vec{b}; \vec{c}).A} \text{ Lemma 46}}{G' \mid \vdash \vec{a}; \vec{b}.\Gamma_1, (\vec{a}; \vec{b})\vec{a}.B, (\vec{b}; \vec{c})(\vec{a}; \vec{b}).A} \text{ M}}{G' \mid \vdash \Gamma_1, \vec{a}.B, \vec{b}; \vec{c}.A} \text{ Lemma 45}$$

□

2.2.5 M–elimination

In this section, we will show the M elimination theorem. Recall that the M elimination theorem states

if a hypersequent G is derivable, then it has a M–free derivation.

However, since this result will be used in the proof of the CAN elimination theorem, we have to ensure that the M elimination theorem does not add any instance of the CAN rule. Thus we will show the slightly different result

if a hypersequent G is CAN–free derivable, then it has a CAN–free M–free derivation.

To do so, we will show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free derivations d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there exists also a CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$.

Recall that the idea of the proof is to combine d_1 and d_2 step-by-step. First we take the derivation d_1 and we modify it into a CAN–free and M–free prederivation (i.e., an open derivation) of

$$G \mid \ G \mid \vdash \Gamma, \Delta$$

where all the leaves in the prederivation are either terminated (by the INIT axiom) or non–terminated and of the form:

$$G \mid \vdash \vec{r}.\Delta$$

for some vector $\vec{r}$ of scalars. Then we use the derivation d_2 to construct a CAN–free and M–free derivation of each

$$G \mid \vdash \vec{r}.\Delta$$

hence completing the prederivation of

$$G \mid \ G \mid \vdash \Gamma, \Delta$$

into a full derivation. From this it is possible to obtain the desired CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$ using several times the C rule:

$$\frac{G \mid \ G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta} \text{ C}^*$$

In what follows, the first step is formalised as Lemma 53 and the second step as Lemma 54.

Lemma 53. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ and let H be a hypersequent and Δ be a sequent. Then there exists a CAN–free M–free prederivation of*

$$G \mid \ H \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are of the form $H \mid \vdash \vec{r}.\Delta$ for some vector $\vec{r}$.

Proof. This is an instance of the slightly more general statement of Lemma 56 below where:

- $[\vdash \Gamma_i]_{i=1}^{n-1} = G$ and $\Gamma_n = \Gamma$.
- $\vec{r}_i = \emptyset$ for $1 \leq i < n$ and $\vec{r}_n = 1$.

□

Lemma 54. *Let d_2 be CAN–free and M–free derivation of $H \mid \vdash \Delta$. Then, for every vector $\vec{r}$, there exists a CAN–free and M–free derivation of*

$$H \mid \vdash \vec{r}.\Delta$$

Proof. This is an instance of the slightly more general statement of Lemma 57 below where:

- $[\vdash \Delta_i]_{i=1}^{n-1} = G$ and $\Delta_n = \Delta$.
- $\vec{r}_i = 1$ for $1 \leq i < n$ and $\vec{r}_n = \vec{r}$.

□

Before proving Lemmas 56 and57, we show how to remove one instance of the M rule and then the M-elimination theorem.

Lemma 55. *If $G \mid \vdash \Gamma$ and $H \mid \vdash \Delta$ have CAN–free M–free derivations, then so does $G \mid\ H \mid \vdash \Gamma, \Delta$.*

Proof. By using Lemma 53, we have a prederivation of $G \mid\ H \mid \vdash \Gamma, \Delta$ where all non–terminated leaves are of the form $H \mid \vdash \vec{r}.\Delta$ for some vector $\vec{r}$.

To conclude, we have to show that every non-terminated leaf is derivable, which can be done using Lemma 54. □

Theorem 2.2.3 (M elimination)**.** *If G is CAN-free derivable, then G is CAN-free M-free derivable.*

Proof. We prove the result by induction on the derivation of G. The only interesting case if the M rule, i.e., if the derivation finishes with

$$\frac{G \mid \vdash \Gamma \quad G \mid \vdash \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{M}$$

then by induction hypothesis, $G \mid \vdash \Gamma$ and $G \mid \vdash \Delta$ have a CAN-free M-free derivation.

By using Lemma 55, we have a CAN-free M-free derivation of $G \mid\ G \mid \vdash \Gamma, \Delta$. The derivation is then

$$\frac{G \mid\ G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{C}^*$$

□

Lastly, we prove the technical version of Lemmas 53 and 54.

Lemma 56. *Let d_1 be a CAN–free and M–free derivation of $[\vdash \Gamma_i]_{i=1}^n$ and let H be a hypersequent and Δ be a sequent. Then for every sequence of vectors $\vec{r_i}$, there exists a CAN–free M–free prederivation of*

$$H \mid\ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^n$$

where all non-terminated leaves are of the form $H \mid\ \vec{r}.\Delta$ for some vector $\vec{r}$.

Proof. By induction on d_1. We will show the case of the INIT axiom and the S rule, the other cases are similar.

- If the derivation finishes with

$$\frac{}{\vdash}\ \text{INIT}$$

  Let n be a natural number. The prederivation of $H \mid \vdash \vec{r_1}.\Delta$ is simply the leaf

$$H \mid \vdash \vec{r_1}.\Delta$$

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k}{[\vdash \Gamma_i]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1} \mid \vdash \Gamma_k}\ \text{S}$$

  Let n_i be a sequence of natural numbers. By induction hypothesis, there exists a CAN–free M–free prederivation of

$$H \mid\ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k, (\vec{r}_{k-1}; \vec{r}_k).\Delta$$

where all non-terminated leaves are of the form $H \mid \Delta^n$ for some n. We then continue this prederivation with

$$\frac{H \mid [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \Gamma_k, (\vec{r}_{k-1}; \vec{r_k}).\Delta}{H \mid [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^{k-2} \mid \vdash \Gamma_{k-1}, \vec{r}_{k-1}.\Delta \mid \vdash \Gamma_k, \vec{r_k}.\Delta} \text{ S}$$

□

Lemma 57. *If* $[\vdash \Delta_i]_{i=1}^n$ *has a CAN–free M–free derivation then for all* $\vec{r_i}$*, there is a CAN–free M–free derivation of* $[\vdash \vec{r_i}.\Delta_i]_{i=1}^n$.

Proof. By induction on the derivation of $[\vdash \Delta_i]_{i=1}^n$. We show the only nontrivial case:

- If the derivation finishes with

$$\frac{[\vdash \Delta_i]_{i=3}^n \mid \vdash \Delta_1, \Delta_2}{[\vdash \Delta_i]_{i=3}^n \mid \vdash \Delta_1 \mid \vdash \Delta_2} \text{ S}$$

  By induction hypothesis there is CAN–free derivation of

$$[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1, (\vec{r_1}\vec{r_2}).\Delta_2$$

  If $\vec{r_1} = \emptyset$ or $\vec{r_2} = \emptyset$, we have the empty sequent which is derivable. Otherwise,

$$\cfrac{\cfrac{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1, (\vec{r_1}\vec{r_2}).\Delta_2}{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1 \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_2} \text{ S}}{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash \vec{r_1}.\Delta_1 \mid \vdash \vec{r_2}.\Delta_2} \text{ Lemma 45}$$

□

Remark 24. The careful readers may notice that the T rule is used in this proof, which contradicts with our statement in the introduction of Section 2.2.4 where we claim that the T rule would not be necessary in this section. However, if we carefully inspect the proof of the lemmas, we notice that if both derivations of $G \mid \vdash \Gamma$ and $G \mid \vdash \Delta$ are *T–free* M–free, then all scalars in the vectors $\vec{r_i}$ are equal to 1 and thus the resulting derivation of $G \mid \vdash \Gamma, \Delta$ is also *T–free* M–free.

Thus if the CAN–free invertibility could be proven without the T rule, then the T rule would indeed not be necessary for the CAN elimination theorem.

2.2.6 CAN–elimination

The CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}}{G \mid \vdash \Gamma} \text{ CAN}, \sum \vec{r} = \sum \vec{s}$$

We prove the CAN-elimination theorem by showing that if the hypersequent $G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$ has a M–free CAN–free derivation then the hypersequent $G \mid \vdash \Gamma$ also has a M–free CAN–free derivation.

Our proof proceeds by induction on the complexity of the term A. The base case is given by $A = x$ (or equivalently $A = \overline{x}$) for some variable x. The following lemma proves this base case.

Lemma 58. *If there is a M–free CAN–free derivation of* $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$*, where* $\sum \vec{r} = \sum \vec{s}$ *then there exists a M–free CAN–free derivation of* $G \mid \vdash \Gamma$*.*

Proof. The statement follows as a special case of Lemma 60 below, a stronger version of Lemma 58 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$, where:

- $[\vdash \Gamma_i]_{i=1}^{n-1} = G$ and $\Gamma_n = \Gamma$.
- $\vec{r_i} = \vec{r'_i} = \vec{s_i} = \vec{s'_i} = \emptyset$ for $1 \leq i < n$.
- $\vec{r_n} = \vec{r}$, $\vec{s_n} = \vec{s}$ and $\vec{r'_n} = \vec{s'_n} = \emptyset$.

□

For complex terms A, we proceed by using the CAN–free invertibility lemmas of Section 2.2.4.

Lemma 59. *If there is a M–free CAN–free derivation of* $G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$ *where* $\sum \vec{r} = \sum \vec{s}$*, then there exists a M–free CAN–free derivation of* $G \mid \vdash \Gamma$*.*

Proof. We proceed by induction on A.

- If $A = x$, we are in the base case of Lemma 58.
- If $A = 0$, we can conclude with the CAN–free invertibility of the 0 rule and the M–elimination theorem.
- If $A = B + C$, since the + rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{r}.C, \vec{s}.\overline{B}, \vec{s}.\overline{C}$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = r'B$, since the $\times$ rule is CAN–free invertible, $G \mid \vdash \Gamma, (r'\vec{r}).B, (r'\vec{s}).\overline{B}$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma$ by invoking the inductive hypothesis on the simpler term B.
- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.(\overline{B} \sqcap \overline{C}) \mid \vdash \Gamma, \vec{r}.C, \vec{s}.(\overline{B} \sqcap \overline{C})$ has a M–free CAN–free derivation. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler terms B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:

  $$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ C$$

- If $A = B \sqcap C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.(B \sqcap C), \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.(B \sqcap C), \vec{s}.\overline{C}$ has a M–free CAN–free derivation. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}$ has a M–free CAN–free derivation. Therefore we can have a M–free CAN–free derivation of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler terms B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:

$$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ C$$

□

We can now prove the CAN elimination theorem by a straightforward induction on the derivation of the hypersequent, the CAN rule being dealt with using Lemma 59 above.

Theorem 2.2.4 (CAN elimination)**.** *For all hypersequent G, if $\triangleright_{\boldsymbol{HR}} G$ then $\triangleright_{\boldsymbol{HR}\backslash\{CAN\}} G$.*

Proof. We proceed by induction on the derivation of G. We only show the case of the CAN rule and one example for the other rules since the other cases are all similar.

- If the derivation finishes with:
$$\frac{G}{G \mid \vdash \Gamma}\ \mathrm{W}$$
then $\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G$ and by using the W rule, we obtain a CAN-free derivation of $G \mid \vdash \Gamma$.
- If the derivation finishes with
$$\frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}}{G \mid \vdash \Gamma}\ \mathrm{CAN}, \sum \vec{r} = \sum \vec{s}$$
then $\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$ and we can conclude with Lemma 59.

□

We now prove Lemma 60, the stronger version of Lemma 58.

Lemma 60. *If there is a CAN–free and M–free derivation of the hypersequent*

$$[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i=1}^n$$

then for all $\vec{r_i'}$ and $\vec{s_i'}$, with $1 \leq i \leq n$, such that $\sum \vec{r_i} - \sum \vec{s_i} = \sum \vec{r_i'} - \sum \vec{s_i'}$, there is a CAN–free, M–free derivation of

$$\left[\vdash \Gamma_i, \vec{r_i'}.x, \vec{s_i'}.\overline{x}\right]_{i=1}^n$$

Proof. By induction on the derivation of $[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i=1}^n$. Most cases are trivial, we just describe the most interesting one.

- If the derivation finishes with:
$$\frac{[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i\geq 2} \mid \vdash \Gamma_1, \vec{c}.x, \vec{c'}.\overline{x}}{[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{a}; \vec{b}; \vec{c}).x, (\vec{a'}; \vec{b'}; \vec{c'}).\overline{x}}\ \mathrm{ID}, \sum(\vec{a}; \vec{b}) = \sum(\vec{a'}; \vec{b'})$$
with $\vec{r_1} = \vec{b}; \vec{c}$ and $\vec{s_1} = \vec{b'}; \vec{c'}$. We want to show that
$$\triangleright_{\mathbf{HR}\backslash\{\mathrm{CAN}\}} \left[\vdash \Gamma_i, \vec{r_i'}.x, \vec{s_i'}.\overline{x}\right]_{i\geq 2} \mid\ \Gamma_1, (\vec{a}; \vec{r_1'}).x, (\vec{a'}; \vec{s_1'}).\overline{x}$$

We will now prove that $\sum \vec{c} - \sum \vec{c'} = \sum \vec{r'}_1 + \sum \vec{a} - (\sum \vec{s'}_1 + \sum \vec{a'})$ to be able to conclude with the induction hypothesis.

$$\begin{aligned}
\sum \vec{c} - \sum \vec{c'} &= (\sum \vec{r}_1 - \sum \vec{b}) - (\sum \vec{s}_1 - \sum \vec{b'}) \\
&= (\sum \vec{r}_1 - \sum \vec{s}_1) + (\sum \vec{b'} - \sum \vec{b}) \\
&= (\sum \vec{r'}_1 - \sum \vec{s'}_1) + (\sum \vec{a} - \sum \vec{a'}) \\
&= \sum \vec{r'}_1 + \sum \vec{a} - (\sum \vec{s'}_1 + \sum \vec{a'})
\end{aligned}$$

so by induction hypothesis, we have

$$\triangleright_{\mathbf{HR}\backslash\{\text{CAN}\}} \left[\vdash \Gamma_i, \vec{r'_i}.x, \vec{s'_i}.\overline{x}\right]_{i\geq 2} \mid \Gamma_1, (\vec{a}; \vec{r'_1}).x, (\vec{a'}; \vec{s'_1}).\overline{x}$$

which is the result we want.

□

2.2.7 Algebraic property

The algebraic property of the system **GA**‖ (see Section 2.1.7) can be adapted to the system **HR**. Recall that the algebraic property mostly states that to derive an atomic hypersequent, one could only use the structural rules (S,M,C,W and ID rules) and thus could only multiply the sequents by natural numbers before using the ID rule. However, in the system **HR**, there is one more structural rule, namely the T rule. Thus the algebraic property now states that we can only multiply the sequents by *real* numbers before using the ID rule.

Remark 25. The main difference between Lemma 49 and the algebraic property below is indeed the presence of the T rule. Lemma 49 requires the derivation to be T–free and thus the sequents can only be multiplied by natural numbers and not real numbers.

Theorem 2.2.5. *For all atomic hypersequents G, built using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$, of the form*

$$\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_k}$, the following are equivalent:

1. *G has a derivation.*
2. *there exist numbers $t_1, ..., t_m \in \mathbb{R}_{\geq 0}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

   $$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

   i.e., the scaled (by the numbers $t_1 \ldots t_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in front of the covariable $\overline{x_j}$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the derivation of G. By using Theorem 2.2.4, we can assume that the derivation of G is CAN–free. We will only deal with the case of T rule since every other case is similar to the proof of Theorem 2.1.5. If the derivation finishes with

$$\frac{\vdash \Gamma_1 \mid \ \dots \mid \vdash r.\Gamma_m}{\vdash \Gamma_1 \mid \ \dots \mid \vdash \Gamma_m} \text{ T}$$

then by induction hypothesis there are $t_1, ..., t_m \in \mathbb{R}$ such that :

- there exists $i \in [1..m]$ such that $t_i \neq 0$.
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=0}^{m-1} t_i. \sum \vec{r}_{i,j} + t_m. \sum r\vec{r}_{m,j} = \sum_{i=0}^{m-1} t_i. \sum \vec{s}_{i,j} + t_m. \sum r\vec{s}_{m,j}$.

so $t_1, \dots, t_{m-1}, rt_m$ satisfies the property.

The other way ((2) $\Rightarrow$ (1)) is also very similar to Theorem 2.1.5, only using the T rule instead of the C and S rules. If there exist numbers $t_1, ..., t_m \in \mathbb{R}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $t_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

then we can use the W rule to remove the sequents corresponding to the numbers $t_i = 0$, and use the T rule on the ith sequent to multiply it by t_i. If we assume that there is a natural number l such that $t_i = 0$ for all $i > l$ and $t_i \neq 0$ for all $i \leq l$, then the CAN–free derivation is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{\vdash} \text{ INIT}}{\vdash t_1.\Gamma_1, \dots, t_l.\Gamma_l} \text{ ID}^*}{\vdash t_1.\Gamma_1 \mid \ \dots \mid \vdash t_l.\Gamma_l} \text{ S}^*}{\vdash \Gamma_1 \mid \ \dots \mid \vdash \Gamma_l} \text{ T}^*}{\vdash \Gamma_1 \mid \ \dots \mid \ \Gamma_m} \text{ W}^*$$

□

2.3 Hypersequent calculus MGA

In this section we add the $\Diamond$ operator to the system **GA**$\|$ as well as the constant 1, thus obtaining the system **MGA** used to derive positive *modal* Abelian l–group terms.

We start by adapting the definitions and conventions used for the system **GA**$\|$ to the system **MGA**.

Definition 2.3.1. A *term* is a formal expression A where A is modal Abelian l–group terms in NNF.

We use the notation A^n for the multiset $A, ..., A$ consisting of n copies of A, and the notation Γ^n for the multiset $\Gamma, ..., \Gamma$ consisting of the concatenation of n copies of Γ.

Definition 2.3.2. A *sequent* is a formal expression of the form $\vdash \Gamma$.

If $\Gamma = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 2.3.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \dots | \vdash \Gamma_n$.

We use the letter G, H to range over hypersequents.

We now give two notions of "simple" hypersequents, *atomic* hypersequents already introduced in Section 2.1 and *basic* hypersequents.

Definition 2.3.4. A hypersequent is said *atomic* if it only contains atoms, i.e., formulas of the form x or $\overline{x}$.

Definition 2.3.5. A hypersequent is said *basic* if it only contains atoms (formulas of the form x or $\overline{x}$), $\Diamond$ formulas (formulas of the form $\Diamond A$), 1 formulas and $\overline{1}$ formulas.

The notion of atomic hypersequent is no longer sufficient in the presence of the $\Diamond$ operator, as we will see that is not as easy to reduce the complexity of a $\Diamond$ formula as it is for the other formulas. As a result, some of the inductive proofs will now have the following pattern: reduce the complexity of all non-$\Diamond$ formulas until we reach *basic* hypersequents, remove all the atoms then remove one $\Diamond$ application and start over until we reach a hypersequent with only 1 and $\overline{1}$ formulas.

We now describe how sequents and hypersequents can be interpreted by modal Abelian l–group terms.

Definition 2.3.6 (Interpretation)**.** We interpret sequents $\vdash \Gamma$ and hypersequents G as the modal Abelian l–group terms $(\!|\vdash \Gamma|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Sequents	$\vdash A_1, \dots, A_n$	$(\!|A_1|\!) + \dots + (\!|A_n|\!)$
Hypersequents	$\vdash \Gamma_1 | \dots | \vdash \Gamma_n$	$(\!|\vdash \Gamma_1|\!) \sqcup \dots \sqcup (\!|\vdash \Gamma_n|\!)$

Hence a sequent is interpreted as sum ($\sum$) and a hypersequent is interpreted as a join of sums ($\bigsqcup \sum$).

Example 19. The interpretation of the hypersequent:

$$\vdash x, (y \sqcap z) \mid \vdash (\overline{x} \sqcap y)$$

is the Riesz term:

$$\big(x + (y \sqcap z)\big) \sqcup \big((\overline{x} \sqcap y)\big).$$

The hypersequent calculus **MGA** is a deductive system for deriving hypersequents whose interpretation is positive, i.e., the hypersequents G such that $\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash 0 \leq (\!|G|\!)$. The rules of **MGA** consist of the rules of the system **GA**$\|$(see Figure 2.1) with the additional rules of Figure 2.9 below.

The 1 rule is quite similar to the ID rule but it reflects the axiom $0 \leq 1$ of modal Abelian l–groups, and thus the side condition expresses an inequality, rather than an equality. The $\Diamond$ rule, as we will show in the soundness and completeness theorems below (Theorem 2.3.1 and Theorem 2.3.2), is remarkably capturing in one single rule all three axioms regarding the ($\Diamond$) modality

$$\Diamond(x - y) = \Diamond(x) - \Diamond(y) \quad 0 \leq \Diamond(x \sqcup 0) \quad 1 \leq \Diamond(1)$$

Modal rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, 1^n, \overline{1}^n}\; 1, n \geq m \qquad \frac{\vdash \Gamma, 1^n}{\vdash \Diamond\Gamma, 1^n}\; \Diamond$$

Figure 2.9: Inference rules of **MGA**.

Remark 26. Note how the $\Diamond$ rule imposes strong constraints on the shape of its (single) premise and conclusion. First, both the conclusion and the premise are required to be hypersequents consisting of exactly one sequent. Furthermore, in the conclusion, all terms, except those of the form 1 need to be of the form $\Diamond A$ for some term A. These constraints determine the main difficulties when trying to adapt the proofs of Section 2.1 for the system **MGA**, but they are necessary. Indeed, for example, the following two alternative relaxed rules, while more natural looking, are in fact not sound:

$$\frac{G \mid \vdash \Gamma, 1^n}{G \mid \vdash \Diamond\Gamma, 1^n}\; \Diamond_1 \qquad\qquad \frac{\vdash \Gamma, A^n}{\vdash \Gamma, \Diamond A^n}\; \Diamond_2$$

Indeed, the hypersequent $\vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(x) \sqcup \Diamond(y)$ would be derivable (see below) while $\mathcal{A}^{\Diamond}_{\text{l-groups}} \not\models \Diamond(-x \sqcap -y) + \big(\Diamond(x) \sqcup \Diamond(y)\big) \geq 0$ or, equivalently, $\mathcal{A}^{\Diamond}_{\text{l-groups}} \not\models \Diamond(x \sqcup y) \leq \Diamond(x) \sqcup \Diamond(y)$ (see Example 10).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{x}, x}\,\text{ID}}{\vdash \overline{x}, x \mid \vdash \overline{x} \sqcap \overline{y}, y}\,\text{W} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{y}, y}\,\text{ID}}{\vdash \overline{x}, \overline{y}, x, y}\,\text{ID}}{\vdash \overline{y}, x \mid \vdash \overline{x}, y}\,\text{S} \quad \dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{y}, y}\,\text{ID}}{\vdash \overline{y}, x \mid \vdash \overline{y}, y}\,\text{W}}{\vdash \overline{y}, x \mid \vdash \overline{x} \sqcap \overline{y}, y}\,\sqcap}{\vdash \overline{x} \sqcap \overline{y}, x \mid \vdash \overline{x} \sqcap \overline{y}, y}\,\sqcap}{\vdash \overline{x} \sqcap \overline{y}, x \mid \vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(y)}\,\Diamond_1}{\vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(x) \mid \vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(y)}\,\Diamond_1}{\vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(x) \sqcup \Diamond(y)}\,\sqcup$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{x}, x}\,\text{ID}}{\vdash \overline{x}, \Diamond(x)}\,\Diamond_2}{\vdash \overline{x}, \Diamond(x) \mid \vdash \overline{x} \sqcap \overline{y}, \Diamond(y)}\,\text{W} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{y}, y}\,\text{ID}}{\vdash \overline{x}, \overline{y}, x, y}\,\text{ID}}{\vdash \overline{x}, \overline{y}, x, \Diamond(y)}\,\Diamond_2}{\vdash \overline{x}, \overline{y}, \Diamond(x), \Diamond(y)}\,\Diamond_2}{\vdash \overline{y}, \Diamond(x) \mid \vdash \overline{x}, \Diamond(y)}\,\text{S} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash \overline{y}, y}\,\text{ID}}{\vdash \overline{y}, \Diamond(y)}\,\Diamond_2}{\vdash \overline{y}, \Diamond(x) \mid \vdash \overline{y}, \Diamond(y)}\,\text{W}}{\vdash \overline{y}, \Diamond(x) \mid \vdash \overline{x} \sqcap \overline{y}, \Diamond(y)}\,\sqcap}{\vdash \overline{x} \sqcap \overline{y}, \Diamond(x) \mid \vdash \overline{x} \sqcap \overline{y}, \Diamond(y)}\,\sqcap}{\vdash (\overline{x} \sqcap \overline{y}), \Diamond(x) \sqcup \Diamond(y)}\,\sqcup}{\vdash \Diamond(\overline{x} \sqcap \overline{y}), \Diamond(x) \sqcup \Diamond(y)}\,\Diamond_2$$

Moreover, if we replace the $\Diamond$ rule with the rule:

$$\frac{\vdash \Gamma}{\vdash \Diamond\Gamma}\; \Diamond_3$$

the system is not complete as the axiom $\Diamond 1 \leq 1$ is no longer derivable.

The notion of complexity of a hypersequent introduced for the system **GA**$\|$ needs to be adapted to the particularities of the $\Diamond$ operator and the constant 1. The complexity of a hypersequent is used to ensure that we can only apply a finite number of logical rules before reaching an atomic hypersequent. To do so, we show that applying a logical rule acting on a maximal sequent in the hypersequent decreases the complexity of the hypersequent.

However, if the $\Diamond$ operator is included in the computation of the complexity of the sequents, it may not be possible to use a logical rule on a maximal sequent even if the hypersequent is not atomic nor basic. Take for instance the hypersequent

$$\vdash 1.x \sqcap y \mid \vdash 1.\Diamond x, 1.\Diamond y$$

The hypersequent is not atomic nor basic, but no logical rule can be applied on the maximal sequent $\vdash 1.\Diamond x, 1.\Diamond y$, and using the $\sqcap$ rule will not decrease the complexity of the hypersequent.

Therefore, the $\Diamond$ operator and the constant 1 must be kept out of the computation of the complexity of a term, as shown in Definition 1.3.4, to ensure that the process of applying logical rules until we reach a basic hypersequent always terminates. Finally, to make the application of a $\Diamond$ rule also decrease the complexity of the hypersequent, the modal depth is added to the complexity.

Definition 2.3.7 (Modal depth of a hypersequent)**.** We define the modal depth of a sequent, noted $d^{\Diamond}(\vdash \Gamma)$, has the maximal modal depth of a term in Γ, i.e., if $\Gamma = A_1, ..., A_n$, $d^{\Diamond}(\vdash \Gamma) = \max\limits_{i \in [1..n]} d^{\Diamond}(A_i)$.

The modal depth of a hypersequent G, noted $d^{\Diamond}(G)$, is then the maximal modal depth of the sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $d^{\Diamond}(G) = \max\limits_{i \in [1..n]} d^{\Diamond}(\vdash \Gamma_i)$.

Definition 2.3.8 (Complexity)**.** We define the complexity of a sequent $\vdash \Gamma$, noted $c(\vdash \Gamma)$, as the sum of the operators which are not under a $\Diamond$ operator or the constant 1 used in the terms of Γ (see Definition 1.3.4), i.e., if $\Gamma = A_1, ..., A_n$, $c(\vdash \Gamma) = \sum_{i=1}^{n} c^{\Diamond}(A_i)$.

The complexity of a hypersequent G, noted $c(G)$, is then defined as the triplet $c(G) = (a, b, c)$ where

- a is the modal depth of the hypersequent G, and
- b is the maximum complexity of a sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $b = \max\limits_{i \in [1..n]} c(\vdash \Gamma_i)$, and
- c is the number of sequents in G having a complexity of b, i.e., $c = \#\{\vdash \Gamma_i \mid c(\vdash \Gamma_i) = b\}$.

We say that a sequent $\vdash \Gamma$ of G is maximal if $c(\vdash \Gamma) = b$.

Remark 27. As in Remark 14, the premises of one of the $+, \times, \sqcup, \sqcap$ and 0 logical rules acting on a maximal sequent have a strictly lower complexity than the conclusion of the logical rule with regard to the lexicographic order.

The $\Diamond$ rule has a very peculiar place in the system **MGA**: informally, a **MGA** derivation can be seen as a sequence of **GA**$\|$ derivations separated by a $\Diamond$ rule as illustrated in the Figure 2.10 below.

Some results on the system **MGA** can then be proven by induction on the number of $\Diamond$ rules appearing in a branch of the derivations, which we call the *modal depth* of the derivation: the basic case is very similar to a **GA**$\|$ derivation – we just add the 1 rule and the proofs for the system **GA**$\|$ can be easily adapted to deal with this additional rule.

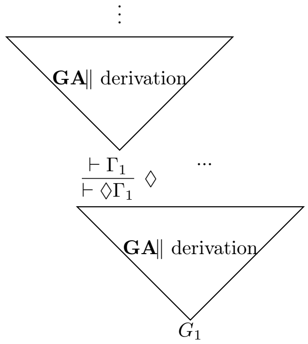


Figure 2.10: A **MGA** derivation can be seen as a sequence of **GA**‖ derivations.

Definition 2.3.9 (Modal depth of a derivation)**.** The *modal depth* of a derivation is the maximal number of $\Diamond$ rules used in a branch of the derivation.

Remark 28. Note that the modal depth of a derivation is not necessarily the same as the modal depth of the end hypersequent. Indeed, the derivation could introduce terms with $\Diamond$ operators by using the CAN rule, and thus can make the modal depth of the derivation greater than the modal depth of the end hypersequent.

2.3.1 Preliminary lemmas

As in Section 2.1, we start by proving a few technical lemmas that are used in this section.

Our first lemma states that the following variant of the ID rule (see Figure 2.1) where general terms A are considered rather than just variables, is admissible in the proof system **MGA**.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, A^n, \overline{A}^n} \text{ ID}$$

Formally, we prove the admissibility of a slightly more general rule which can act on several sequents of the hypersequent at the same time.

Lemma 61. *For all terms A*

$$\textit{if } \triangleright_{\textbf{MGA}} [\vdash \Gamma_i]_{i=1}^k \textit{ then } \triangleright_{\textbf{MGA}} [\vdash \Gamma_i, A^{n_i}, \overline{A}^{n_i}]_{i=1}^k$$

Proof. We prove the result by double induction on A and the derivation of $\triangleright_{\mathbf{MGA}}[\vdash \Gamma_i]_{i=1}^k$.

- If A is a variable, we simply use the ID rule m-times.
- If $A = 0$, we use the 0 rule k-times.
- If $A = B + C$, we use the $+$ rule $2k$-times (for $A + B$ and $\overline{A} + \overline{B}$) and conclude with the induction hypothesis.
- For the case $A = B \sqcap C$ or $A = B \sqcup C$, we first use the $\sqcap$ rule $2^n - 1$ times – one time on the conclusion, then again on the two premises, then on the four premises and so forth until we used the $\sqcap$–rule for all sequents – and then the $\sqcup$ rule n times on each premise and

the W rule n times on each premise to remove the sequents with both B and C in them. We can then conclude with the induction hypothesis.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\vdash \Gamma_i]_{i=0}^n}{[\vdash \Gamma_i, B^{n_i}, \overline{B}^{n_i}]_{i=0}^k \mid [\vdash \Gamma_i, C^{n_i}, \overline{C}^{n_i}]_{i=k+1}^n}\ \text{IH}^2}{[\vdash \Gamma_i, B^{n_i}, \overline{B}^{n_i}]_{i=0}^k \mid [\vdash \Gamma_i, B^{n_i}, \overline{C}^{n_i}]_{i=0}^k \mid [\vdash \Gamma_i, C^{n_i}, \overline{B}^{n_i}]_{i=k+1}^n \mid [\vdash \Gamma_i, C^{n_i}, \overline{C}^{n_i}]_{i=k+1}^n}\ \text{W}^n}{[\vdash \Gamma_i, B^{n_i}, (\overline{B} \sqcup \overline{C})^{n_i}]_{i=0}^k \mid [\vdash \Gamma_i, C^{n_i}, (\overline{B} \sqcup \overline{C})^{n_i}]_{i=k+1}^n}\ \sqcup^n \quad \ddots \quad \vdots \quad \iddots}{[\vdash \Gamma_i, (B \sqcap C)^{n_i}, (\overline{B} \sqcup \overline{C})^{n_i}]_{i=1}^n}\ \sqcap$$

Note that the premises obtained after applying the $\sqcap$–rule can have a different shape than the displayed premise in the derivation above, where B is chosen for the first k sequents and C for the remaining ones. Indeed, the general shape of the premise can be any combination of B and C appearing in the sequents.

- For the case $A = \Diamond B$, we can not use the $\Diamond$ rule because of the constraints on the hypersequent in the rule. Instead we will now work on the derivation of $\triangleright_{\mathbf{MGA}} [\vdash \Gamma_i]_{i=1}^k$ until we reach a $\Diamond$ rule to conclude. We will only show the case of the $\Diamond$ rule and one other rule, the remaining of the rules are similar.
  - if the derivation of $\triangleright_{\mathbf{MGA}} [\vdash \Gamma_i]_{i=1}^k$ finishes with

  $$\dfrac{[\vdash \Gamma_i]_{i=3}^k \mid \vdash \Gamma_1, \Gamma_2}{[\vdash \Gamma_i]_{i=3}^k \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S}$$

  then by induction hypothesis on the premise

  $$\triangleright_{\mathbf{MGA}} [\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=3}^k \mid \vdash \Gamma_1, \Gamma_2, \Diamond B^{n_1+n_2}, \Diamond \overline{B}^{n_1+n_2}$$

  and thus

  $$\dfrac{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=3}^k \mid \vdash \Gamma_1, \Gamma_2, \Diamond B^{n_1+n_2}, \Diamond \overline{B}^{n_1+n_2}}{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=3}^k \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1} \mid \vdash \Gamma_2, \Diamond B^{n_2}, \Diamond \overline{B}^{n_2}}\ \text{S}$$

  - if the derivation of $\triangleright_{\mathbf{MGA}} [\vdash \Gamma_i]_{i=1}^k$ finishes with

  $$\dfrac{\vdash \Gamma_1, 1^n}{\vdash \Diamond \Gamma_1, 1^n}\ \Diamond$$

  then by induction hypothesis on B

  $$\triangleright_{\mathbf{MGA}} \vdash \Gamma_1, B^{n_1}, \overline{B}^{n_1}, 1^n$$

  thus

  $$\dfrac{\vdash \Gamma_1, B^{n_1}, \overline{B}^{n_1}, 1^n}{\vdash \Diamond \Gamma_1, \Diamond B^{n_1}, \overline{B}^{n_1}, 1^n}\ \Diamond$$

□

We can see in this lemma that it is difficult to reduce the complexity of a $\Diamond$ formula. For any other operations, we simply can use the corresponding logical rule (or the invertibility of the corresponding logical rule) but because of the constraints on the $\Diamond$ rule, we can not do the same to reduce the $\Diamond$ formula and thus we need to work by induction on the derivation. As often when proving a result by induction on the derivation, we prove a more general result.

The next result states that derivability in the **MGA** system is preserved by substitution of terms for variables.

Lemma 62. *For all hypersequents G and terms A, if $\triangleright_{\mathbf{MGA}} G$ then $\triangleright_{\mathbf{MGA}} G[A/x]$.*

Proof. We prove the result by induction on the derivation of G. Most cases are quite straightforward, we simply use the induction hypothesis on the premises and then use the same rule. For instance, if the derivation finishes with

$$\frac{G \mid \vdash \Gamma, B^n, C^n}{G \mid \vdash \Gamma, (B+C)^n} +$$

by induction hypothesis $\triangleright_{\mathbf{MGA}} G[A/x] \mid \vdash \Gamma[A/x], B[A/x]^n, C[A/x]^n$ so

$$\frac{G[A/x] \mid \vdash \Gamma[A/x], B[A/x]^n, C[A/x]^n}{G[A/x] \mid \vdash \Gamma[A/x], (B+C)[A/x]^n} +$$

The only tricky case is when the ID rule is used on the variable x, where we conclude using Lemma 61. □

The next lemma states that the $\{0, +, \sqcup, \sqcap\}$-logical rules are invertible using the CAN rule, meaning that if the conclusion is derivable, then the premises are also derivable. Unlike a stronger result we will prove later in Section 2.3.4 where we prove the CAN–free version of this lemma, the derivations of the premises may contain CAN rules and thus this result is not sufficient to imply the CAN elimination theorem.

Lemma 63. *The $\{0, +, \sqcup, \sqcap\}$ logical rules are invertible.*

Proof. We simply use the CAN rule to introduce the operators. We will show the two most interesting cases, the other cases are trivial.

- The $\sqcap$ rule: we assume that $G \mid \vdash \Gamma, (A \sqcap B)^n$ is derivable. The derivation of $G \mid \vdash \Gamma, A^n$ is then:

$$\dfrac{\dfrac{G \mid \vdash \Gamma, (A \sqcap B)^n \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash A^n, \overline{A}^n}\text{ Lemma 61}}{\vdash A^n, \overline{A}^n \mid \vdash A^n, \overline{B}^n}\text{ W}}{\vdash A^n, (\overline{A} \sqcup \overline{B})^n}\ \sqcup}{G \mid \vdash A^n, (\overline{A} \sqcup \overline{B})^n}\text{ W}^*}{G \mid \vdash \Gamma, A^n, (A \sqcap B)^n, (\overline{A} \sqcup \overline{B})^n}\text{ M}}{G \mid \vdash \Gamma, A^n}\text{ CAN}$$

The derivation of $G \mid \vdash \Gamma, B^n$ is similar.

- The $\sqcup$ rule: we assume that $G \mid \vdash \Gamma, (A \sqcup B)^n$ is derivable. The derivation of $G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n$ is then:

$$
\dfrac{\dfrac{G \mid \vdash \Gamma, (A \sqcup B)^n}{G \mid \vdash \Gamma, (A \sqcup B)^n \mid \vdash \Gamma, B^n}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash}\,\text{INIT}}{\vdash A^n, \overline{A}^n}\,\text{Lemma 61}}{G \mid \vdash A^n, \overline{A}^n \mid \vdash \Gamma, B^n}\,\text{W}^* \qquad \dfrac{\Pi}{G \mid \vdash A^n, \overline{B}^n \mid \vdash \Gamma, B^n}}{G \mid \vdash A^n, (\overline{A} \sqcap \overline{B})^n \mid \vdash \Gamma, B^n}\,\sqcap}{\dfrac{G \mid \vdash \Gamma, A^n, (A \sqcup B)^n, (\overline{A} \sqcap \overline{B})^n \mid \vdash \Gamma, B^n}{G \mid \vdash \Gamma, A^n \mid \vdash \Gamma, B^n}\,\text{CAN}}\,\text{M}
$$

where Π is the following derivation:

$$
\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, (A \sqcup B)^n}{G \mid \vdash A^n, \overline{B}^n \mid \vdash \Gamma, (A \sqcup B)^n}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash}\,\text{INIT}}{\vdash B^n, \overline{B}^n}\,\text{Lemma 61}}{\vdash A^n, \overline{B}^n, B^n, \overline{A}^n}\,\text{Lemma 61}}{\vdash A^n, \overline{B}^n \mid \vdash B^n, \overline{A}^n}\,\text{S} \qquad \dfrac{\dfrac{\overline{\vdash}\,\text{INIT}}{\vdash B^n, \overline{B}^n}\,\text{Lemma 61}}{\vdash A^n, \overline{B}^n \mid \vdash B^n, \overline{B}^n}\,\text{W}}{\vdash A^n, \overline{B}^n \mid \vdash B^n, (\overline{A} \sqcap \overline{B})^n}\,\sqcap}{G \mid \vdash A^n, \overline{B}^n \mid \vdash B^n, (\overline{A} \sqcap \overline{B})^n}\,\text{W}^*}{G \mid \vdash A^n, \overline{B}^n \mid \vdash \Gamma, B^n, (A \sqcup B)^n, (\overline{A} \sqcap \overline{B})^n}\,\text{M}}{G \mid \vdash A^n, \overline{B}^n \mid \vdash \Gamma, B^n}\,\text{CAN}
$$

□

Remark 29. Note that the $\Diamond$ rule is also invertible, but because of the constraints of the $\Diamond$ rule, we can not use it to reduce the complexity of a $\Diamond$ formula. Thus we do not use its invertibility to prove the CAN elimination theorem.

The next lemmas state that CAN–free derivability in the **MGA** system is preserved by scalar multiplication.

Lemma 64. *Let $n > 0$ be a natural number and G a hypersequent. If $\triangleright_{\boldsymbol{MGA} \backslash \{CAN\}} G \mid \vdash \Gamma^n$ then $\triangleright_{\boldsymbol{MGA} \backslash \{CAN\}} G \mid \vdash \Gamma$.*

Proof. We simply use the C and S rules :

$$
\dfrac{\dfrac{G \mid \vdash \Gamma^n}{G \mid \vdash \Gamma \mid \ \ldots \mid \vdash \Gamma}\,\text{S}^{n-1}}{G \mid \vdash \Gamma}\,\text{C}^{n-1}
$$

□

Lemma 65. *Let $n > 0$ be a natural number and G a hypersequent. If $\triangleright_{\boldsymbol{MGA} \backslash \{CAN\}} G \mid \vdash \Gamma$ then $\triangleright_{\boldsymbol{MGA} \backslash \{CAN\}} G \mid \vdash \Gamma^n$.*

Proof. We simply use the M rule $n - 1$ times. □

2.3.2 Soundness

We need to prove that if there exists a **MGA** derivation of a hypersequent G then $(\!|G|\!) \geq 0$ is derivable in equational logic (written $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$). This is done in a straightforward way by showing that each deduction rule of the system **MGA** is sound. Notice that the soundness of the rules already present in **MGA** is proved in the exact same way so we will only show the soundness of the new rules.

Theorem 2.3.1 (Soundness of **MGA**). *For all hypersequent G, if $\rhd_{\boldsymbol{MGA}} G$ then $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$.*

Proof. By induction on the derivation of G. We only show the $\Diamond$ and 1 rules since the other are similar to Theorem 2.1.1.

- For the rule

$$\frac{\vdash \Gamma, 1^n}{\vdash \Diamond\Gamma, 1^n}\ \Diamond$$

  the hypothesis is $(\!|\vdash \Gamma, 1^n|\!) \geq 0$ so

$$\begin{array}{rcl} (\!|\vdash \Diamond\Gamma, 1^n|\!) & \geq & (\!|\vdash \Diamond\Gamma, \Diamond 1^n|\!) \text{ since } \Diamond 1 \leq 1 \\ & = & \Diamond((\!|\vdash \Gamma, 1^n|\!)) \text{ by linearity of } \Diamond \\ & \geq & 0 \text{ by the hypothesis and the monotonicity of } \Diamond. \end{array}$$

- For the rule

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, 1^n, \overline{1}^m}\ 1, n \geq m$$

  the hypothesis is $(\!|G \mid \vdash \Gamma|\!) \geq 0$ so

$$\begin{array}{rcl} (\!|G \mid \vdash \Gamma, 1^n, \overline{1}^m|\!) & \geq & (\!|G \mid \vdash \Gamma|\!) \text{ since } n \geq m \text{ and } 0 \leq 1 \\ & \geq & 0 \end{array}$$

□

2.3.3 Completeness

In order to prove the completeness of the system **MGA**, i.e. that if $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ then $\rhd_{\mathbf{MGA}} G$, we first prove an equivalent result (Lemma 66 below) stating that if $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash A = B$ then the hypersequents $\vdash A, \overline{B}$ and $\vdash B, \overline{A}$ are both derivable.

From Lemma 66 one indeed obtains Theorem 2.3.2 as a corollary.

Theorem 2.3.2 (Completeness of **MGA**). *For all hypersequent G, if $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ then $\rhd_{\boldsymbol{MGA}} G$.*

Proof. Recall that $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ is a shorthand for $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash 0 = (\!|G|\!) \sqcap 0$. Hence, from the hypothesis $\mathcal{A}^{\Diamond}_{\text{l-groups}} \vdash (\!|G|\!) \geq 0$ we can deduce, by using Lemma 66, that $\rhd_{\mathbf{MGA}} \vdash 0 \sqcap (\!|G|\!), 0$.

From this we can show that $\rhd_{\mathbf{MGA}} G$ by invoking Lemma 63. Indeed, if G is $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ then $(\!|G|\!) = (\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)$ and

1. by using the invertibility of the 0 rule, $\vdash (0 \sqcap ((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)))$ is derivable,

2. by using the invertibility of the $\sqcap$ rule, $\vdash ((\!\vdash \Gamma_1\!) \sqcup ... \sqcup (\!\vdash \Gamma_n\!))$ is derivable,
3. by using the invertibility of the $\sqcup$ rule $n-1$ times, $\vdash (\!\vdash \Gamma_1\!) \mid \; ... \mid \vdash (\!\vdash \Gamma_n\!)$ is derivable,
4. and finally, by using the invertibility of the $+$ rule, $\vdash \Gamma_1 \mid \; ... \mid \vdash \Gamma_n$ is derivable.

∎

Lemma 66. *If* $\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash A = B$ *then* $\vdash A, \overline{B}$ *and* $\vdash B, \overline{A}$ *are derivable.*

Proof. We prove this result by induction on the derivation in equational logic (see Definition 1.1.2) of $\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash A = B$. Most cases are the same as in Lemma 30, we will only show some of the interesting axioms.

- If the derivation finishes with

$$\dfrac{}{\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash \Diamond 1 \sqcap 1 = \Diamond 1}\ \text{ax}$$

  then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 1, \overline{1}}\ 1}{\vdash \Diamond 1, \Diamond \overline{1}}\ \Diamond \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 1, \overline{1}}\ 1}{\vdash 1, \Diamond \overline{1}}\ \Diamond}{\vdash \Diamond 1 \sqcap 1, \Diamond \overline{1}}\ \sqcap$$

  and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 1, \overline{1}}\ 1}{\vdash \Diamond 1, \Diamond \overline{1}}\ \Diamond}{\vdash \Diamond 1, \overline{1} \mid \vdash \Diamond 1, \Diamond \overline{1}}\ \text{W}}{\vdash \Diamond 1, \overline{1} \sqcup \Diamond \overline{1}}\ \sqcup$$

- If the derivation finishes with

$$\dfrac{}{\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash 0 \sqcap \Diamond(0 \sqcup x) = 0}\ \text{ax}$$

  then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 0}\ 0 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 0}\ 0}{\vdash 0 \mid \vdash x}\ \text{W}}{\vdash 0 \sqcup x}\ \sqcup}{\vdash \Diamond(0 \sqcup x)}\ \Diamond}{\vdash 0 \sqcap \Diamond(0 \sqcup x)}\ \sqcap}{\vdash 0 \sqcap \Diamond(0 \sqcup x), 0}\ 0$$

  and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 0}\ 0}{\vdash 0 \mid \vdash \Diamond(0 \sqcap \overline{x})}\ \text{W}}{\vdash 0 \sqcup \Diamond(0 \sqcap \overline{x})}\ \sqcup}{\vdash 0, 0 \sqcup \Diamond(0 \sqcap \overline{x})}\ 0$$

∎

2.3.4 CAN–free invertibility

In this section, we prove that the $\{0, +, \sqcup, \sqcap\}$–logical rules are CAN-free invertible, i.e., that if the conclusion of a logical rule has a CAN-free derivation, then so do the premises. As for the system **GA**‖, it allows us to reduce the complexity of the formulas in an hypersequent in the proof of the CAN elimination theorem, and thus it is important that we do not add any CAN rule in the proofs of invertibility. For this reason, the CAN–free invertibility result is stronger than Lemma 63 of Section 2.3.1.

Like in the previous Section 2.1.4, we will prove the CAN-free invertibility of more general rules as those rules are more convenient for the induction process.

Logical rules:

$$\frac{[\vdash \Gamma_i]_{i=1}^n}{[\vdash \Gamma_i, 0^{n_i}]_{i=1}^n}\ 0 \qquad \frac{[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^n}\ +$$

$$\frac{[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=1}^n}\ \sqcup \qquad \frac{[\vdash \Gamma_i, A^{n_i}]_{i=1}^n \quad [\vdash \Gamma_i, B^{n_i}]_{i=1}^n}{[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^n}\ \sqcap$$

Figure 2.11: Generalised logical rules

The proof steps dealing with the rules already present in **GA**‖ are the same as in Section 2.1.4. In what follows we just show the details of the proof steps associated with the new cases associated with the $\Diamond$–rule and 1–rule of **MGA**.

We conceptually divide the logical rules in three categories:

Type 1 The rule with only one premise but that adds one sequent to the hypersequent: the $\sqcup$ rule.

Type 2 The rules with only one premise and that do not change the number of sequents: the $0, +$ rules.

Type 3 The rule with two premises: the $\sqcap$ rule.

Because of the similarities of the rules in each of these categories, we just prove the CAN–free invertibility of one rule in each category by means of a sequence of lemmas.

Lemma 67 (Type 1)**.** *If* $[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=1}^k$ *has a CAN-free derivation then* $[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=1}^k$ *has a CAN–free derivation.*

Proof. By induction on the derivation of $[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=1}^k$. We only show the cases of the $\Diamond$ rule and the 1 rule.

- If the derivation finishes with

$$\frac{\vdash \Gamma_1, 1^n}{\vdash \Diamond\Gamma_1, 1^n}\ \Diamond$$

then $k = 1$ and $n_1 = 0$. Thus we want to derive $\vdash \Diamond\Gamma_1, 1^n \mid \vdash \Diamond\Gamma_1, 1^n$. The derivation is

$$\frac{\dfrac{\vdash \Gamma_1, 1^n}{\vdash \Diamond\Gamma_1, 1^n}\ \Diamond}{\vdash \Diamond\Gamma_1, 1^n \mid \vdash \Diamond\Gamma_1, 1^n}\ \text{C}$$

- If the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A \sqcup B)^{n_1}}{[\vdash \Gamma_i, (A \sqcup B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A \sqcup B)^{n_1}, 1^n, 1^m}\ 1, n \geq m$$

then by induction hypothesis

$$\triangleright_{\mathbf{MGA}}[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_1, B^{n_1}$$

and thus the derivation is

$$\dfrac{\dfrac{[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1} \mid \vdash \Gamma_1, B^{n_1}}{[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, 1^n, 1^m \mid \vdash \Gamma_1, B^{n_1}}\ 1, n \geq m}{[\vdash \Gamma_i, A^{n_i} \mid \vdash \Gamma_i, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, 1^n, 1^m \mid \vdash \Gamma_1, B^{n_1}, 1^n, 1^m}\ 1, n \geq m$$

□

Lemma 68 (Type 2)**.** *If* $[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^k$ *has a CAN–free derivation then* $[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=1}^k$ *has a CAN–free derivation.*

Proof. By induction on the derivation of $[\vdash \Gamma_i, (A+B)^{n_i}]_{i=1}^k$. We only show the cases of the $\Diamond$ rule and the 1 rule.

- If the derivation finishes with

$$\dfrac{\vdash \Gamma_1, 1^n}{\vdash \Diamond\Gamma_1, 1^n}\ \Diamond$$

then $k = 1$ and $n_1 = 0$. Thus $\vdash \Diamond\Gamma_1, 1^n, A^{n_1}, B^{n_1} = \vdash \Diamond\Gamma_1, 1^n$ which is derivable.

- If the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, (A+B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A+B)^{n_1}}{[\vdash \Gamma_i, (A+B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A+B)^{n_1}, 1^n, 1^m}\ 1, n \geq m$$

then by induction hypothesis

$$\triangleright_{\mathbf{MGA}}[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, B^{n_1}$$

and thus the derivation is

$$\dfrac{[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, B^{n_1}}{[\vdash \Gamma_i, A^{n_i}, B^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, B^{n_1}, 1^n, 1^m}\ 1, n \geq m$$

□

Lemma 69 (Type 3)**.** *If* $[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, A^{n_i}]_{i=1}^n$ *and* $[\vdash \Gamma_i, B^{n_i}]_{i=1}^n$ *have a CAN–free derivation.*

Proof. By induction on the derivation of $[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=1}^k$. We only show the cases of the $\Diamond$ rule and the 1 rule, and that $[\vdash \Gamma_i, A^{n_i}]_{i=1}^k$ is derivable.

- If the derivation finishes with

$$\dfrac{\vdash \Gamma_1, 1^n}{\vdash \Diamond\Gamma_1, 1^n}\ \Diamond$$

then $k = 1$ and $n_1 = 0$. Thus $\vdash \Diamond\Gamma_1, 1^n, A^{n_1} = \vdash \Diamond\Gamma_1, 1^n$ which is derivable.

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A \sqcap B)^{n_1}}{[\vdash \Gamma_i, (A \sqcap B)^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, (A \sqcap B)^{n_1}, 1^n, 1^m} \; 1, n \geq m$$

then by induction hypothesis

$$\triangleright_{\mathbf{MGA}} [\vdash \Gamma_i, A^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}$$

and thus the derivation is

$$\frac{[\vdash \Gamma_i, A^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}}{[\vdash \Gamma_i, A^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, A^{n_1}, 1^n, 1^m} \; 1, n \geq m$$

□

2.3.5 M–elimination

In this section, we will show the M elimination theorem. Recall that the M elimination theorem states

if a hypersequent G is derivable, then it has a M–free derivation.

However, since this result will be used in the proof of the CAN elimination theorem, we have to ensure that the M elimination theorem does not add any instance of the CAN rule. Thus we will show the slightly different result

if a hypersequent G is CAN–free derivable, then it has a CAN–free M–free derivation.

Following the same pattern of Section 2.1.5, we need to show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free derivations d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there also exists a CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$.

The general idea presented in Section 2.1.5 is to combine the derivations d_1 and d_2 in a sequential way, first constructing a prederivation d_1' of $G \mid \; G \mid \vdash \Gamma, \Delta$ (using d_1) whose leaves are either axioms or hypersequents of the form $G \mid \vdash \Delta^n$, and then by completing this prederivation into a derivation (using d_2). Finally, $G \mid \; G \mid \vdash \Gamma, \Delta$ can be easily turned into a derivation of $G \mid \vdash \Gamma, \Delta$ as desired.

However, this technique cannot be directly applied in the context of the system **MGA** due to the constraints imposed on the shape of the hypersequent by the $\Diamond$ rule. Indeed an application of the $\Diamond$ rule in d_1 acting on some hypersequent of the form

$$\vdash \Diamond\Gamma', 1^m$$

cannot turned into an application of the $\Diamond$ rule on

$$G \mid \vdash \Delta^n, \Diamond\Gamma', 1^m$$

because this hypersequent cannot be the conclusion of a $\Diamond$ rule as it does not satisfy the constraints. To deal with the $\Diamond$ rule, we will expand the construction of Section 2.1.5 by induction on the modal depth of the derivation d_1.

Indeed, when constructing the prederivation d_1' inductively from d_1, we stop at the applications of the $\Diamond$–rule. Hence, the inductive procedure takes the derivation d_1 and produces a CAN–free and M–free prederivation d_1' of

$$G \mid \; G \mid \vdash \Gamma, \Delta$$

where all the leaves in the prederivation are either:

1. terminated, or
2. non–terminated and having the shape
$$G \mid \vdash \Delta^n$$
which can then be completed using the derivation d_2 in the exact same way explained in Section 2.1.5, or
3. non–terminated and having the shape:
$$G \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$$
for some sequent Γ' and natural numbers n and m. For each of these leaves there is a corresponding derivation of
$$\vdash \Gamma', 1^m \tag{2.2}$$

To obtain derivations of the leaves of d_1' of the third type, and thus complete the derivation, we proceed as follows. First, we use the derivation d_2 to construct a CAN–free and M–free derivation $d_{2,n}$
$$G \mid \ \ \Delta^n$$
for each natural number n in the leaves. We then modify each derivation $d_{2,n}$ into a prederivation d_2' of
$$G \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$$
using the exact same inductive procedure (which stops when reaching applications of $\Diamond$ terms) introduced above for producing d_1' from d_1. Note that in this case, the leaves of the third kind in d_2' are of the form:
$$\vdash (\Diamond\Gamma', 1^m)^{n'}, \Diamond\Delta', 1^{m'}$$
and have associated derivations of
$$\vdash \Delta', 1^{m'} \tag{2.3}$$
Therefore, we can legitimately apply the $\Diamond$ rule (Lemma 70 below ensures that the proviso of the rule is respected) and reduce these leaves to leaves of the form
$$\vdash (\Gamma', 1^m)^{n'}, \Delta', 1^{m'}$$
which, importantly, have a lower modal depth compared to the conclusion $G \mid \vdash \Gamma$ of the derivation d_1 we started with above.

In order the produce a derivation for the leaves $\vdash (\Gamma', 1^m)^{n'}, \Delta', 1^{m'}$, and thus conclude the completion of d_1' into a full derivation, it is sufficient to re–apply the whole process using the derivations of Equation 2.2 and Equation 2.3 above. This process is well founded and eventually terminates because the modal depth is decreasing. We summarise the different steps in the Figure 2.12 below.

For instance, removing the M rule in the following derivation
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{INIT}}{\vdash x, \overline{x}}\text{ID}}{\vdash \Diamond x, \Diamond\overline{x}}\Diamond}{\vdash \Diamond x \mid \vdash \Diamond\overline{x}}\text{S}}{\vdash \Diamond x \sqcup \Diamond\overline{x}}\sqcup \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{INIT}}{\vdash y, \overline{y}}\text{ID}}{\vdash \Diamond y, \Diamond\overline{y}}\Diamond}{\vdash \Diamond y \mid \vdash \Diamond\overline{y}}\text{S}}{\vdash \Diamond y \sqcup \Diamond\overline{y}}\sqcup}{\vdash \Diamond x \sqcup \Diamond\overline{x}, \Diamond y \sqcup \Diamond\overline{y}}\text{M}$$

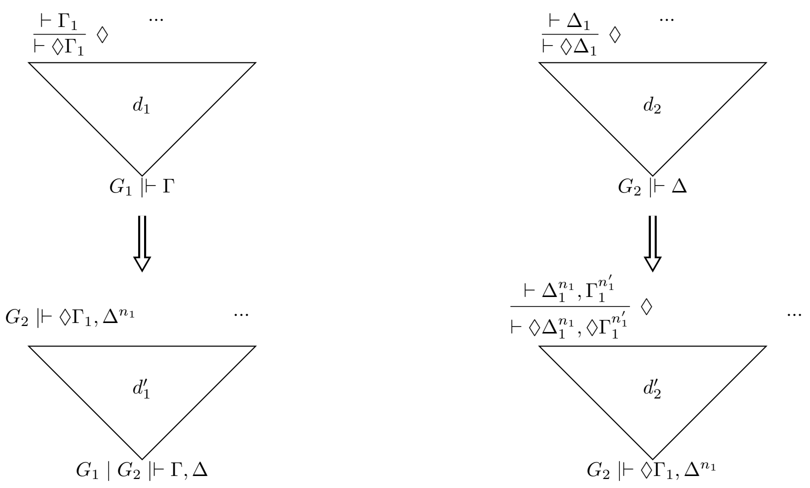


Figure 2.12: Sequentially composing d_1 and d_2 in the M elimination proof.

gives the following M free derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash y^2, \overline{y}^2}\text{ ID}| \; d_1 \text{ is over, we finish } d_2}{\vdash x^2, \overline{x}^2, y^2, \overline{y}^2}\text{ ID}| \text{ we start again with } d_1}{\vdash (\Diamond x)^2, (\Diamond \overline{x})^2, (\Diamond y)^2, (\Diamond \overline{y})^2}\Diamond| \; d_2 \text{ reached a } \Diamond \text{ rule so the } \Diamond \text{ rule is now valid}}{\vdash \Diamond x, \Diamond \overline{x}, (\Diamond y)^2 \mid \vdash \Diamond x, \Diamond \overline{x}, (\Diamond \overline{y})^2}\text{ S}}{\vdash \Diamond x, \Diamond \overline{x}, (\Diamond y \sqcup \Diamond \overline{y})^2}\sqcup| \text{ we reached a } \Diamond \text{ rule so we switch to } d_2}{\vdash \Diamond x, \Diamond y \sqcup \Diamond \overline{y} \mid \vdash \Diamond \overline{x}, \Diamond y \sqcup \Diamond \overline{y}}\text{ S}}{\vdash \Diamond x \sqcup \Diamond \overline{x}, \Diamond y \sqcup \Diamond \overline{y}}\sqcup| \text{ start with } d_1$$

We now proceed with the technical statements.

Lemma 70. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ and let H be a hypersequent and Δ be a sequent. Then there exists a prederivation of*

$$G \mid \; H \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are either of the form $H \mid \vdash \Delta^n$ or of the form $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ for some sequent Γ' and natural numbers n and m such that $\vdash \Gamma', 1^m$ has a derivation d_1' with a strictly lower modal depth than d_1.

Proof. This is an instance of the slightly more general statement of Lemma 73 below where:

- $[\vdash \Gamma_i]_{i=1}^{k-1} = G$ and $\Gamma_k = \Gamma$.
- $n_i = 0$ for $1 \leq i < k$ and $n_k = 1$. □

Remark 30. Following Remark 31, if the derivation of $G \mid \vdash \Gamma$ does not use any $\Diamond$ rule then all unfinished leaves are of the form $H \mid \vdash \Delta^n$.

Lemma 71. *Let d_2 be CAN–free and M–free derivation of $H \mid \vdash \Delta$. Then, for any natural number n, there exists a CAN–free and M–free derivation of*

$$H \mid \vdash \Delta^n$$

with a modal depth lower or equal to d_2.

Proof. This is an instance of the slightly more general statement of Lemma 74 below where:

- $[\vdash \Delta_i]_{i=1}^{k-1} = H$ and $\Delta_k = \Delta$.
- $n_i = 1$ for $1 \leq i < n$ and $n_k = n$. □

We now show how to remove one instance of the M rule and then the M-elimination theorem.

Lemma 72. *If $G \mid \vdash \Gamma$ and $H \mid \vdash \Delta$ have CAN–free M–free derivations, then so does $G \mid\ H \mid \vdash \Gamma, \Delta$.*

Proof. We show the lemma by induction on the modal depth of the derivation d of $G \mid \vdash \Gamma$.

If the modal depth of d is 0, then we proceed as in Theorem 2.1.3, i.e., we use Lemma 70 to have a prederivation of $G \mid\ H \mid \vdash \Delta, \Gamma$ where all leaves are of the form $H \mid \vdash \Delta^n$, and we finish the prederivation by using Lemma 71.

Otherwise d uses some $\Diamond$ rule. We do the following:

- we use Lemma 70 to have a prederivation of $G \mid\ H \mid \vdash \Gamma, \Delta$ where all non–terminated leaves are either of the form $H \mid \vdash \Delta^n$ or of the form $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ for some sequent Γ' and natural numbers n and m such that $\vdash \Gamma', 1^m$ has a derivation d_1' with a strictly lower modal depth than d_1.
- We show that all leaves of the form $H \mid \vdash \Delta^n$ are derivable using Lemma 71.
- We conclude by showing that all leaves of the form $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ are derivable. Let's show how to derive them.
  - We show that $H \mid \vdash \Delta^n$ are derivable using Lemma 71.
  - Then we build a prederivation of $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ using Lemma 70 where all non-terminated leaves are either of the form $\vdash (\Diamond\Gamma', 1^m)^{n'}$ or of the form $\vdash (\Diamond\Gamma', 1^m)^{n'}, \Diamond\Delta', 1^{m'}$ such that $\vdash \Delta', 1^{m'}$ has a CAN-free M-free derivation.
  - The leaves of the form $\vdash (\Diamond\Gamma', 1^m)^{n'}$ can be terminated using the $\Diamond$ rule and Lemma 71.
  - For the leaves of the form $\vdash (\Diamond\Gamma', 1^m)^{n'}, \Diamond\Delta', 1^{m'}$, we will show that $\vdash (\Gamma', 1^m)^{n'}, \Delta', 1^{m'}$ has a CAN-free M-free derivation and we can conclude using the $\Diamond$ rule. Recall that $\vdash \Gamma', 1^m$ has a CAN-free M-free derivation with strictly lower modal depth than d_1. We use Lemma 71 to have a derivation of $\vdash (\Gamma', 1^m)^{n'}$ with strictly lower depth than d_1.
  - We can then use the induction hypothesis since the derivation of $\vdash (\Gamma', 1^m)^{n'}$ has a strictly lower depth than d_1, thus building a derivation of $\vdash (\Gamma', 1^m)^{n'}, \Delta', 1^{m'})$ to conclude the proof.

□

Theorem 2.3.3 (M elimination)**.** *If G is CAN-free derivable, then G is CAN-free M-free derivable.*

Proof. We prove the result by induction on G. The only interesting case if the M rule, i.e., if the derivation finishes with

$$\frac{G \mid \vdash \Gamma \quad G \mid \vdash \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{M}$$

then by induction hypothesis $G \mid \vdash \Gamma$ and $G \mid \vdash \Delta$ have CAN-free M-free derivation.

By using Lemma 72, we have a CAN-free M-free derivation of $G \mid \ G \mid \vdash \Gamma, \Delta$. The derivation is then

$$\frac{G \mid \ G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{C}^*$$

□

We now prove the technical version of Lemmas 70 and 71.

Lemma 73. *Let d_1 be a CAN–free and M–free derivation of $[\vdash \Gamma_i]_{i=1}^k$ and let H be a hypersequent and Δ be a sequent. Then for every sequence of natural numbers n_i, there exists a prederivation of*

$$H \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^k$$

where all non–terminated leaves are either of the form $H \mid \vdash \Delta^n$ or of the form $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ for some sequent Γ' and natural numbers n and m such that $\vdash \Gamma', 1^m$ has a derivation d_1' with a strictly lower modal depth than d_1.

Proof. We prove the result by induction on d_1. We will only show the $\Diamond$ and the 1 rules, since all other cases are done in the same way as in Lemma 37.

- if d_1 finishes with:

$$\frac{[\vdash \Gamma_i]_{i=2}^k \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^k \mid \vdash \Gamma_1, 1^{n'}, \overline{1}^{m'}}\ 1, n' \geq m'$$

  then by induction hypothesis, there is a prederivation of $G \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, \Delta^{n_1}$ where all non–terminated leaves are either of the form $H \mid \vdash \Delta^n$ or of the form $H \mid \vdash \Diamond\Gamma', \Delta^n, 1^m$ for some sequent Γ' and natural numbers n and m such that $\vdash \Gamma', 1^m$ has a derivation d_1' with a strictly lower modal depth than d_1. We continue the prederivation with

$$\frac{G \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=2}^n \mid \vdash \Gamma_1, \Delta^{n_1}}{G \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=2}^k \mid \vdash \Gamma_1, \Delta^{n_1}, 1^{n'}, \overline{1}^{m'}}\ 1, n' \geq m'$$

- If d_1 finishes with:

$$\frac{\vdash \Gamma_1, 1^m}{\vdash \Diamond\Gamma_1, 1^m}\ \Diamond$$

  then the prederivation is simply the leaf $H \mid \vdash \Diamond\Gamma_1, \Delta^n, 1^m$. □

Remark 31. Notice that if the derivation $[\vdash \Gamma_i]_{i=1}^k$ does not use any $\Diamond$ rule, then all leaves of the prederivation of $H \mid \ [\vdash \Gamma_i, \Delta^{n_i}]_{i=1}^k$ are of the form $H \mid \ \Delta^n$.

Lemma 74. *If d_2 is a CAN–free M–free derivation of $[\vdash \Delta_i]_{i=1}^k$ then for all n_i, there is a CAN–free M–free derivation of $[\vdash \Delta_i^{n_i}]_{i=1}^k$ with a modal depth lower or equal than d_2.*

Proof. We will only show the $\Diamond$ and 1 rules, the other cases being similar to Lemma 38 – and so do not introduce any new $\Diamond$ rule.

- if d_2 finishes with:

$$\dfrac{[\vdash \Delta_i]_{i=2}^k \mid \vdash \Delta_1}{[\vdash \Delta_i]_{i=2}^k \mid \vdash \Delta_1, 1^n, \overline{1}^m}\ 1, n \geq m$$

  then by induction hypothesis, there is a CAN–free M–free derivation of $[\vdash \Delta_i^{n_i}]_{i=2}^k \mid \vdash \Delta_1^{n_1}$ with a modal depth lower or equal than d_2. We continue the derivation with

$$\dfrac{[\vdash \Delta_i^{n_i}]_{i=2}^k \mid \vdash \Delta_1^{n_1}}{[\vdash \Delta_i^{n_i}]_{i=2}^k \mid \vdash \Delta_1^{n_1}, 1^{nn_1}, \overline{1}^{mn_1}}\ 1, n_1 n \geq n_1 m$$

  which does not increase the modal depth of the derivation.

- If d_2 finishes with:

$$\dfrac{\vdash \Delta_1, 1^n}{\vdash \Diamond\Delta_1, 1^n}\ \Diamond$$

  by induction hypothesis, there is a derivation of $\vdash \Delta_1^{n_1}, 1^{nn_1}$ with a modal depth strictly less than d_2. We continue the derivation with

$$\dfrac{\vdash \Delta_1^{n_1}, 1^{nn_1}}{\vdash \Diamond\Delta_1^{n_1}, 1^{nn_1}}\ \Diamond$$

  which gives a derivation with a modal depth less or equal than d_2. □

2.3.6 CAN–elimination

Recall that the CAN rule has the following form:

$$\dfrac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma}\ \text{CAN}$$

As in Section 2.1.6, we prove Theorem 2.1.4 by showing that if the hypersequent $G \mid \vdash \Gamma, A^n, \overline{A}^n$ has a M–free CAN–free derivation, then so does the hypersequent $G \mid \vdash \Gamma$.

The CAN elimination theorem follows the same pattern as in the system **GA**‖: we reduce the complexity of the formula introduced by the CAN rule until we reach an atom (or a 1 formula). However, because of the constraint of $\Diamond$ rule, we can not invoke its invertibility to reduce the complexity of the a $\Diamond$ formula since the hypersequent $G \mid \vdash \Gamma, \vec{r}.\Diamond A, \vec{s}.\Diamond\overline{A}$ is, in general, not the conclusion of an application of the $\Diamond$ rule.

To circumvent this issue, we prove the slightly more general Lemma 77 by double induction on both the term A and the derivation of $G \mid \vdash \Gamma, A^n, \overline{A}^n$.

We first prove the two basic cases where $A = x$ (or equivalently $A = \overline{x}$) in Lemma 75 and $A = 1$ (or equivalently $A = \overline{1}$) in Lemma 76, and the general case in Lemma 77.

Lemma 75. *If there is a M–free CAN–free derivation of $G \mid \vdash \Gamma, x^n, \overline{x}^n$ then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. The statement follows as a special case of Lemma 78 below, a stronger version of Lemma 75 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, x^n, \overline{x}^n$, where:

- $[\vdash \Gamma_i]_{i=1}^{k-1} = G$ and $\Gamma_k = \Gamma$.
- $n_i = m_i = n'_i = m'_i = 0$ for $1 \leq i < k$.

- $n_k = m_k = n$ and $n'_k = m'_k = 0$. □

Lemma 76. *If there is a M-free CAN–free derivation of $G \mid \vdash \Gamma, 1^n, \overline{1}^n$ then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. The statement follows as a special case of Lemma 79 below, a stronger version of Lemma 76 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, 1^n, \overline{1}^n$, where:

- $[\vdash \Gamma_i]_{i=1}^{k-1} = G$ and $\Gamma_k = \Gamma$.
- $n_i = m_i = n'_i = m'_i = 0$ for $1 \leq i < k$.
- $n_k = m_k = n$ and $n'_k = m'_k = 0$. □

We are now ready to prove the general case.

Lemma 77. *For all terms A and numbers $k > 0$ and for all sequents Γ_i and natural numbers n_i,*

$$\text{if } [\vdash \Gamma_i, A^{n_i}, \overline{A}^{n_i}]_{i=1}^k \text{ has a M-free CAN–free derivation, then so does } [\vdash \Gamma_i]_{i=1}^k.$$

Proof. For the basic cases $A = x$, $A = \overline{x}$, $A = 1$ and $A = \overline{1}$, we use Lemmas 75 and 76. For complex terms A which are not $\Diamond$ terms, we proceed by invoking the CAN–free invertibility of the logical rules proven in Section 2.3.4 as follows:

- If $A = 0$, we can conclude with the CAN–free invertibility of the rule 0.
- If $A = B+C$, since the $+$ rule is CAN–free invertible, $\left[\vdash \Gamma_i, B^{n_i}, C^{n_i}, \overline{B}^{n_i}, \overline{C}^{n_i}\right]$ has a CAN–free, M–free derivation. Therefore we can have a CAN–free derivation of the hypersequent $[\vdash \Gamma_i]_{i=1}^k$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible,
$$\left[\vdash \Gamma_i, B^{n_i}, (\overline{B} \sqcap \overline{C})^{n_i}\right] \mid \left[\vdash \Gamma_i, C^{n_i}, (\overline{B} \sqcap \overline{C})^{n_i}\right]$$
has a CAN–free, M–free derivation. Then since the $\sqcap$ is CAN–free invertible,
$$\left[\vdash \Gamma_i, B^{n_i}, \overline{B}^{n_i}\right] \mid \left[\vdash \Gamma_i, C^{n_i}, \overline{C}^{n_i}\right]$$
has a CAN–free, M–free derivation. Therefore we can obtain a CAN–free derivation of the hypersequent $[\vdash \Gamma_i]_{i=1}^k$ by invoking the induction hypothesis twice on the simpler terms B and C.
- If $A = B \sqcap C$, we proceed in a similar way as for the case $A = B \sqcup C$.
- Finally, if $A = \Diamond B$, we distinguish two cases:
  1. the derivation ends with an application of the $\Diamond$ rule which simplifies $A = \Diamond B$ to B. In this case we can simply conclude by invoking the induction hypothesis on B and the $\Diamond$ rule.

2. The derivation ends with some other rule (recall that no CAN rules and no M rules appear in the derivation). In this case we decrease the complexity of the derivation, keeping $\Diamond B$ as the CAN term, and then invoke the induction hypothesis on the derivation having reduced complexity. This proof step is rather long to prove, as it requires analysing all possible cases. We just illustrate the two cases when the derivation ends with a logical rule (+) and a structural rule (C) to illustrate the general method.
   - if the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=2}^{k} \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1}, C^n, D^n}{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=2}^{k} \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1}, (C+D)^n}\ +$$

by induction hypothesis, there is a CAN–free M–free derivation of

$$[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1, C^n, D^n$$

We continue the derivation with

$$\dfrac{[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1, C^n, D^n}{[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1, (C+D)^n}\ +$$

   - if the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=2}^{k} \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1} \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1}}{[\vdash \Gamma_i, \Diamond B^{n_i}, \Diamond \overline{B}^{n_i}]_{i=2}^{k} \mid \vdash \Gamma_1, \Diamond B^{n_1}, \Diamond \overline{B}^{n_1}}\ \text{C}$$

by induction hypothesis, there is a CAN–free M–free derivation of

$$[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1 \mid \vdash \Gamma_1$$

We continue the derivation with

$$\dfrac{[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1 \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^{k} \mid \vdash \Gamma_1}\ \text{C}$$

□

Remark 32. Note it is important for the derivation to be M-free in the proof of Lemma 77 above. Indeed, when dealing with the case $A = \Diamond B$, the M rule would be critically difficult to deal with, as this rule breaks the symmetry between the $\Diamond B$ and $\Diamond \overline{B}$ of the CAN rule. For instance, we do not know how to deal with the following instance of the M rule:

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, \Diamond B, \Diamond B, \Diamond \overline{B} \quad G \mid \vdash \Gamma_2, \Diamond \overline{B}}{G \mid \vdash \Gamma_1, \Gamma_2, \Diamond B, \Diamond B, \Diamond \overline{B}, \Diamond \overline{B},}\ \text{M}}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{CAN}$$

since we cannot use the induction hypothesis on the two premises. Thus, even though the M elimination theorem is not used to prove the CAN elimination theorem in the system without the $\Diamond$ operator like the system **GA**||, it becomes crucial in its presence.

We now have all necessary tools to prove the CAN elimination theorem.

Theorem 2.3.4 (CAN elimination)**.** *For all hypersequents G, if $\triangleright_{\boldsymbol{MGA}} G$ then $\triangleright_{\boldsymbol{MGA}\backslash\{CAN\}} G$.*

Proof. We want to prove that if G has a derivation, then G has a CAN–free derivation. We prove this result by induction on the derivation of G:

- If the derivation finishes with an application of a rule that is not the CAN–rule, then by induction, the premises have CAN–free derivations and we can conclude by using the exact same rule to obtain a CAN–free derivation of G.
- If the derivation finishes with
$$\frac{G \mid \vdash \Gamma, A^n, \overline{A}^n}{G \mid \vdash \Gamma} \text{ CAN}$$
then by induction $G \mid \vdash \Gamma, A^n, \overline{A}^n$ has a CAN–free derivation. By invoking the M–elimination Theorem 2.3.3, $G \mid \vdash \Gamma, A^n, \overline{A}^n$ has a CAN–free M–free derivation and we can conclude by using Lemma 77. □

Finally, we prove Lemma 78 and Lemma 79, the stronger versions of Lemma 75 and Lemma 76.

Lemma 78. *If there is a CAN–free and M–free derivation of the hypersequent*
$$[\vdash \Gamma_i, x^{n_i}, \overline{x}^{m_i}]_{i=1}^k$$
then for all n_i', m_i' such that $n_i - m_i = n_i' - m_i'$ for all $i \in [1..k]$, there is a CAN–free, M–free derivation of
$$\left[\vdash \Gamma_i, x^{n_i'}, \overline{x}^{m_i'}\right]_{i=1}^k$$

Proof. The proof is done by induction on the derivation and is similar to the proof of Lemma 41. □

Lemma 79. *If there is a CAN–free and M–free derivation of the hypersequent*
$$\left[\vdash \Gamma_i, 1^{n_i}, \overline{1}^{m_i}\right]_{i=1}^k$$
then for all n_i', m_i' such that $n_i - m_i \leq n_i' - m_i'$ for all $i \in [1..k]$, there is a CAN–free, M–free derivation of
$$\left[\vdash \Gamma_i, 1^{n_i'}, \overline{1}^{m_i'}\right]_{i=1}^k$$

Proof. By induction on the derivation of $\left[\vdash \Gamma_i, 1^{n_i}, \overline{1}^{m_i}\right]_{i=1}^k$. Most cases are trivial, we just describe the most interesting one.

- If the derivation finishes with:
$$\frac{\left[\vdash \Gamma_i, 1^{n_i}, \overline{1}^{m_i}\right]_{i\geq 2} \mid \vdash \Gamma_1, 1^{n_c}, \overline{1}^{m_c}}{\left[\vdash \Gamma_i, 1^{n_i}, \overline{1}^{m_i}\right]_{i\geq 2} \mid \vdash \Gamma_1, 1^{n_a+n_b+n_c}, \overline{1}^{m_a+m_b+m_c}} \text{ 1}$$
with $n_1 = n_b + n_c$, $m_1 = m_b + m_c$ and $n_a + n_b = m_a + m_b$.
We want to show that
$$\rhd_{\mathbf{MGA}\backslash\{\text{CAN}\}} \left[\vdash \Gamma_i, 1^{n_i'}, \overline{1}^{m_i'}\right]_{i\geq 2} \mid \vdash \Gamma_1, 1^{n_a}, 1^{n_1'}, \overline{1}^{m_a}, \overline{1}^{m_1'}$$
We will now prove that $n_c - m_c \leq n_1' + n_a - (m_1' + m_a)$ to be able to conclude with the induction hypothesis.
$$\begin{aligned} n_c - m_c &= (n_1 - n_b) - (m_1 - m_b) \\ &= (n_1 - m_1) + (m_b - n_b) \\ &\leq (n_1' - m_1') + (n_a - m_a) \\ &= (n_1' + n_a) - (m_1' + m_a) \end{aligned}$$

so by induction hypothesis, we have

$$\triangleright_{\mathbf{MGA}\backslash\{\mathrm{CAN}\}} \left[\vdash \Gamma_i, 1^{n'_i}, \overline{1}^{m'_i}\right]_{i\geq 2} \mid \vdash \Gamma_1, 1^{n_a}, 1^{n'_1}, \overline{1}^{m_a}, \overline{1}^{m'_1}$$

which is the result we want.

□

2.3.7 Algebraic property

We will now adapt the algebraic property of the system **GA**‖ (see Section 2.1.7) to the system **MGA**. However, instead of characterising the derivability of atomic hypersequents, we will give an algebraic characterisation for *basic* hypersequents, i.e., hypersequents with only atoms, 1, $\overline{1}$ and $\Diamond$ terms.

The main difference with the algebraic property of the system **GA**‖ is that instead of reducing the question of derivability to just real numbers, we also reduce it to the derivability of a simpler hypersequent, where we remove the outer applications of the $\Diamond$ operator.

Theorem 2.3.5. *For all basic hypersequents G, built using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$, of the form*

$$\vdash \Gamma_1, \Diamond\Delta_1, 1^{c_1}, \overline{1}^{d_1} \mid \ldots \mid \vdash \Gamma_m, \Diamond\Delta_m, 1^{c_m}, \overline{1}^{d_m}$$

where $\Gamma_i = x_1^{a_{i,1}}, ..., x_k^{a_{i,k}}, \overline{x_1}^{b_{i,1}}, ..., \overline{x_k}^{b_{i,k}}$, the following are equivalent:

1. *G has a derivation.*
2. *there exist numbers $t_1, ..., t_m \in \mathbb{N}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*
   $$\sum_{i=1}^{m} t_i(\sum a_{i,j}) = \sum_{i=1}^{m} t_i(\sum b_{i,j})$$
   i.e., the scaled (by the numbers $t_1 \ldots t_m$) sum of the number of the variable x_j is equal to the scaled sum of the number of the covariable $\overline{x_j}$.
   - *$\sum_{i=1}^{m} t_i(\sum c_i) \geq \sum_{i=1}^{m} t_i(\sum d_i)$, i.e., there are more 1 than $\overline{1}$, and*
   - *the hypersequent*
   $$\vdash \Delta_1^{t_1}, ..., \Delta_m^{t_m}, 1^n$$
   where $n = \sum_{i=1}^{m} t_i(\sum c_i) - \sum_{i=1}^{m} t_i(\sum d_i)$ has a derivation, i.e., the hypersequent obtained after cancelling every atoms and using the $\Diamond$ rule is derivable.

Proof. We prove (1) ⇒ (2) by induction on the derivation of G. By using Theorem 2.3.4, we can assume that the derivation of G is CAN–free. We will only deal with the case of $\Diamond$ rule since every other case is similar to the proof of Theorem 2.1.5. If the derivation finishes with

$$\frac{\vdash \Delta_1, 1^{n_1}}{\vdash \Diamond\Delta_1, 1^{n_1}}\ \Diamond$$

then $t_1 = 1$ satisfies the property.

The other way ((2) ⇒ (1)) is also very similar to Theorem 2.1.5, only finishing with the $\Diamond$ rule. If there exist numbers $t_1, ..., t_m \in \mathbb{N}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum a_{i,j}) = \sum_{i=1}^{m} t_i(\sum b_{i,j})$$

- $\sum_{i=1}^{m} t_i(\sum c_i) \geq \sum_{i=1}^{m} t_i(\sum d_i)$, and
- the hypersequent

$$\vdash \Delta_1^{t_1}, ..., \Delta_m^{t_m}, 1^n$$

  where $n = \sum_{i=1}^{m} t_i(\sum c_i) - \sum_{i=1}^{m} t_i(\sum d_i)$ has a derivation

then we can use the W rule to remove the sequents corresponding to the numbers $t_i = 0$, and use the C and S rule on the ith sequent to multiply it by t_i. If we assume that there is a natural number l such that $t_i = 0$ for all $i > l$ and $t_i \neq 0$ for all $i \leq l$, then the CAN–free derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Delta_1^{t_1}, \ldots, \Delta_l^{t_l}, 1^n}{\vdash \Diamond\Delta_1^{t_1}, \ldots, \Diamond\Delta_l^{t_l}, 1^n}\ \Diamond}{\vdash \Diamond\Delta_1^{t_1}, 1^{t_1 c_1}, \overline{1}^{t_1 d_1}, \ldots, \Diamond\Delta_l^{t_l}, 1^{t_l c_l}, \overline{1}^{t_l d_l}}\ 1}{\vdash \Gamma_1^{t_1}, \Diamond\Delta_1^{t_1}, 1^{t_1 c_1}, \overline{1}^{t_1 d_1}, \ldots, \Gamma_l^{t_l}, \Diamond\Delta_l^{t_l}, 1^{t_l c_l}, \overline{1}^{t_l d_l}}\ \text{ID}^*}{\vdash \Gamma_1^{t_1}, \Diamond\Delta_1^{t_1}, 1^{t_1 c_1}, \overline{1}^{t_1 d_1} \mid \ldots \mid \vdash \Gamma_l^{t_l}, \Diamond\Delta_l^{t_l}, 1^{t_l c_l}, \overline{1}^{t_l d_l}}\ \text{S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, 1^{c_1}, \overline{1}^{d_1} \mid \ldots \mid \vdash \Gamma_l, \Diamond\Delta_l, 1^{c_l}, \overline{1}^{d_l}}\ \text{C}^* - \text{S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, 1^{c_1}, \overline{1}^{d_1} \mid \ldots \mid \vdash \Gamma_m, \Diamond\Delta_m, 1^{c_m}, \overline{1}^{d_m}}\ \text{W}^*$$

where $n = \sum_{i=1}^{m} t_i(\sum c_i) - \sum_{i=1}^{m} t_i(\sum d_i)$ and since $\vdash \Delta_1^{t_1}, \ldots, \Delta_l^{t_l}, 1^n$ is derivable, we can complete the derivation. ☐

Remark 33. Notice that if the hypersequent is atomic, the algebraic property of **MGA** is exactly the same as for **GA**$\|$.

2.4 Hypersequent calculus HMR

In this section, we will merge both the system **HR**, dealing with scalar and Riesz terms, and the system **MGA**, regarding the $\Diamond$ operator and the 1 constant. Since the two modifications are quite orthogonal, merging the two systems goes without any additional difficulty. Most of the difficulties were already seen and resolved in the two previous sections.

As in the previous sections, we start by recalling the definitions and conventions used in the system **HMR**.

Definition 2.4.1. A *weighted term* is a formal expression $r.A$ where $r \in \mathbb{R}_{>0}$ and A is a term in NNF.

Recall that the scalars appearing in these terms in NNF are all strictly positive and are ranged over by the letters $r, s, t \in \mathbb{R}_{>0}$. From now on, the term scalar should always be understood as strictly positive scalar.

We remind the readers of the following notations used in the system **HR**:

- Given a sequence $\vec{r} = (r_1, \ldots r_n)$ of scalars and a term A, we denote with $\vec{r}.A$ the multiset $[r_1.A, \ldots, r_n.A]$. When $\vec{r}$ is empty, the multiset $\vec{r}.A$ is also empty.
- Given a multiset $\Gamma = [r_1.A_1, \ldots, r_n.A_n]$ and a scalar $s > 0$, we denote with $s.\Gamma$ the multiset $[s.r_1.A_1, \ldots, s.r_n.A_n]$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars and a multiset Γ, we denote with $\vec{s}.\Gamma$ the multiset $s_1.\Gamma, \ldots, s_n.\Gamma$.
- Given two sequences $\vec{r} = (r_1, \ldots r_n)$ and $\vec{s} = (s_1, \ldots s_m)$ of scalars, we denote $\vec{r}; \vec{s}$ the concatenation of the two sequences, i.e. the sequence $(r_1, \ldots r_n, s_1, \ldots s_m)$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars and a scalar r, we denote $(r\vec{s})$ the sequence $(rs_1, \ldots rs_n)$.
- Given two sequences $\vec{r} = (r_1, \ldots r_n)$ and $\vec{s} = (s_1, \ldots s_m)$ of scalars, we denote $\vec{r}\vec{s}$ the sequence $r_1\vec{s}; \ldots; r_n\vec{s}$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars, we denote $\sum \vec{s}$ the sum of all scalars in $\vec{s}$, i.e. the scalar $\sum_{i=1}^{n} s_i$.

Definition 2.4.2. A *sequent* is a formal expression of the form $\vdash \Gamma$.

If $\Gamma = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 2.4.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \ldots | \vdash \Gamma_n$.

We use the letter G, H to range over hypersequents.

As in the system **MGA**, we will use two notions of "simple" hypersequents since atomic hypersequents are not sufficient in the presence of the $\Diamond$ operator.

Definition 2.4.4. A hypersequent is said *atomic* if it only contains atoms, i.e., formulas of the form x or $\overline{x}$.

Definition 2.4.5. A hypersequent is said *basic* if it only contains atoms (formulas of the form x or $\overline{x}$), $\Diamond$ formulas (formulas of the form $\Diamond A$), 1 formulas and $\overline{1}$ formulas.

We now describe how sequents and hypersequents can be interpreted by modal Riesz terms.

Definition 2.4.6 (Interpretation)**.** We interpret sequents $\vdash \Gamma$ and hypersequents G as the modal Riesz terms $(\!|\vdash \Gamma|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Weighted terms	$r.A$	rA
Sequents	$\vdash r_1.A_1, \ldots, r_n.A_n$	$(\!|r_1.A_1|\!) + \cdots + (\!|r_n.A_n|\!)$
Hypersequents	$\vdash \Gamma_1 | \ldots | \vdash \Gamma_n$	$(\!|\vdash \Gamma_1|\!) \sqcup \cdots \sqcup (\!|\vdash \Gamma_n|\!)$

Hence a sequent is interpreted as sum ($\sum$) and a hypersequent is interpreted as a join of sums ($\bigsqcup \sum$).

The hypersequent calculus **HMR** is a deductive system for deriving hypersequents whose interpretation is positive, i.e., the hypersequents G such that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq (\!|G|\!)$. The rules of **HMR** consist of the rules of the system **HR** (see Figure 2.5) with the additional rules of Figure 2.13 below.

Modal rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}\ 1, \sum \vec{r} \geq \sum \vec{s} \qquad \frac{\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}}\ \Diamond, \sum \vec{r} \geq \sum \vec{s}$$

Figure 2.13: Inference rules of **HMR**

Notice that the 1 rule is very similar to the system **MGA** modulo the usual translation $A^n \leftrightarrow \vec{r}.A$. However, the $\Diamond$ is a little more complicated, there are no longer only 1 but also $\overline{1}$. While the system with the following $\Diamond$ rule

$$\frac{\vdash \Gamma, \vec{r}.1}{\vdash \Diamond\Gamma, \vec{r}.1}\ \Diamond_4$$

is both sound and complete, it does not satisfy the CAN-elimination theorem. Indeed, the hypersequent $\vdash 1.\Diamond\overline{1}, 2.1, 1.\overline{1}$ has the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 2.1, 2.\overline{1}}\ 1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash 1.\overline{1}, 1.1}\ 1}{\vdash 1.\Diamond\overline{1}, 1.1}\ \Diamond_4}{\vdash 1.\Diamond\overline{1}, (1,1).1, 1.\overline{1}}\ 1}{\vdash 1.\Diamond\overline{1}, 2.1, 1.\overline{1}, (1,1).1, 2.\overline{1}}\ \text{M}}{\vdash 1.\Diamond\overline{1}, 2.1, 1.\overline{1}}\ \text{CAN}, 1+1=2$$

but would not have a CAN-free derivation with the rule above. The idea is that the structural rules can only be used to multiply the sequent by a real number, and so the only useful rule we can use is the 1 rule, whose premise can only be $\vdash 1.\Diamond\overline{1}$ which is not derivable.

Definition 2.4.7 (Modal depth of a hypersequent)**.** We define the modal depth of a sequent, noted $d^{\Diamond}(\vdash \Gamma)$, has the maximal modal depth of a term in Γ, i.e., if $\Gamma = r_1.A_1, ..., r_n.A_n$, $d^{\Diamond}(\vdash \Gamma) = \max_{i \in [1..n]} d^{\Diamond}(A_i)$.

The modal depth of a hypersequent G, noted $d^{\Diamond}(G)$, is then the maximal modal depth of the sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $d^{\Diamond}(G) = \max_{i \in [1..n]} d^{\Diamond}(\vdash \Gamma_i)$.

Definition 2.4.8 (Complexity)**.** We define the complexity of a sequent $\vdash \Gamma$, noted $c(\vdash \Gamma)$, as the sum of the operators which are not under a $\Diamond$ operator or the constant 1 used in the terms of Γ (see Definition 1.3.4), i.e., if $\Gamma = r_1.A_1, ..., r_n.A_n$, $c(\vdash \Gamma) = \sum_{i=1}^{n} c^{\Diamond}(A_i)$.

The complexity of a hypersequent G, noted $c(G)$, is then defined as the triplet $c(G) = (a, b, c)$ where

- a is the modal depth of the hypersequent G, and
- b is the maximum complexity of a sequent in G, i.e., if $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$, then $b = \max_{i \in [1..n]} c(\vdash \Gamma_i)$, and
- c is the number of sequents in G having a complexity of b, i.e., $c = \#\{\vdash \Gamma_i \mid c(\vdash \Gamma_i) = b\}$.

Definition 2.4.9 (Modal depth)**.** The *modal depth* of a derivation is the maximal number of $\Diamond$ rules used in a branch of the derivation.

2.4.1 Preliminary lemmas

We proved the usual technical lemmas used in this section. The results and proofs are very similar to the one for the system **MGA** where we replace the occurrences of A^n and Γ^n with $\vec{r}.A$ and $\vec{r}.\Gamma$.

Our first lemma states that the following variant of the ID rule (see Figure 2.5) where general terms A are considered rather than just variables, is admissible in the proof system **HMR**.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}} \text{ ID}, \sum \vec{r} = \sum \vec{s}$$

Lemma 80. *For all $A, \vec{r_i}, \vec{s_i}$ such that $\sum \vec{r_i} = \sum \vec{s_i}$, it holds that:*

$$\textit{if } \rhd_{\boldsymbol{HMR}} [\vdash \Gamma_i]_{i=1}^n \textit{ then } \rhd_{\boldsymbol{HMR}} \left[\vdash \Gamma_i, \vec{r_i}.A, \vec{s_i}.\overline{A}\right]_{i=1}^n.$$

Proof. Let d be a derivation of $\rhd_{\mathbf{HMR}} [\vdash \Gamma_i]_{i=1}^n$. We prove the result by double induction on (A, d). If A is not a $\Diamond$ term, we prove the result as in Lemma 42 – which decreases the complexity of the term each time. Otherwise $A = \Diamond B$. We prove the result by induction on the derivation d. We will only show three cases: the other cases are similar to the $+$ case.

- If d finishes with
$$\frac{}{\vdash} \text{ INIT}$$
then by induction hypothesis on B, $\rhd_{\mathbf{HMR}} \vdash \vec{r_1}.B, \vec{s_1}.\overline{B}$ so
$$\frac{\vdash \vec{r_1}.B, \vec{s_1}.\overline{B}}{\vdash \vec{r_1}.\Diamond(B), \vec{s_1}.\Diamond(\overline{B})} \Diamond$$

- If d finishes with
$$\frac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.C, \vec{s}.D}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.(C+D)} +$$
then by induction hypothesis on the subderivation
$$\rhd_{\mathbf{HMR}} [\vdash \Gamma_i, \vec{r_i}.\Diamond(B), \vec{s_i}.\Diamond(\overline{B})]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.C, \vec{s}.D, \vec{r_1}.\Diamond(B), \vec{s_1}.\Diamond(\overline{B})$$
so
$$\frac{[\vdash \Gamma_i, \vec{r_i}.\Diamond(B), \vec{s_i}.\Diamond(\overline{B})]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.C, \vec{s}.D, \vec{r_1}.\Diamond(B), \vec{s_1}.\Diamond(\overline{B})}{[\vdash \Gamma_i, \vec{r_i}.\Diamond(B), \vec{s_i}.\Diamond(\overline{B})]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.(C+D), \vec{r_1}.\Diamond(B), \vec{s_1}.\Diamond(\overline{B})} +$$

- If d finishes with
$$\frac{\vdash \Gamma_1, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}} \Diamond$$
then by induction hypothesis on B
$$\rhd_{\mathbf{HMR}} \vdash \Gamma_1, \vec{r_1}.B, \vec{s_1}.\overline{B}, \vec{r}.1, \vec{s}.\overline{1}$$
so
$$\frac{\vdash \Gamma_1, \vec{r_1}.B, \vec{s_1}.\overline{B}, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond(\Gamma_1), \vec{r_1}.\Diamond(B), \vec{s_1}.\Diamond(\overline{B}), \vec{r}.1, \vec{s}.\overline{1}} \Diamond$$
□

The next lemma states that if G is provable then the hypersequent obtained by substituting an atom for a term in G is also provable.

Lemma 81. *If* $\triangleright_{\boldsymbol{HMR}} G$ *then for all terms* A, $\triangleright_{\boldsymbol{HMR}} G[A/x]$.

Proof. Similar to the proof of Lemma 43. □

The next lemma states that the $\{0, +, \times, \sqcup, \sqcap\}$-logical rules are invertible using the CAN rule, meaning that if the conclusion is derivable, then the premises are also derivable. Unlike a stronger result we will prove later in Section 2.4.4 where we prove the CAN–free version of this lemma, the derivations of the premises may contain CAN rules and thus this result is not sufficient to imply the CAN elimination theorem.

Lemma 82. *All the logical rules* $\{0, +, \times, \sqcup, \sqcap\}$ *are invertible.*

Proof. Similar to Lemma 44. □

Remark 34. The proof of Lemma 82 does not introduce any new T rule, so if the conclusion of one of the logical rules $\{0, +, \times, \sqcup, \sqcap\}$ has a T–free derivation, then the premises also have T–free derivations.

Remark 35. As in the system **MGA**, note that the $\Diamond$ rule is also invertible, but because of the constraints of the $\Diamond$ rule, we can not use it when we would need it, and thus we leave this result to Chapter 3.

The next lemmas state that CAN–free derivability in the **HMR** system is preserved by scalar multiplication.

Lemma 83. *Let* $\vec{r} \in \mathbb{R}_{>0}$ *be a non-empty vector and* G *a hypersequent. If* $\triangleright_{\boldsymbol{HMR}\backslash\{CAN\}} G \mid \vdash \vec{r}.\Gamma$ *then* $\triangleright_{\boldsymbol{HMR}\backslash\{CAN\}} G \mid \vdash \Gamma$.

Proof. Similar to Lemma 45. □

Lemma 84. *Let* $\vec{r} \in \mathbb{R}_{>0}$ *be a vector and* G *a hypersequent. If* $\triangleright_{\boldsymbol{HMR}\backslash\{CAN\}} G \mid \vdash \Gamma$ *then* $\triangleright_{\boldsymbol{HMR}\backslash\{CAN\}} G \mid \vdash \vec{r}.\Gamma$.

Proof. Similar to Lemma 46. □

2.4.2 Soundness

We need to prove that if there exists a **HMR** derivation of a hypersequent G then $(\!|G|\!) \geq 0$ is derivable in equational logic (written $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$). This is done in a straightforward way by showing that each deduction rule of the system **HMR** is sound. Notice that the soundness of the rules already present in **HR** is proved in the exact same way so we will only show the soundness of the new rules.

Theorem 2.4.1 (Soundness of **HMR**)**.** *For all hypersequent* G*, if* $\triangleright_{\boldsymbol{HMR}} G$ *then* $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$.

Proof. By induction on the derivation of G. We only show the $\Diamond$ and 1 rules since the other are similar to Theorem 2.2.1.

- For the rule

$$\frac{\vdash \Gamma, 1^n}{\vdash \Diamond\Gamma, 1^n}\ \Diamond$$

the hypothesis is $(\!|\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}|\!) \geq 0$ so

$$\begin{array}{rcl}(\!|\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}|\!) & = & (\!|\vdash \Diamond\Gamma, (\sum \vec{r} - \sum \vec{s}).1|\!) \text{ by distributivity} \\ & \geq & (\!|\vdash \Diamond\Gamma, (\sum \vec{r} - \sum \vec{s}).\Diamond 1|\!) \text{ since } \Diamond 1 \leq 1 \\ & = & \Diamond((\!|\vdash \Gamma, (\sum \vec{r} - \sum \vec{s}).1|\!)) \text{ by linearity of } \Diamond \\ & = & \Diamond((\!|\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1})|\!) \text{ by distributivity} \\ & \geq & 0 \text{ by the hypothesis and the monotonicity of } \Diamond.\end{array}$$

- For the rule

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}\ 1, \sum \vec{r} \geq \sum \vec{s}$$

the hypothesis is $(\!|G \mid \vdash \Gamma|\!) \geq 0$ so

$$\begin{array}{rcl}(\!|G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}|\!) & \geq & (\!|G \mid \vdash \Gamma|\!) \text{ since } \sum \vec{r} \geq \sum \vec{s} \text{ and } 0 \leq 1 \\ & \geq & 0\end{array}$$

□

2.4.3 Completeness

In order to prove the completeness of the system **HMR**, i.e. that if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ then $\triangleright_{\mathbf{HMR}} G$, we first prove an equivalent result (Lemma 85 below) stating that if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ then the hypersequents $\vdash r.A, r.\overline{B}$ and $\vdash r.B, r.\overline{A}$ are both derivable for all $r > 0$.

From Lemma 85 one indeed obtains Theorem 2.4.2 as a corollary.

Theorem 2.4.2 (Completeness of **HMR**). *For all hypersequent G, if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ then $\triangleright_{\boldsymbol{HMR}} G$.*

Proof. Recall that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ is a shorthand for $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 = (\!|G|\!) \sqcap 0$. Hence, from the hypothesis $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$ we can deduce, by using Lemma 85, that $\triangleright_{\mathbf{HMR}} \vdash 1.0 \sqcap (\!|G|\!), 1.0$.

From this we can show that $\triangleright_{\mathbf{HMR}} G$ by invoking Lemma 82. Indeed, if G is $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ then $(\!|G|\!) = (\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)$ and

1. by using the invertibility of the 0 rule, $\vdash 1.(0 \sqcap ((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!)))$ is derivable,
2. by using the invertibility of the $\sqcap$ rule, $\vdash 1.((\!|\vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\vdash \Gamma_n|\!))$ is derivable,
3. by using the invertibility of the $\sqcup$ rule $n-1$ times, $\vdash 1.(\!|\vdash \Gamma_1|\!) \mid \ ... \mid \vdash 1.(\!|\vdash \Gamma_n|\!)$ is derivable,
4. and finally, by using the invertibility of the $+$ and $\times$ rules, $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ is derivable.

□

Lemma 85. *If $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ then for all $r > 0$, $\vdash r.A, r.\overline{B}$ and $\vdash r.B, r.\overline{A}$ are derivable.*

Proof. Since the other cases are proven in the exact same way as in Theorem 2.2.2, we will only derive the new axioms.

- For the axiom $0 \leq 1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.0}\,0 \qquad \dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.1}\,1, r \geq 0}{\vdash r.(0 \sqcap 1)}\,\sqcap}{\vdash r.(0 \sqcap 1), r.0}\,0$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.0}\,0}{\vdash r.0 \mid \vdash r.\overline{1}}\,\text{W}}{\vdash r.(0 \sqcup \overline{1})}\,\sqcup}{\vdash r.0, r.(0 \sqcup \overline{1})}\,0$$

- For the axiom $\Diamond(1) \leq 1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.1, r.\overline{1}}\,1}{\vdash r.\Diamond(1), r.\Diamond(\overline{1})}\,\Diamond \qquad \dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.1, r.\overline{1}}\,1}{\vdash r.1, r.\Diamond(\overline{1})}\,\Diamond}{\vdash r.(\Diamond(1) \sqcap 1), r.\Diamond(\overline{1})}\,\sqcap$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r.1, r.\overline{1}}\,1}{\vdash r.\Diamond(1), r.\Diamond(\overline{1})}\,\Diamond}{\vdash r.\Diamond(1), r.\Diamond(\overline{1}) \mid \vdash r.\Diamond(1), r.\overline{1}}\,\text{W}}{\vdash r.\Diamond(1), r.(\Diamond(\overline{1}) \sqcup \overline{1})}\,\sqcup$$

- For the axiom $\Diamond(r_1 x + r_2 y) = r_1 \Diamond(x) + r_2 \Diamond(y)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r_1 r.x, r_2 r.y, r_1 r.\overline{x}, r_2 r.\overline{y}}\,\text{ID}^2}{\vdash r.r_1 x, r.r_2 y, r_1 r.\overline{x}, r_2 r.\overline{y}}\,\times^2}{\vdash r.(r_1 x + r_2 y), r_1 r.\overline{x}, r_2 r.\overline{y}}\,+}{\vdash r.\Diamond(r_1 x + r_2 y), r_1 r.\Diamond(\overline{x}), r_2 r.\Diamond(\overline{y})}\,\Diamond}{\vdash r.\Diamond(r_1 x + r_2 y), r.r_1 \Diamond(\overline{x}), r.r_2 \Diamond(\overline{y})}\,\times^2}{\vdash r.\Diamond(r_1 x + r_2 y), r.(r_1 \Diamond(\overline{x}) + r_2 \Diamond(\overline{y}))}\,+$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\,\text{INIT}}{\vdash r_1 r.x, r_2 r.y, r_1 r.\overline{x}, r_2 r.\overline{y}}\,\text{ID}^2}{\vdash r_1 r.x, r_2 r.y, r.r_1 \overline{x}, r.r_2 \overline{y}}\,\times^2}{\vdash r_1 r.x, r_2 r.y, r.(r_1 \overline{x} + r_2 \overline{y})}\,+}{\vdash r_1 r.\Diamond(x), r_2 r.\Diamond(y), r.\Diamond(r_1 \overline{x} + r_2 \overline{y})}\,\Diamond}{\vdash r.r_1 \Diamond(x), r.r_2 \Diamond(y), r.\Diamond(r_1 \overline{x} + r_2 \overline{y})}\,\times^2}{\vdash r.(r_1 \Diamond(x) + r_2 \Diamond(y)), r.\Diamond(r_1 \overline{x} + r_2 \overline{y})}\,+$$

- For the axiom $0 \leq \Diamond(0 \sqcup x)$.

$$
\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash r.0}\;0 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash r.0}\;0}{\vdash r.0 \mid \vdash r.x}\;\text{W}}{\vdash r.(0 \sqcup x)}\;\sqcup}{\vdash r.\Diamond(0 \sqcup x)}\;\Diamond}{\dfrac{\vdash r.(0 \sqcap \Diamond(0 \sqcup x))}{\vdash r.(0 \sqcap \Diamond(0 \sqcup x)), r.0}\;0}\;\sqcap
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash r.0}\;0}{\vdash r.0 \mid \vdash r.\Diamond(0 \sqcap \overline{x})}\;\text{W}}{\vdash r.(0 \sqcup \Diamond(0 \sqcap \overline{x}))}\;\sqcup}{\vdash r.0, r.(0 \sqcup \Diamond(0 \sqcap \overline{x}))}\;0
$$

□

Remark 36. By inspecting the proof of Lemma 85 it is possible to verify that the T rule is never used in the construction of $\triangleright_{\mathbf{HMR}} G$. This, together with the similar Remark 34 regarding Lemma 82, implies that the T rule is never used in the proof of the completeness Theorem 2.4.2. From this we get the following corollary.

Corollary 3. *The T rule is admissible in the system* ***HMR****.*

However, as in the system **HR**, there is no hope of eliminating both the T rule and the CAN rule from the **HMR** system.

Lemma 86. *Let r_1 and r_2 be two irrational numbers that are incommensurable (so there is no $q \in \mathbb{Q}$ such that $qr_1 = r_2$). Then the atomic hypersequent G*

$$\vdash r_1.x \mid \vdash r_2.\overline{x}$$

does not have a CAN–free and T–free derivation.

Proof. By analysing a potential CAN–free T–free derivation of $\vdash r_1.x \mid \vdash r_2.\overline{x}$, we can show that only the S, M, C, W, ID and INIT rules can be used (the logical rules do nothing since there are only atoms in the hypersequent).

Therefore, if $\vdash r_1.x \mid \vdash r_2.\overline{x}$ has a CAN–free T–free derivation in the system **HMR**, it also has one in the system **HR** which contradicts Lemma 48. □

2.4.4 CAN–free invertibility

In this section, we prove the $0, +, \times, \sqcup$ and $\sqcap$ rules are CAN-free invertible, i.e., that if the conclusion of one of those logical rules has a CAN-free derivation, then so do the premises. As for the previous systems, it allows us to reduce the complexity of the formulas in an hypersequent in the proof of the CAN elimination theorem, and thus it is important that we do not add any CAN rule in the proofs of invertibility. For this reason, the CAN–free invertibility result is stronger than Lemma 82 of Section 2.4.1.

As usual, we prove the CAN–free invertibility of more general rules. The general rules are the same as the ones in the system **HR**:

Logical rules:

$$\frac{[\vdash \Gamma_i]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.0]_{i=1}^n}\ 0 \quad \frac{[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A+B)]_{i=1}^n}\ + \quad \frac{[\vdash \Gamma_i, (s\vec{r}_i).A]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(sA)]_{i=1}^n}\ \times$$

$$\frac{[\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=1}^n}\ \sqcup \quad \frac{[\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n \quad [\vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=1}^n}\ \sqcap$$

Figure 2.14: Generalised logical rules

We will prove that those rules are CAN–free invertible by induction on the derivation of the conclusion. The proof steps dealing with the rules already present in **HR** are the same as in Section 2.2.4. In what follows we just show the details of the proof steps associated with the new cases associated with the $\Diamond$–rule and 1–rule of **HMR**.

We conceptually divide the logical rules in three categories:

Type 1 The rule with only one premise but that adds one sequent to the hypersequent: the $\sqcup$ rule.

Type 2 The rules with only one premise and that do not change the number of sequents: the $0, +$ rules.

Type 3 The rule with two premises: the $\sqcap$ rule.

Because of the similarities of the rules in each of these categories, we just prove the CAN–free invertibility of one rule in each category by means of a sequence of lemmas.

Lemma 87 (Type 1)**.** *If* $[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n$ *has a CAN–free derivation.*

Proof. By induction on the derivation.

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B)}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B), \vec{r}.1, \vec{s}.\overline{1}}\ 1$$

  then by induction hypothesis on the CAN–free derivation of the premise we have that

$$\triangleright_{\mathbf{HMR}\backslash\{\mathrm{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \vec{r}_1.B$$

  so

$$\frac{G' \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \vec{r}_1.B}{G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{r}.1, \vec{s}.\overline{1}}\ 1^*$$

  with $G' = [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=2}^n$

- If the derivation finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

  with $\vec{r_1} = \emptyset$ so the hypersequent

$$\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Diamond\Gamma_1, \vec{r_1}.B, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

  is CAN–free derivable using the C rule. □

Lemma 88 (Type 2)**.** *If* $[\vdash \Gamma_i, \vec{r_i}(A+B)]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, \vec{r_i}A, \vec{r_i}B]_{i=1}^n$ *has a CAN–free derivation.*

Proof. By induction on the derivation.

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i, \vec{r_i}.(A+B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.(A+B)}{[\vdash \Gamma_i, \vec{r_i}.(A+B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.(A+B), \vec{r}.1, \vec{s}.\overline{1}}\ 1$$

  then by induction hypothesis on the CAN–free derivation of the premise we have that

$$\triangleright_{\mathbf{HMR}\setminus\{\text{CAN}\}}[\vdash \Gamma_i, \vec{r_i}.A, \vec{r_i}.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A, \vec{r_1}.B$$

  so

$$\frac{[\vdash \Gamma_i, \vec{r_i}.A, \vec{r_i}.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A, \vec{r_1}.B}{[\vdash \Gamma_i, \vec{r_i}.A, \vec{r_i}.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A, \vec{r_1}.B, \vec{r}.1, \vec{s}.\overline{1}}\ 1$$

- If the derivation finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

  with $\vec{r_1} = \emptyset$ so the hypersequent $\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r_1}.B, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$ is derivable. □

Lemma 89 (Type 3)**.** *If* $\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=1}^n$ *has a CAN–free derivation then* $[\vdash \Gamma_i, \vec{r_i}.A]_{i=1}^n$ *and* $[\vdash \Gamma_i, \vec{r_i}.B]_{i=1}^n$ *have CAN–free derivations.*

Proof. By induction on the derivation. We will only show that $\triangleright_{\mathbf{HMR}\setminus\{\text{CAN}\}}[\vdash \Gamma_i, \vec{r_i}.A]_{i=1}^n$, the other case is similar.

- If the derivation finishes with

$$\frac{[\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.(A \sqcap B)}{[\vdash \Gamma_i, \vec{r_i}.(A \sqcap B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.(A \sqcap B), \vec{r}.1, \vec{s}.\overline{1}}\ 1$$

  then by induction hypothesis on the CAN–free derivation of the premise we have that

$$\triangleright_{\mathbf{HMR}\setminus\{\text{CAN}\}}[\vdash \Gamma_i, \vec{r_i}.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A$$

  so

$$\frac{[\vdash \Gamma_i, \vec{r_i}.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A}{[\vdash \Gamma_i, \vec{r_i}.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.A, \vec{r}.1, \vec{s}.\overline{1}}\ 1$$

- If the derivation finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

  with $\vec{r_1} = \emptyset$ so the hypersequent $\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$ is derivable. □

2.4.5 M–elimination

In this section, we will show the M elimination theorem. Recall that the M elimination theorem states

if a hypersequent G is derivable, then it has a M–free derivation.

However, since this result will be used in the proof of the CAN elimination theorem, we have to ensure that the M elimination theorem does not add any instance of the CAN rule. Thus we will show the slightly different result

if a hypersequent G is CAN–free derivable, then it has a CAN–free M–free derivation.

Following the same pattern of Section 2.3.5, we need to show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free derivations d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there also exists a CAN–free and M–free derivation of $G \mid \vdash \Gamma, \Delta$.

The general idea is similar to the one presented for the system **MGA**. The results and proofs can be easily adapted to the system **HMR** with the usual translation $A^n \leftrightarrow \vec{r}.A$. We recall the different steps of the proof (see Section 2.3.5 for more details).

- we first construct a prederivation d_1' of $G \mid\ G \mid \vdash \Gamma, \Delta$ (using d_1) whose unfinished leaves are hypersequents of the form $G \mid \vdash \vec{r}.\Delta, \Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1}$,
- then we construct a derivation $d_{2,\vec{r}}$ of $G \mid \vdash \vec{r}.\Delta$ using d_2,
- and a prederivation d_2' of $G \mid \vdash \vec{r}.\Delta, \Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1}$ (using $d_{2,\vec{r}}$) whose unfinished leaves are hypersequents of the form $\vdash \vec{r'}.(\Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1}), \Diamond\Delta', \vec{s'}.1, \vec{t'}.\overline{1}$,
- and finally we use the $\Diamond$ rule and we conclude by induction on modal depth of the derivations.

We now proceed with the technical statements.

Lemma 90. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ using the $\Diamond$ rule and let H be a hypersequent and Δ be a sequent. Then there exists a prederivation of*

$$G \mid\ H \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are either of the form $H \mid \vdash \vec{r}.\Delta$ or of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that

- $\sum \vec{s} \geq \sum \vec{t}$ *and*
- $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ *has a derivation d_1' with a strictly lower modal depth than d_1.*

Proof. This is an instance of the slightly more general statement of Lemma 93 below where:

- $[\vdash \Gamma_i]_{i=1}^{n-1} = G$ and $\Gamma_n = \Gamma$.
- $\vec{r}_i = \emptyset$ for $1 \leq i < n$ and $\vec{r}_n = 1$. ☐

Remark 37. Following Remark 38, if the derivation of $G \mid \vdash \Gamma$ uses no $\Diamond$ rule, then all unfinished leaves are of the form $H \mid \vdash \vec{r}.\Delta$.

Lemma 91. *Let d_2 be CAN–free and M–free derivation of $H \mid \vdash \Delta$. Then, for every vector $\vec{r}$, there exists a CAN–free and M–free derivation of*

$$H \mid \vdash \vec{r}.\Delta$$

with a modal depth lower or equal than d_2.

Proof. This is an instance of the slightly more general statement of Lemma 94 below where:

- $[\vdash \Delta_i]_{i=1}^{n-1} = H$ and $\Delta_n = \Delta$.
- $\vec{r_i} = 1$ for $1 \leq i < n$ and $\vec{r_n} = \vec{r}$. □

We now show how to remove one instance of the M rule and then the M-elimination theorem.

Lemma 92. *If $G \mid \vdash \Gamma$ and $H \mid \vdash \Delta$ have CAN–free M–free derivations, then so does $G \mid\ H \mid \vdash \Gamma, \Delta$.*

Proof. We show the lemma by induction on the modal depth of the derivation d of $G \mid \vdash \Gamma$.

If the modal depth of d is 0, then we proceed as in Theorem 2.2.3, i.e., we use Lemma 90 to have a prederivation of $G \mid\ H \mid \vdash \Delta, \Gamma$ where all leaves are of the form $H \mid \vdash \vec{r}.\Delta$, and we finish the prederivation by using Lemma 91.

Otherwise d uses some $\Diamond$ rule. We do the following:

- we use Lemma 90 to have a prederivation of $G \mid\ H \mid \vdash \Gamma, \Delta$ where all non–terminated leaves are either of the form $H \mid \vdash \vec{r}.\Delta$ or of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that
  * $\sum \vec{s} \geq \sum \vec{t}$ and
  * $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ has a derivation d_1' with a strictly lower modal depth than d_1.
- We show that all leaves of the form $H \mid \vdash \vec{r}.\Delta$ are derivable using Lemma 91.
- We conclude by showing that all leaves of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ are derivable. Let's show how to derive them.
  - We show that $H \mid \vdash \vec{r}.\Delta$ are derivable using Lemma 91.
  - Then we build a prederivation of $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ using Lemma 90 where all non-terminated leaves are either of the form $\vdash \vec{r'}.(\Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1})$ or of the form $\vdash \vec{r'}.(\Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1}), \Diamond\Delta', \vec{s'}.1, \vec{t'}.1$ such that
    * $\sum \vec{s'} \geq \sum \vec{t'}$ and
    * $\vdash \Delta', \vec{s'}.1, \vec{t'}.\overline{1}$ has a derivation.
  - The leaves of the form $\vdash \vec{r'}.(\Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1})$ can be terminated using the $\Diamond$ rule and Lemma 91.
  - For the leaves of the form $\vdash \vec{r'}.(\Diamond\Gamma', \vec{s}.1, \vec{t}.\overline{1}), \Diamond\Delta', \vec{s'}.1, \vec{t'}.1$, we will show that $\vdash \vec{r'}.(\Gamma', \vec{s}.1, \vec{t}.\overline{1}), \Delta', \vec{s'}.1, \vec{t'}.1$ has a CAN-free M-free derivation and we can conclude using the $\Diamond$ rule. Recall that $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ has a CAN-free M-free derivation with strictly lower modal depth than d_1. We use Lemma 91 to have a derivation of $\vdash \vec{r'}.(\Gamma', \vec{s}.1, \vec{t}.\overline{1})$ with strictly lower depth than d_1.
  - We can then use the induction hypothesis since the derivation of $\vdash \vec{r'}.(\Gamma', \vec{s}.1, \vec{t}.\overline{1})$ has a strictly lower depth than d_1, thus building a derivation of $\vdash \vec{r'}.(\Gamma', \vec{s}.1, \vec{t}.\overline{1}), \Delta', \vec{s'}.1, \vec{t'}.1$ to conclude the proof.

□

Theorem 2.4.3 (M elimination)**.** *If G is CAN-free derivable, then G is CAN-free M-free derivable.*

Proof. We prove the result by induction on G. The only interesting case if the M rule, i.e., if the derivation finishes with

$$\frac{G \mid \vdash \Gamma \quad G \mid \vdash \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{M}$$

then by induction hypothesis $G \mid \vdash \Gamma$ and $G \mid \vdash \Delta$ have CAN-free M-free derivation.

By using Lemma 92, we have a CAN-free M-free derivation of $G \mid \ G \mid \vdash \Gamma, \Delta$. The derivation is then

$$\frac{G \mid \ G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta}\ \text{C}^*$$

□

We now prove the technical version of Lemmas 90 and 91.

Lemma 93. *Let d_1 be a CAN–free and M–free derivation of $[\vdash \Gamma_i]_{i=1}^n$ and let H be a hypersequent and Δ be a sequent. Then for every sequence of vectors $\vec{r_i}$, there exists a prederivation of*

$$H \mid \ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^n$$

where all non–terminated leaves are either of the form $H \mid \vdash \vec{r}.\Delta$ or of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that

- $\sum \vec{s} \geq \sum \vec{t}$ *and*
- $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ *has a derivation d_1' with a strictly lower modal depth than d_1.*

Proof. We prove the result by induction on d_1. We will only show the $\Diamond$ and the 1 rules, since all other cases are done in the same way as in Lemma 56.

- if d_1 finishes with:

$$\frac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}}\ 1, \sum \vec{s} \geq \sum \vec{t}$$

  then by induction hypothesis, there is a prederivation of $H \mid \ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Delta$ where all non–terminated leaves are either of the form $H \mid \vdash \vec{r}.\Delta$ or of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that

  - $\sum \vec{s} \geq \sum \vec{t}$ and
  - $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ has a derivation d_1' with a strictly lower modal depth than d_1.

  We continue the prederivation with

$$\frac{H \mid \ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Delta}{H \mid \ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Delta, \vec{s}.1, \vec{t}.\overline{1}}\ 1, \sum \vec{s} \geq \sum \vec{t}$$

- If d_1 finishes with:

$$\frac{\vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}}{\vdash \Diamond\Gamma_1, \vec{s}.1, \vec{t}.\overline{1}}\ \Diamond, \sum \vec{s} \geq \sum \vec{t}$$

  then the prederivation is simply the leaf $H \mid \vdash \Diamond\Gamma_1, \vec{r_1}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ which satisfies both

– $\sum \vec{s} \geq \sum \vec{t}$ and

– $\vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}$ is derivable using strictly less $\Diamond$ rule than in d_1. $\square$

Remark 38. Since the leaves of the form $H \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ appear only in the case of the $\Diamond$ rule, if the derivation of $[\vdash \Gamma_i]_{i=1}^n$ does not use any $\Diamond$ rule, all leaves are of the form $H \mid \vdash \vec{r}.\Delta$.

Lemma 94. *If d_2 is a CAN–free M–free derivation of $[\vdash \Delta_i]_{i=1}^n$ then for all $\vec{r_i}$, there is a CAN–free M–free derivation of $[\vdash \vec{r_i}.\Delta_i]_{i=1}^n$ with a modal depth lower or equal to d_2.*

Proof. We will only show the $\Diamond$ and 1 rules, the other cases being similar to Lemma 57 – and so do not introduce any new $\Diamond$ rule.

- if d_2 finishes with:
$$\frac{[\vdash \Delta_i]_{i=2}^n \mid \vdash \Delta_1}{[\vdash \Delta_i]_{i=2}^n \mid \vdash \Delta_1, \vec{s}.1, \vec{t}.\overline{1}}\ 1, \sum \vec{s} \geq \sum \vec{t}$$
then by induction hypothesis, there is a CAN–free M–free derivation of $[\vdash \vec{r_i}.\Delta_i]_{i=2}^n \mid \vdash \vec{r_1}.\Delta_1$ with a modal depth lower or equal than d_2. We continue the derivation with
$$\frac{[\vdash \vec{r_i}.\Delta_i]_{i=2}^n \mid \vdash \vec{r_1}.\Delta_1}{[\vdash \vec{r_i}.\Delta_i]_{i=2}^n \mid \vdash \vec{r_1}.\Delta_1, (\vec{r_1}\vec{s}).1, (\vec{r_1}\vec{t}).\overline{1}}\ 1, \sum \vec{r_1}\vec{s} \geq \sum \vec{r_1}\vec{t}$$
which does not increase the modal depth of the derivation.

- If d_2 finishes with:
$$\frac{\vdash \Delta_1, \vec{s}.1, \vec{t}.\overline{1}}{\vdash \Diamond\Delta_1, \vec{s}.1, \vec{t}.\overline{1}}\ \Diamond, \sum \vec{s} \geq \sum \vec{t}$$
by induction hypothesis, there is a derivation of $\vdash \vec{r_1}.\Delta_1, (\vec{r_1}\vec{s}).1, (\vec{r_1}\vec{t}).\overline{1}$ with a modal depth strictly less than d_2. We continue the derivation with
$$\frac{\vdash \vec{r_1}.\Delta_1, (\vec{r_1}\vec{s}).1, (\vec{r_1}\vec{t}).\overline{1}}{\vdash \vec{r_1}.\Diamond\Delta_1, (\vec{r_1}\vec{s}).1, (\vec{r_1}\vec{t}).\overline{1}}\ \Diamond, \sum \vec{r_1}\vec{s} \geq \sum \vec{r_1}\vec{t}$$
which gives a derivation with a modal depth less or equal than d_2. $\square$

2.4.6 CAN–elimination

Recall that the CAN rule has the following form:
$$\frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}}{G \mid \vdash \Gamma}\ \text{CAN}, \sum \vec{r} = \sum \vec{s}$$

As in the previous sections, we prove Theorem 2.4.4 by showing that if the hypersequent $G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$ has a M–free CAN–free derivation, then so does the hypersequent $G \mid \vdash \Gamma$.

The proof follows the same pattern as in the system **MGA**: we first prove the result when $A = x$ (or equivalently $A = \overline{x}$) and $A = 1$, and by double induction on both the formula A and the derivation for complex formulas.

Lemma 95. *If there is a M–free CAN–free derivation of $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$, where $\sum \vec{r} = \sum \vec{s}$, then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. The statement follows as a special case of Lemma 98 below, a stronger version of Lemma 95 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$, where:

- $[\vdash \Gamma_i]_{i=1}^{n-1} = G$ and $\Gamma_n = \Gamma$.
- $\vec{r_i} = \vec{r_i'} = \vec{s_i} = \vec{s_i'} = \emptyset$ for $1 \leq i < n$.
- $\vec{r_n} = \vec{r}$, $\vec{s_n} = \vec{s}$ and $\vec{r_n'} = \vec{s_n'} = \emptyset$. □

Lemma 96. *If there is a M-free CAN–free derivation of $G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}$, where $\sum \vec{r} = \sum \vec{s}$ then there exists a M–free CAN–free derivation of $G \mid \vdash \Gamma$.*

Proof. The statement follows as a special case of Lemma 99 below, a stronger version of Lemma 96 that allows for a simpler proof by induction on the structure of the derivation of $G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}$, where:

- $[\vdash \Gamma_i]_{i=1}^{n-1} = G$ and $\Gamma_n = \Gamma$.
- $\vec{r_i} = \vec{r_i'} = \vec{s_i} = \vec{s_i'} = \emptyset$ for $1 \leq i < n$.
- $\vec{r_n} = \vec{r}$, $\vec{s_n} = \vec{s}$ and $\vec{r_n'} = \vec{s_n'} = \emptyset$. □

We are now ready to prove the general case.

Lemma 97. *For all terms A and numbers $n > 0$ and for all sequents Γ_i and vectors $\vec{r_i}, \vec{s_i}$ such that $\sum \vec{r_i} = \sum \vec{s_i}$, for $1 \leq i \leq n$,*

$$\textit{if } [\vdash \Gamma_i, \vec{r_i}.A, \vec{s_i}.\overline{A}]_{i=1}^{n} \textit{ has a M–free CAN–free derivation, then so does } [\vdash \Gamma_i]_{i=1}^{n}.$$

Proof. For the basic cases $A = x$, $A = \overline{x}$, $A = 1$ and $A = \overline{1}$, we use Lemmas 95 and 96. For complex terms A which are not $\Diamond$ terms, we proceed by invoking the CAN–free invertibility proven in Section 2.4.4 as follows:

- If $A = 0$, we can conclude with the CAN–free invertibility of the rule 0.
- If $A = B + C$, since the $+$ rule is CAN–free invertible, $\left[\vdash \Gamma_i, \vec{r_i}.B, \vec{r_i}.C, \vec{s_i}.\overline{B}, \vec{s_i}.\overline{C}\right]$ has a CAN–free, M–free derivation. Therefore we can have a CAN–free derivation of the hypersequent $[\vdash \Gamma_i]_{i=1}^{n}$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = r'B$, since the $\times$ rule is CAN–free invertible, $\left[\vdash \Gamma_i, (r'\vec{r_i}.).B, (r'\vec{s_i}.).\overline{B}\right]$ has a CAN–free, M–free derivation. Therefore we can have a CAN–free derivation of the hypersequent $[\vdash \Gamma_i]_{i=1}^{n}$ by invoking the induction hypothesis on the simpler term B.
- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible,

  $$\left[\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.(\overline{B} \sqcap \overline{C})\right] \mid \left[\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.(\overline{B} \sqcap \overline{C})\right]$$

  has a CAN–free, M–free derivation. Then since the $\sqcap$ is CAN–free invertible,

  $$\left[\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.\overline{B}\right] \mid \left[\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.\overline{C}\right]$$

  has a CAN–free, M–free derivation. Therefore we can obtain a CAN–free derivation of the hypersequent $[\vdash \Gamma_i]_{i=1}^{n}$ by invoking the induction hypothesis twice on the simpler terms B and C.

- If $A = B \sqcap C$, we proceed in a similar way as for the case $A = B \sqcup C$.
- Finally, if $A = \Diamond B$, we distinguish two cases:
  1. the derivation ends with an application of the $\Diamond$ rule which simplifies $A = \Diamond B$ to B. In this case we can simply conclude by invoking the induction hypothesis on B.
  2. The derivation ends with some other rule (recall that no CAN rules and no M rules appear in the derivation). In this case we decrease the complexity of the derivation, keeping $\Diamond B$ as the CAN term, and then invoke the induction hypothesis on the derivation having reduced complexity. This proof step is rather long to prove, as it requires analysing all possible cases. We just illustrate the two cases when the derivation ends with a logical rule (+) and a structural rule (C) to illustrate the general method.
     - if the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, \vec{r_i}.\Diamond B, \vec{s_i}.\Diamond \overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Diamond B, \vec{s_1}.\Diamond \overline{B}, \vec{r'}.C, \vec{r'}.D}{[\vdash \Gamma_i, \vec{r_i}.\Diamond B, \vec{s_i}.\Diamond \overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Diamond B, \vec{s_1}.\Diamond \overline{B}, \vec{r'}.(C+D)} +$$

by induction hypothesis, there is a CAN–free M–free derivation of

$$[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.C, \vec{r'}.D$$

We continue the derivation with

$$\dfrac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.C, \vec{r'}.D}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.(C+D)} +$$

     - if the derivation finishes with

$$\dfrac{[\vdash \Gamma_i, \vec{r_i}.\Diamond B, \vec{s_i}.\Diamond \overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Diamond B, \vec{s_1}.\Diamond \overline{B} \mid \vdash \Gamma_1, \vec{r_1}.\Diamond B, \vec{s_1}.\Diamond \overline{B}}{[\vdash \Gamma_i, \vec{r_i}.\Diamond B, \vec{s_i}.\Diamond \overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Diamond B, \vec{s_1}.\Diamond \overline{B}} \text{ C}$$

by induction hypothesis, there is a CAN–free M–free derivation of

$$[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1 \mid \vdash \Gamma_1$$

We continue the derivation with

$$\dfrac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1 \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1} \text{ C}$$

□

We now have all necessary tools to prove the CAN–elimination theorem.

Theorem 2.4.4 (CAN elimination). *For all hypersequents G, if $\rhd_{\boldsymbol{HMR}} G$ then $\rhd_{\boldsymbol{HMR}\backslash\{CAN\}} G$.*

Proof. We want to prove that if G has a derivation, then G has a CAN–free derivation. We prove this result by induction on the derivation of G:

- If the derivation finishes with an application of a rule that is not the CAN–rule, then by induction, the premises have CAN–free derivations and we can conclude by using the exact same rule to obtain a CAN–free derivation of G. For

- If the derivation finishes with

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma} \text{ CAN}, \sum \vec{r} = \sum \vec{s}$$

  then by induction $G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}$ has a CAN–free derivation. By invoking the M–elimination Theorem 2.4.3, $G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}$ has a CAN–free M–free derivation and we can conclude by using Lemma 97. □

Finally, we prove Lemma 98 and Lemma 99, the stronger versions of Lemma 95 and Lemma 96.

Lemma 98. *If there is a CAN–free, M–free derivation of* $[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i=1}^n$ *then for all* $\vec{r_i'}$. *and* $\vec{s_i'}$. *such that for all* $i, \sum \vec{r}_i - \sum \vec{s}_i = \sum \vec{r'}_i - \sum \vec{s'}_i$, *there is a CAN–free, M–free derivation of* $\left[\vdash \Gamma_i, \vec{r_i'}.x, \vec{s_i'}.\overline{x}\right]_{i=1}^n$.

Proof. The proof is done by induction on the derivation and is similar to the proof of Lemma 60. □

Lemma 99. *If there is a CAN–free, M–free derivation of* $\left[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}\right]_{i=1}^n$ *then for all* $\vec{r_i'}$. *and* $\vec{s_i'}$. *such that for all* $i, \sum \vec{r}_i - \sum \vec{s}_i \leq \sum \vec{r'}_i - \sum \vec{s'}_i$, *there is a CAN–free, M–free derivation of* $\left[\vdash \Gamma_i, \vec{r_i'}.1, \vec{s_i'}.\overline{1}\right]_{i=1}^n$.

Proof. By induction on derivation. We show only the non–trivial case.

- If the derivation finishes with:

$$\frac{\left[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, \vec{c}1, \vec{c'}\overline{1}}{\left[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{a}; \vec{b}; \vec{c}).1, (\vec{a'}; \vec{b'}; \vec{c'}).\overline{1}} \; 1, \sum \vec{a} + \sum \vec{b} \geq \sum \vec{a'} + \vec{b'}$$

  with $\vec{r_1} = \vec{b}; \vec{c}$ and $\vec{s_1} = \vec{b'}; \vec{c'}$. We want to show that

$$\rhd_{\mathbf{HMR}} \left[\vdash \Gamma_i, \vec{r_i'}.1, \vec{s_i'}.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{r_1'}; \vec{a}).1, (\vec{s_1'}; \vec{a'}).\overline{1}$$

  We will now prove that $\sum \vec{c} - \sum \vec{c'} \leq \sum \vec{r'}_1 + \sum \vec{a} - (\sum \vec{s'}_1 + \sum \vec{a'})$ to be able to conclude with the induction hypothesis.

$$\begin{aligned}
\sum \vec{c} - \sum \vec{c'} &= (\sum \vec{r}_1 - \sum \vec{b}) - (\sum \vec{s}_1 - \sum \vec{b'}) \\
&= (\sum \vec{r}_1 - \sum \vec{s}_1) + (\sum \vec{b'} - \sum \vec{b}) \\
&\leq (\sum \vec{r'}_1 - \sum \vec{s'}_1) + (\sum \vec{a} - \sum \vec{a'}) \\
&= \sum \vec{r'}_1 + \sum \vec{a} - (\sum \vec{s'}_1 + \sum \vec{a'})
\end{aligned}$$

  so by induction hypothesis

$$\rhd_{\mathbf{HMR}} \left[\vdash \Gamma_i, \vec{r_i'}.1, \vec{s_i'}.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{r_1'}; \vec{a}).1, (\vec{s_1'}; \vec{a'}).\overline{1}$$

  which is the result we want. □

2.4.7 Algebraic property

The algebraic property of the system **HMR** can be obtained by merging both the algebraic properties of the systems **HR** and **MGA**. Indeed, because of the presence of the $\Diamond$ operator, the algebraic property is about basic hypersequent instead of atomic hypersequent, as in the system **MGA**. Moreover, because of the T rule, the sequents can be multiplied by real numbers instead of only natural numbers, as in the system **HR**. Thus the algebraic property of the system **HMR** is very similar to the one of the system **MGA**, but with real numbers.

Theorem 2.4.5. *For all basic hypersequents G, built using the variables and negated variables $x_1, \overline{x_1}, \dots, x_k, \overline{x_k}$, of the form*

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \ ... \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_k}$, the following are equivalent:

1. *G has a derivation.*
2. *there exist numbers $t_1, ..., t_m \in \mathbb{R}_{\geq 0}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

     $$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

     i.e., the scaled (by the numbers $t_1 \dots t_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in front of the covariable $\overline{x_j}$.
   - *$\sum_{i=1}^{n} t_i \sum \vec{s'}_i \leq \sum_{i=1}^{n} t_i \sum \vec{r'}_i$, i.e, there are more 1 than $\overline{1}$ and,*
   - *the hypersequent consisting of only one sequent*

     $$\vdash t_1.\Delta_1, ..., t_m.\Delta_m, (t_1\vec{r'}_1).1, ..., (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, ..., (t_m\vec{s'}_m).\overline{1}$$

     has a derivation, where the notation $0.\Gamma$ means $\emptyset$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the derivation of G. By using Theorem 2.4.4, we can assume that the derivation of G is CAN–free. We will only deal with the case of $\Diamond$ rule since every other case is similar to the proof of Theorem 2.2.5. If the derivation finishes with

$$\frac{\vdash \Delta_1, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Delta_1, \vec{r}.1, \vec{s}.\overline{1}} \ \Diamond, \sum \vec{r} \geq \sum \vec{s}$$

then $t_1 = 1$ satisfies the property.

The other way ((2) $\Rightarrow$ (1)) is also very similar to Theorem 2.2.5, only finishing with the $\Diamond$ rule. If there exist numbers $t_1, ..., t_m > 0$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) \geq \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

- $\sum_{i=1}^{m} t_i(\sum \vec{r'}_i) = \sum_{i=1}^{m} t_i(\sum \vec{s'}_i)$, and
- the hypersequent

  $$\vdash t_1.\Delta_1, ..., t_m.\Delta_m, (t_1\vec{r'}_1).1, ..., (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\bar{1}, ..., (t_m\vec{s'}_m).\bar{1}$$

  has a derivation,

then we can use the W rule to remove the sequents corresponding to the numbers $t_i = 0$, and use the C and S rule on the ith sequent to multiply it by t_i. If we assume that there is a natural number l such that $t_i = 0$ for all $i > l$ and $t_i \neq 0$ for all $i \leq l$, then the CAN–free derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash t_1.\Delta_1, \ldots, t_l.\Delta_l, (t_1\vec{r'}_1).1, ..., (t_l\vec{r'}_l).1, (t_1\vec{s'}_1).\bar{1}, ..., (t_l\vec{s'}_l).\bar{1}}{\vdash t_1.\Diamond\Delta_1, \ldots, t_l.\Diamond\Delta_l, (t_1\vec{r'}_1).1, ..., (t_l\vec{r'}_l).1, (t_1\vec{s'}_1).\bar{1}, ..., (t_l\vec{s'}_l).\bar{1}}\ \Diamond}{\vdash t_1.\Gamma_1, t_1.\Diamond\Delta_1, (t_1\vec{r'}_1).1, (t_1\vec{s'}_1).\bar{1}, \ldots, t_l.\Gamma_l, t_l.\Diamond\Delta_l, (t_l\vec{r'}_l).1, (t_l\vec{s'}_l).\bar{1}}\ \text{ID}^*}{\vdash t_1.\Gamma_1, t_1.\Diamond\Delta_1, (t_1\vec{r'}_1).1, (t_1\vec{s'}_1).\bar{1} \mid \ \ldots \mid \vdash t_l.\Gamma_l, t_l.\Diamond\Delta_l, (t_l\vec{r'}_l).1, (t_l\vec{s'}_l).\bar{1}}\ \text{S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\bar{1} \mid \ \ldots \mid \vdash \Gamma_l, \Diamond\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\bar{1}}\ \text{T}^*}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\bar{1} \mid \ \ldots \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_1.1, \vec{s'}_1.\bar{1}}\ \text{W}^*$$

and since $\vdash t_1.\Delta_1, ..., t_l.\Delta_l, (t_1\vec{r'}_1).1, ..., (t_l\vec{r'}_l).1, (t_1\vec{s'}_1).\bar{1}, ..., (t_l\vec{s'}_l).\bar{1}$ is derivable, we can complete the derivation. □

Remark 39. Notice that if the hypersequent is atomic, the algebraic property of **HMR** is exactly the same as for **HR**.

Chapter 3

Applications

In this chapter, we use the machinery of the hypersequent calculus developed previously to study (modal) Riesz spaces. Most specifically, we will prove results concerning free modal Riesz spaces over a set of generators X, noted $\mathbb{F}^{\Diamond}_{\mathrm{Riesz}}(X)$. We will also explore the connections between the different equational theories mentioned in this thesis. More formally, we will show the following results:

1. $\Diamond$ **injectivity:** for all set X and $A, B \in \mathbb{F}^{\Diamond}_{\mathrm{Riesz}}(X)$, if $\Diamond A = \Diamond B$ then $A = B$,
2. **Weak unit:** for all set X, $1 \in \mathbb{F}^{\Diamond}_{\mathrm{Riesz}}(X)$ is a weak unit,
3. **Conservativity regarding scalar multiplication:** for all Abelian l–group terms A, B, $\mathcal{A}_{\mathrm{l\text{–}groups}} \vdash A = B$ if and only if $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$,
4. **Conservativity regarding the $\Diamond$ operator:** for all Riesz space terms A, B, $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$ if and only if $\mathcal{A}^{\Diamond}_{\mathrm{Riesz}} \vdash A = B$
5. **Decidability:** the equational theory of modal Riesz spaces is decidable, i.e., there is an algorithm to check whether or not $\mathcal{A}^{\Diamond}_{\mathrm{Riesz}} \vdash A = B$ for all modal Riesz space terms A, B, and
6. **Archimedean:** for all set X, $\mathbb{F}^{\Diamond}_{\mathrm{Riesz}}(X)$ is Archimedean.

Note that, to the best of our knowledge, apart from the third result which is already known as folklore in the theory of Riesz spaces, all the results are completely new.

The first two properties are examples of properties easy to prove using the hypersequent calculus **HMR**, but to the best of our knowledge, not easy without **HMR**. Note also that the constant 1 was shown to be a strong unit in the free modal Riesz spaces over the empty set of generators (see [FMM20]), but this result does not hold in the presence of generators since if x is a generator, then $x \not\leq n1$ for all n.

The two next properties are the conservativity results known regarding the different algebras considered in this thesis. Note that a similar proof of the conservativity regarding the $\Diamond$ operator could be used to prove that the theory of modal Abelian l–groups if a conservative extension of the theory of Abelian l–groups, but it is an open problem whether or not the theory of modal Riesz spaces is a conservative extension of the theory of modal Abelian l–groups (see Remark 41 below).

The last two results are quite more complicated, and actually require the machinery of *parametrized* hypersequent calculus that will be introduced later on. Recall that the Archimedean property of free modal Riesz spaces was a problem left open in [FMM20].

This chapter will be divided in three sections: the first one concerning the properties of the new operations in modal Riesz spaces, i.e., the injectivity of the $\Diamond$ operator and the fact that 1 is a weak unit. The second section will show the conservativity results, and finally the last section will deal with the parametrized hypersequent calculus and the last two properties.

3.1 Injectivity and unit

3.1.1 Injectivity of the $\Diamond$ operator

We start by proving the injectivity of the $\Diamond$ operator. By using Corollary 1, we have to show that if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash \Diamond A = \Diamond B$ then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$. Note that even though we prove the result for free modal Riesz spaces, a similar proof can be used for free modal Abelian l–groups.

It is sufficient to show the following result: for all A,

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq \Diamond A \Rightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq A \tag{3.1}$$

Indeed using Lemma 16 and the linearity of the $\Diamond$ operator, we have

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash \Diamond A = \Diamond B \Leftrightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq \Diamond(A - B) \text{ and } \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq \Diamond(B - A)$$

and

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B \Leftrightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq A - B \text{ and } \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq B - A$$

and we can conclude using the implication (3.1) to show the injectivity statement (1).

Now using the soundness and completeness of the system **HMR**, the implication (3.1) is equivalent to the following result

$$\rhd_{\mathbf{HMR}} \vdash 1.\Diamond A \Rightarrow \rhd_{\mathbf{HMR}} \vdash 1.A$$

which is a direct corollary of the invertibility of the $\Diamond$ rule proved in the following Lemma 100.

Lemma 100. *If* $[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n$ *has a derivation, then* $[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n$ *has a derivation.*

Proof. By using the CAN elimination Theorem 2.4.4, we can assume that the derivation of $[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n$ is CAN free. We can then proceed by induction on the CAN–free derivation of $[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n$.

- if the derivation finishes the INIT axiom, then $n = 1$, $\Gamma_1 = \emptyset$ and $\vec{r_1} = \vec{s_1} = \emptyset$ so the result is trivial.
- We will only show one of the structural rules, the others are similar. If the derivation finishes with

$$\frac{[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n}{[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^{n+1}}\text{ W}$$

  then by induction hypothesis

$$\rhd_{\mathbf{HMR}} [\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n$$

  and thus

$$\frac{[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^n}{[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=1}^{n+1}}\text{ W}$$

- If the derivation finishes with

$$\dfrac{\vdash \Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}}{\vdash \Diamond\Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}} \ \Diamond, \sum \vec{s_1} \leq \sum \vec{r_1}$$

then $\vdash \Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}$ is derivable which is the result we want.

- If the derivation finishes with

$$\dfrac{[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=2}^n \mid \vdash \Diamond\Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}}{[\vdash \Diamond\Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=2}^n \mid \vdash \Diamond\Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}, \vec{r}.1, \vec{s}.\overline{1}} \ 1, \sum \vec{s} \leq \sum \vec{r}$$

then by induction hypothesis

$$\triangleright_{\mathbf{HMR}} [\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}$$

and thus

$$\dfrac{[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}}{[\vdash \Gamma_i, \vec{r_i}.1, \vec{s_i}.\overline{1}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.1, \vec{s_1}.\overline{1}, \vec{r}.1, \vec{s}.\overline{1}} \ 1, \sum \vec{s} \leq \sum \vec{r}$$

□

We can now prove the injectivity of the $\Diamond$ operator.

Theorem 3.1.1 ($\Diamond$ injectivity)**.** *For all term A and B, if $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash \Diamond A = \Diamond B$ then $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash A = B$.*

Proof. Since $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash \Diamond A = \Diamond B$ then by using Lemma 16, we have

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq \Diamond(A - B)$$

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq \Diamond(B - A)$$

and thus by the completeness Theorem 2.4.2 of the system **HMR** and the invertibility of the 0 rule,

$$\triangleright_{\mathbf{HMR}} \vdash 1.\Diamond(A + \overline{B})$$

$$\triangleright_{\mathbf{HMR}} \vdash 1.\Diamond(B + \overline{A})$$

We can now use Lemma 100 to obtain

$$\triangleright_{\mathbf{HMR}} \vdash 1.(A + \overline{B})$$

$$\triangleright_{\mathbf{HMR}} \vdash 1.(B + \overline{A})$$

and by the soundness Theorem 2.4.1 of the system **HMR**

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq A + \overline{B}$$

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq B + \overline{A}$$

and we can conclude by using Lemma 16 again to obtain

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash A = B$$

□

3.1.2 Weak unit

We will show that 1 is a weak unit in free modal Riesz spaces. Using Corollary 1 of Section 1.2, it is sufficient to show that for all A such that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq A$, if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \sqcap 1 = 0$ then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = 0$.

To do so we will prove the slightly different result stating that if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -(A \sqcap 1)$ then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -A$. This implies that 1 is a weak unit since if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \sqcap 1 = 0$, then by using Lemma 16 we have $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -(A \sqcap 1)$ and thus $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -A$. We can then conclude using the same lemma that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = 0$ since $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq A$.

Using the soundness and completeness of the system **HMR**, we can rephrase the previous property as follows

$$\rhd_{\mathbf{HMR}} \vdash 1.\overline{A \sqcap 1} \Rightarrow \rhd_{\mathbf{HMR}} \vdash 1.\overline{A}$$

or, using the invertibility of the $\sqcup$ rule,

$$\rhd_{\mathbf{HMR}} \vdash 1.\overline{A} \mid \vdash 1.\overline{1} \Rightarrow \rhd_{\mathbf{HMR}} \vdash 1.\overline{A} \tag{3.2}$$

We will first prove a more general version of the implication (3.2), which is better suited for a proof by induction.

Lemma 101. *For all hypersequent G, if $\rhd_{\mathbf{HMR}} G \mid \vdash 1.\overline{1}$ then $\rhd_{\mathbf{HMR}} G$.*

Proof. We prove this result by induction on the complexity of the hypersequent G. We distinguish three cases:

- If G is basic of the form

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \ \dots \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}$$

  since $\rhd_{\mathbf{HMR}} G \mid \vdash 1.\overline{1}$, there $t_1, \dots, t_m, t_{m+1}$ satisfying the conditions of the algebraic property 2.4.5 for the hypersequent $G \mid \vdash 1.\overline{1}$. One of the condition is

$$\rhd_{\mathbf{HMR}} \vdash t_1.\Delta_1, \dots, t_m.\Delta_m, (t_1\vec{r'}_1).1, \dots, (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, \dots, (t_m\vec{s'}_m).\overline{1}, t_{m+1}.\overline{1}$$

  and according to Lemma 102 below with

  - $m = 1$,
  - $\Gamma_1 = t_1.\Delta_1, \dots, t_m.\Delta_m, (t_1\vec{r'}_1).1, \dots, (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, \dots, (t_m\vec{s'}_m).\overline{1}$, and
  - $\vec{r}_1 = t_{m+1}$,

  we can show that

$$\rhd_{\mathbf{HMR}} \vdash t_1.\Delta_1, \dots, t_m.\Delta_m, (t_1\vec{r'}_1).1, \dots, (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, \dots, (t_m\vec{s'}_m).\overline{1}$$

  and thus $t_1, \dots, t_m$ satisfy the algebraic property for the hypersequent G, and we can conclude that G is derivable.

- Otherwise we can use the invertibility of the logical rules to decrease the complexity of the hypersequent G and conclude with the induction hypothesis. For instance, if G has the form

$$H \mid \vdash \Gamma, \vec{r}.A \sqcup B$$

  where $\vdash \Gamma, \vec{r}.A \sqcup B$ is maximal in G. Then by using the invertibility of the $\sqcup$ rule, we can show that

$$\rhd_{\mathbf{HMR}} H \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B \mid \ 1.\overline{1}$$

and by induction hypothesis on $H \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B$,

$$\triangleright_{\mathbf{HMR}} H \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B$$

We can finish the derivation by using the $\sqcup$ rule:

$$\frac{H \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{H \mid \vdash \Gamma, \vec{r}.A \sqcup B} \sqcup$$

□

Notice that because of the constraints of the $\Diamond$ rule that impose to have only one sequent before using the $\Diamond$ rule, we can not simply continue the induction process when the hypersequent G is basic and we require the additional Lemma 102 below.

Lemma 102. *If* $\triangleright_{\boldsymbol{HMR}} [\vdash \Gamma_i, \vec{r}_i.\overline{1}]_{i=1}^m$ *then* $\triangleright_{\boldsymbol{HMR}} [\vdash \Gamma_i]_{i=1}^m$.

Proof. We prove the result by induction on the derivation of $[\vdash \Gamma_i, \vec{r}_i.\overline{1}]_{i=1}^m$. We will only show the case of the 1 rule, the other ones are straightforward.

If the derivation finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.\overline{1}]_{i=2}^m \mid \vdash \Gamma_1, \vec{c}.\overline{1}}{[\vdash \Gamma_i, \vec{r}_i.\overline{1}]_{i=2}^m \mid \vdash \Gamma_1, \vec{r}.1, \vec{a}.\overline{1}, \vec{b}.\overline{1}, \vec{c}.\overline{1}} \ 1, \sum \vec{a}; \vec{b} \leq \sum \vec{r}$$

where $\vec{r}_1 = \vec{b}; \vec{c}$ then by induction hypothesis

$$\triangleright_{\mathbf{HMR}} [\vdash \Gamma_i]_{i=2}^m \mid \vdash \Gamma_1$$

and thus

$$\frac{[\vdash \Gamma_i]_{i=2}^m \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^m \mid \vdash \Gamma_1, \vec{r}.1, \vec{a}.\overline{1}} \ 1, \sum \vec{a} \leq \sum \vec{r}$$

□

We can now prove that 1 is a weak unit in free modal Riesz spaces.

Theorem 3.1.2 (Weak unit)**.** *For all A, if* $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq (\overline{A} \sqcup \overline{1})$ *then* $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq \overline{A}$.

Proof. If $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash 0 \leq (\overline{A} \sqcup \overline{1})$ then by using the completeness Theorem 2.4.2 of the system **HMR** and the invertibility of the 0 rule,

$$\triangleright_{\mathbf{HMR}} \vdash 1.\overline{A} \sqcup \overline{1}$$

and then by invertibility of the $\sqcup$ rule

$$\triangleright_{\mathbf{HMR}} \vdash 1.\overline{A} \mid \vdash 1.\overline{1}$$

We can now use Lemma 101 to show that

$$\triangleright_{\mathbf{HMR}} \vdash 1.\overline{A}$$

and we can conclude using the soundness Theorem 2.4.1 of the system **HMR**. □

3.2 Conservativity

3.2.1 Conservativity over the $\Diamond$ operator

In this section, we will show that modal Riesz spaces are a conservative extension of Riesz spaces, i.e., that for all Riesz terms A, B, $\mathcal{A}_{\text{Riesz}} \vdash A = B$ if and only if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$.

Note that the implication

$$\mathcal{A}_{\text{Riesz}} \vdash A = B \Rightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$$

is trivial since a derivation of $\mathcal{A}_{\text{Riesz}} \vdash A = B$ is also a valid derivation of $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$. Thus, we will focus on the other direction.

A natural option to prove that

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B \Rightarrow \mathcal{A}_{\text{Riesz}} \vdash A = B$$

would be to work by induction on the derivation $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$. However, such derivation could use the transitivity rule

$$\frac{\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = C \quad \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash C = B}{\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B}\ \text{trans}$$

where C can include some $\Diamond$, and thus would not be a Riesz term but a *modal* Riesz term, preventing us from using the induction hypothesis.

To go around this issue, we will use the hypersequent calculus **HMR** since it satisfies the CAN elimination theorem, effectively removing the transitivity rule. Using the soundness and completeness of the system **HMR**, the conservativity result is equivalent to

$$\triangleright_{\mathbf{HMR}} G \Rightarrow \triangleright_{\mathbf{HR}} G$$

where G only contains Riesz terms.

Lemma 103. *Let G be a hypersequent containing only Riesz terms. If $\triangleright_{\boldsymbol{HMR}} G$ then $\triangleright_{\boldsymbol{HR}} G$.*

Proof. By invoking the CAN elimination Theorem 2.4.4, we can assume that the derivation of G is CAN–free. The result is then proven by a straightforward induction on the CAN–free derivation of G. $\square$

Theorem 3.2.1 (Conservativity regarding the $\Diamond$ operator)**.** *For all Riesz terms A, B, $\mathcal{A}_{\text{Riesz}} \vdash A = B$ if and only if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$.*

Proof. As mentioned above, the direction $\mathcal{A}_{\text{Riesz}} \vdash A = B \Rightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ is trivial.

For the other direction, let's assume that $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ then by using Lemma 16, we have

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq A - B$$

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq B - A$$

and thus by the completeness Theorem 2.4.2 of the system **HMR** and the invertibility of the 0 rule,

$$\triangleright_{\mathbf{HMR}} \vdash 1.(A + \overline{B})$$

$$\triangleright_{\mathbf{HMR}} \vdash 1.(B + \overline{A})$$

We can now use Lemma 103 to obtain

$$\triangleright_{\mathbf{HR}} \vdash 1.(A + \overline{B})$$

$$\triangleright_{\mathbf{HR}} \vdash 1.(B + \overline{A})$$

and by the soundness Theorem 2.2.1 of the system **HR**

$$\mathcal{A}_{\text{Riesz}} \vdash 0 \leq A + \overline{B}$$

$$\mathcal{A}_{\text{Riesz}} \vdash 0 \leq B + \overline{A}$$

and we can conclude by using Lemma 7 to obtain

$$\mathcal{A}_{\text{Riesz}} \vdash A = B$$

which concludes the proof. □

Remark 40. The same process can be used to prove that modal Abelian l–groups are a conservative extension of Abelian l–groups, i.e., that for all l-group terms A, B, $\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash A = B$ if and only if $\mathcal{A}_{\text{l–groups}} \vdash A = B$

3.2.2 Conservativity over scalar multiplication

We will now show that Riesz spaces are a conservative extension of Abelian l–groups, i.e., that for all Abelian l–group terms A, B, $\mathcal{A}_{\text{l–groups}} \vdash A = B$ if and only if $\mathcal{A}_{\text{Riesz}} \vdash A = B$. Note that this result is well–known in the theory of Riesz spaces, but we provide a purely syntactic proof and an effective way of translating a derivation in one equational theory to the other.

We will proceed in a similar way as in Section 3.2.1, i.e., we will reduce the theorem to the hypersequent calculi **HR** and **GA**$\|$. However, because of the weights in the hypersequent, a hypersequent G in **HR** is not a valid hypersequent in **GA**$\|$. Thus, for all **HR** hypersequent G, we define the **GA**$\|$ hypersequent $G^\dagger$ by removing the weights, e.g., if $G = \vdash 1.x \mid \vdash 1.x, 1.\overline{y}$ then $G^\dagger = \vdash x \mid \vdash x, \overline{y}$.

Lemma 104. *Let G be a hypersequent containing only Abelian l–group terms and where all weights are equal to* 1*. If $\triangleright_{\boldsymbol{HR}} G$ then $\triangleright_{\boldsymbol{GA}\|} G^\dagger$.*

Proof. By invoking the CAN elimination Theorem 2.2.4, we can assume that the derivation of G is CAN–free.

Moreover, by using the invertibility of the logical rules of Section 2.2.4, we can assume that G is atomic of the form

$$\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_m$$

where $\Gamma_i = \vdash (1.x_1)^{n_{i,1}}, ..., (1.x_k)^{n_{i,k}}, (1.\overline{x_1})^{n'_{i,1}}, ..., (1.\overline{x_k})^{n'_{i,k}}$ (since G contains only Abelian l–group terms, the $\times$ rule is never used and so the weights stay equal to 1).

Then, by application of the algebraic property of **HR** 2.2.5, there are $t_1, ..., t_m$ in $\mathbb{R}_{\geq 0}$ such that

- there exists $i \in [1..m]$ such that $t_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i n_{i,j} = \sum_{i=1}^{m} t_i n'_{i,j}$$

Since all coefficients are rational and the theory of linear arithmetic over $\mathbb{R}$ is an elementary extension of that of linear arithmetic over $\mathbb{Q}$ [FR75], there are $q_1, ..., q_m \in \mathbb{Q}_{\geq 0}$ satisfying the same property of $t_1, ..., t_m$. By multiplying all q_i by the least common multiple of their denominators, we get a solution $k_1, ..., k_m$ in $\mathbb{N}$. So according to the algebraic property of **GA**$\|$ 2.1.5, $G^\dagger$ has a **GA**$\|$ derivation. This concludes the proof. □

Theorem 3.2.2 (Conservativity regarding scalar multiplication)**.** *For all Abelian l–group terms* A, B*,* $\mathcal{A}_{\text{Riesz}} \vdash A = B$ *if and only if* $\mathcal{A}_{\text{l–groups}} \vdash A = B$*.*

Proof. As before, the direction $\mathcal{A}_{\text{l–groups}} \vdash A = B \Rightarrow \mathcal{A}_{\text{Riesz}} \vdash A = B$ is trivial since a derivation of $\mathcal{A}_{\text{l–groups}} \vdash A = B$ is also a derivation of $\mathcal{A}_{\text{Riesz}} \vdash A = B$.

For the other direction, let's assume that $\mathcal{A}_{\text{Riesz}} \vdash A = B$ then by using Lemma 7, we have

$$\mathcal{A}_{\text{Riesz}} \vdash 0 \leq A - B$$

$$\mathcal{A}_{\text{Riesz}} \vdash 0 \leq B - A$$

and thus by the completeness Theorem 2.2.2 of the system **HR** and the invertibility of the 0 rule,

$$\rhd_{\mathbf{HR}} \vdash 1.(A + \overline{B})$$

$$\rhd_{\mathbf{HR}} \vdash 1.(B + \overline{A})$$

We can now use Lemma 104 to obtain

$$\rhd_{\mathbf{GA}\|} \vdash (A + \overline{B})$$

$$\rhd_{\mathbf{GA}\|} \vdash (B + \overline{A})$$

and by the soundness Theorem 2.1.1 of the system **GA**$\|$

$$\mathcal{A}_{\text{l–groups}} \vdash 0 \leq A + \overline{B}$$

$$\mathcal{A}_{\text{l–groups}} \vdash 0 \leq B + \overline{A}$$

and we can conclude by using Lemma 1 to obtain

$$\mathcal{A}_{\text{l–groups}} \vdash A = B$$

which concludes the proof. □

Remark 41 (Open problem). We have not been able to prove or disprove the equivalent of Theorem 3.2.2 in the context of modal Riesz spaces. We leave this as an open problem.

Question. Let A, B be modal Abelian l–group terms, is it true that if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ then $\mathcal{A}^{\Diamond}_{\text{l–groups}} \vdash A = B$?

The proof above can not be adapted to the context of modal Riesz spaces since the algebraic property of **HMR** concerns *polynomial* inequalities instead of *linear* inequalities as in the system **HR**, and thus it is no longer true that if the system has a real solution, then it necessarily has a rational one.

3.3 Decidability and Archimedean property

3.3.1 Parametrized HMR

In this section, we will introduce a generalisation of the hypersequent calculus **HMR** where the weights are not scalars but polynomials. Such a hypersequent can be interpreted as describing the set of possible assignments to these real–valued variables that result in a valid concrete (i.e., where all scalars are numbers and not variables) derivation.

For instance, the derivation

$$\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash r.x, r.\overline{x}}\text{ ID}$$

is valid for any $r \in \mathbb{R}_{>0}$ and, similarly, the derivation

$$\dfrac{\dfrac{}{\vdash}\text{ INIT}}{\vdash r.x, s.x, t.\overline{x}}\text{ ID}, r+s=t$$

is valid for any values of reals $(r, s, t) \in \mathbb{R}^3$ such that $r = s + t$. Lastly, the hypersequent containing two scalar–variables α, β and two concrete scalars s and t

$$\vdash (\alpha^2 - \beta).x, s.\overline{x}, t.\overline{x}$$

is derivable for any assignment of concrete assignments $r_1, r_2 \in \mathbb{R}$ to α and β such that $(r_1)^2 - r_2 > 0$ and $(r_1)^2 - r_2 = s + t$.

The notion of parametrized hypersequent appears naturally when dealing with the algebraic property of the system **HMR** (see Theorem 2.4.5). The algebraic property requires the existence of scalars $t_1, ..., t_m$ satisfying some conditions, one of which is the derivability of the hypersequent H

$$\vdash t_1.\Delta_1, ..., t_m.\Delta_m, (t_1\vec{r'}_1).1, ..., (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, ..., (t_m\vec{s'}_m).\overline{1}$$

Thus to check if a hypersequent satisfies the algebraic property, we have to check whether or not the hypersequent H above is derivable for some $t_1, ..., t_m$. Since the $t_1, ..., t_m$ are not necessarily known and will depend on the possible derivation of the hypersequent H, it is convenient to consider the parametrized hypersequent

$$\vdash \alpha_1.\Delta_1, ..., \alpha_m.\Delta_m, (\alpha_1\vec{r'}_1).1, ..., (\alpha_m\vec{r'}_m).1, (\alpha_1\vec{s'}_1).\overline{1}, ..., (\alpha_m\vec{s'}_m).\overline{1}$$

and then build a parametrized derivation of this hypersequent. One then only has to check if there is an assignment to the variables $\alpha_1, ..., \alpha_m$ that satisfies the other conditions of the algebraic property, and such that the derivation is valid, i.e., all the weights are positive.

We will now give the formal definition of the parametrized hypersequent calculus.

Definition 3.3.1. A *parametrized weighted term* is a formal expression $R.A$ where $R \in \mathbb{R}[\alpha_1, ..., \alpha_k]$ for some variables $\alpha_1, ..., \alpha_k$ and A is a term in NNF.

Example 20. For polynomial variables α, β,

$$(\alpha - 5\beta).x \text{ and } (\alpha^2 + \alpha\beta).x \sqcap \Diamond y$$

are parametrized weighted terms.

Definition 3.3.2. A *parametrized sequent* is a formal expression of the form $\vdash \Gamma$ where Γ is a multiset of parametrized weighted terms.

Example 21. For polynomial variables α, β,

$$\vdash (\alpha - 5\beta).x, (\alpha^2 + \alpha\beta).x \sqcap \Diamond y$$

is a parametrized sequent.

Definition 3.3.3. A *parametrized hypersequent* is a non–empty finite multiset of parametrized sequents, written as $\vdash \Gamma_1 | \dots | \vdash \Gamma_n$.

Example 22. For polynomial variables α, β,

$$\vdash (\alpha^2 - \beta).x, 4.\overline{x}, \beta.\overline{x} \mid \vdash (\alpha - 5\beta).x, (\alpha^2 + \alpha\beta).x \sqcap \Diamond y$$

is a parametrized hypersequent.

We say that a sequent (resp. hypersequent) is parametrized by $\alpha_1, ...\alpha_k$ if $R \in \mathbb{R}[\alpha_1, ..., \alpha_k]$ for all weight R appearing in the sequent (resp. hypersequent). If a sequent $\vdash \Gamma$ is parametrized by $\alpha_1, ..., \alpha_k$, we say that an assignment $t_1, ..., t_k$ of the variables is *valid* if $R(t_1, ..., t_k) \geq 0$ for all weight appearing in $\vdash \Gamma$ (where $R(t_1, ..., t_k)$ is the usual evaluation of polynomials in $\mathbb{R}$). Similarly, if a hypersequent G is parametrized by $\alpha_1, ..., \alpha_k$, we say that an assignment $t_1, ..., t_k$ is *valid* if it is valid for all sequent appearing in G.

Definition 3.3.4. For all sequent $\vdash \Gamma = \vdash R_1.A_1, ..., R_n.A_n$ parametrized by $\alpha_1, ..., \alpha_k$ and valid assignment $t_1, ..., t_k$, $\vdash \Gamma(t_1, ..., t_k)$ is the sequent

$$\vdash R_1(t_1, ..., t_k).A_1, ..., R_n(t_1, ..., t_k).A_n$$

Definition 3.3.5. For all hypersequent $G = \vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ parametrized by $\alpha_1, ..., \alpha_k$ and valid assignment $t_1, ..., t_k$, $G(t_1, ..., t_k)$ is the hypersequent

$$\vdash \Gamma_1(t_1, ..., t_k) \mid \ ... \mid \vdash \Gamma_n(t_1, ..., t_k)$$

Remark 42. Remark that valid assignments evaluate the weights to non-negative scalars instead of positive ones (recall that in Definition 2.4.1 of a weighted term, the weights are positive). This choice was made out of convenience, since it is often useful for the weights to evaluate to 0. In this case, we simply remove the weighted term, e.g., for the parametrized sequent $\vdash \Gamma = \vdash (\alpha_1 - \alpha_2).x, \alpha_1.\overline{x}$, the valid assignments are all t_1, t_2 such that $t_1 \geq t_2 \geq 0$. When $t_1 = t_2$, we have $\vdash \Gamma(t_1, t_2) = \vdash t_1.\overline{x}$ (we removed $0.x$ from the sequent).

Definition 3.3.6. A parametrized derivation is a **HMR** derivation where the T rule can be used with a polynomial instead of a positive scalar, i.e., we use the following T rule

$$\frac{G \mid \vdash R.\Gamma}{G \mid \vdash \Gamma} \ \mathrm{T}, R \in \mathbb{R}[\alpha_1, ..., \alpha_k]$$

We say that a derivation is parametrized by $\alpha_1, ..., \alpha_k$ if all hypersequent appearing in the derivation are parametrized by $\alpha_1, ..., \alpha_k$.

If a derivation is parametrized by $\alpha_1, ..., \alpha_k$, we say that an assignment $t_1, ..., t_k$ is *valid* if the assignment is valid for all hypersequent appearing in the derivation, and if the provisos of the rules are respected for this assignment. For instance, if the derivation contains the following ID rule

$$\frac{\vdash}{\vdash R_1.x, R_2.x, S_1.\overline{x}} \ \mathrm{ID}, R_1 + R_2 = S_1$$

then for the assignment to be valid, we require that $R_1(t_1, ..., t_k) + R_2(t_1, ..., t_k) = S_1(t_1, ..., t_k)$.

For all hypersequent G parametrized by $\alpha_1, ..., \alpha_k$, parametrized derivation of G and valid assignment $t_1, ..., t_k$, we can obtain a derivation of $G(t_1, ..., t_k)$ by simply evaluating all polynomial appearing in the derivation on $t_1, ..., t_k$.

Example 23. A possible parametrized derivation of the hypersequent $G = \vdash (\alpha^2 - \beta).x, s.\overline{x}, t.\overline{x}$ is

$$\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash (\alpha^2 - \beta).x, s.\overline{x}, t.\overline{x}}\ \text{ID}, \alpha^2 - \beta = s + t$$

The valid assignments (r_1, r_2) for such a derivation are exactly the one satisfying

- $r_1^2 - r_2 = s + t$ because of the proviso of the ID rule, and
- $r_1^2 - r_2 \geq 0$ for the assignment to be valid for the hypersquent.

For all assignment satisfying those conditions, we can obtain a derivation of $G(r_1, r_2)$ simply by replacing every occurrences of α by r_1 and β by r_2, obtaining the following **HMR** derivation

$$\dfrac{\dfrac{}{\vdash}\ \text{INIT}}{\vdash (r_1^2 - r_2).x, s.\overline{x}, t.\overline{x}}\ \text{ID}, r_1^2 - r_2 = s + t$$

3.3.2 Decidability

In this section, we use the algebraic property 2.4.5 to introduce an algorithm for deciding if a hypersequent G is derivable in the system **HR**.

The procedure takes a hypersequent G parametrized by $\alpha_1, ..., \alpha_n$, and construct a formula $\phi_G(\vec{\alpha}) \in FO(\mathbb{R}, +, \times, \leq)$ in the language of the first order theory of the reals. The procedure is recursive and terminates because each recursive call decreases the complexity (see Definition 2.4.8) of its input G. The key property is that a sequence of scalars $(s_1, ..., s_n) \in \mathbb{R}$ satisfies ϕ_G if and only if the hypersequent $G(s_1, ..., s_n)$ is derivable in the system **HMR**. The decidability then follows from the well–known fact that the theory $FO(\mathbb{R}, +, \times, \leq)$ admits quantifier elimination and is decidable [Tar51, Gri88].

The algorithm to construct ϕ_G takes as input G and proceeds as follows:

1. if G is not a basic hypersequent (i.e., if it contains any complex term whose outermost connective is not $\Diamond$ or 1 or $\overline{1}$), then the algorithm returns

$$\phi_G = \bigwedge_{i=1}^{n} \phi_{G_i}$$

   where $G_1, \ldots, G_n$ are the basic hypersequents obtained by iteratively applying the logical rules, and ϕ_{G_i} is the formula recursively computed by the algorithm on input G_i.

2. if G has the shape $\vdash \vec{R}.1, \vec{S}.\overline{1}$ then $\phi_G = \sum \vec{S} \leq \sum \vec{R}$.

3. otherwise G is a basic hypersequent which has the shape

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{R'}_1.1, \vec{S'}_1.\overline{1} \mid\ ... \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{R'}_m.1, \vec{S'}_m.\overline{1}$$

   where $\Gamma_i = \vec{R}_{i,1}.x_1, ..., \vec{R}_{i,k}.x_k, \vec{S}_{i,1}.\overline{x_1}, ..., \vec{S}_{i,k}.\overline{x_k}$. We then define:

   - A formula $GZ(\beta_1, ..., \beta_m)$ that states that for all $i \in [1..m]$, $0 \leq \beta_i$.

$$Z(\beta_1, ..., \beta_m) = \bigwedge_{i \in [1..m]} (0 \leq \beta_i)$$

- A formula $NZ(\beta_1, ..., \beta_m)$ that states there is $i \in [1..m]$ such that $\beta_i \neq 0$.

$$NZ(\beta_1, ..., \beta_m) = \bigvee_{i \in [1..m]} \neg(\beta_i = 0)$$

- A formula $A(\beta_1, ..., \beta_m)$ that states that all the atoms cancel each other.

$$A(\beta_1, ..., \beta_m) = \bigwedge_{j=0}^{k} (\sum_{i=1}^{m} \beta_i \sum \vec{R}_{i,j} = \sum_{i=1}^{m} \beta_i \sum \vec{S}_{i,j})$$

- A formula $O(\beta_1, ..., \beta_m)$ that states that there are more 1 than $\overline{1}$,

$$O(\beta_1, ..., \beta_m) = \sum_{i=1}^{m} \beta_i \sum \vec{S'}_i \leq \sum_{i=1}^{m} \beta_i \sum \vec{R'}_i$$

- A hypersequent $H(\beta_1, ..., \beta_m)$ which is the result of cancelling the atoms using $\beta_1, ..., \beta_m$ and then using the $\Diamond$ rule, i.e. is the leaf of the following prederivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \beta_1.\Delta_1, (\beta_1\vec{R'}_1).1, (\beta_1\vec{S'}_1).\overline{1}, ..., \beta_m.\Delta_m, (\beta_m\vec{R'}_m).1, (\beta_m\vec{S'}_m).\overline{1}}{\vdash \beta_1.\Diamond\Delta_1, (\beta_1\vec{R'}_1).1, (\beta_1\vec{S'}_1).\overline{1}, ..., \beta_m.\Diamond\Delta_m, (\beta_m\vec{R'}_m).1, (\beta_m\vec{S'}_m).\overline{1}}\ \Diamond}{\vdash \beta_1.\Gamma_1, \beta_1.\Diamond\Delta_1, (\beta_1\vec{R'}_1).1, (\beta_1\vec{S'}_1).\overline{1}, ..., \beta_m.\Gamma_m, \beta_m.\Diamond\Delta_m, (\beta_m\vec{R'}_m).1, (\beta_m\vec{S'}_m).\overline{1}}\ \text{ID}^*}{\vdash \beta_1.\Gamma_1, \beta_1.\Diamond\Delta_1, (\beta_1\vec{R'}_1).1, (\beta_1\vec{S'}_1).\overline{1} \mid\ ...\mid \vdash \beta_m.\Gamma_m, \beta_m.\Diamond\Delta_m, (\beta_m\vec{R'}_m).1, (\beta_m\vec{S'}_m).\overline{1}}\ \text{S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{R'}_1.1, \vec{S'}_1.\overline{1} \mid\ ...\mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{R'}_m.1, \vec{S'}_m.\overline{1}}\ \text{T}^*$$

- The formula $\phi_{H(\beta_1,...,\beta_m)}$ computed recursively from $H(\beta_1, ..., \beta_m)$ above.

Finally, we return ϕ_G defined as follows:

$$\phi_G = \exists \beta_1, ..., \beta_m, GZ(\beta_1, ..., \beta_m) \wedge NZ(\beta_1, ..., \beta_m) \wedge A(\beta_1, ..., \beta_m) \wedge O(\beta_1, ..., \beta_m) \wedge \phi_{H(\beta_1,...,\beta_m)}$$

The following theorem states the correctness of the above described algorithm.

Theorem 3.3.1. *Let G be a hypersequent parametrized by $\vec{\alpha}$. Let $\phi_G(\vec{\alpha})$ be the formula returned by the algorithm described above on input G. Then, for all valid assignment $\vec{s} \in \mathbb{R}$ of G, the following are equivalent:*

1. *$\phi_G(\vec{s})$ holds in $\mathbb{R}$,*
2. *$G(\vec{s})$ is derivable in **HMR**.*

Proof. By using the CAN–free invertibility of the logical rules, we can assume that G is a basic hypersequent. If G has the shape $\vdash \vec{R}.1, \vec{S}.\overline{1}$, the result is trivial since ϕ_G is the proviso of the 1 rule. Otherwise, the result is a direct corollary of the algebraic property 2.4.5 since the formula NZ corresponds to the first property, the formula A corresponds to the second property, the formula O corresponds to the third one and the formula ϕ_H corresponds to the last one. □

Even though the problem is decidable, the algorithm described previously is non–elementary since the size of the formula ϕ_G can not be bound by a finite tower of exponentials.

Lemma 105. *Let A_n be defined by induction on n as follows:*

- $A_0 = x$ *for some variable* x
- $A_{n+1} = \Diamond A_n \sqcup \Diamond A_n$

For $i \in \mathbb{N}$, *let* $G_i = \vdash 1.A_i$. *Then for all* i, ϕ_{G_i} *has at least* $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{i \text{ times}}$ *existentials (with the convention* $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{0 \text{ times}} = 1$*), i.e., each use of the* $\Diamond$ *rule will add one exponential to the number of existentials.*

Proof. For $i, j, k \in \mathbb{N}$ and $R_1, ..., R_j \in \mathbb{R}[\alpha_1, ..., \alpha_k]$, we define

$$H_{i,j}(R_1, ..., R_j) = \vdash R_1.A_i, ..., R_j.A_i$$

We will show by induction that for all $0 < i$ and $0 < j$, $\phi_{H_{i,j}}$ has at least $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^j}}}}}}_{i \text{ times}}$ existentials.

For all $j \in \mathbb{N}_{>0}$, $k \in \mathbb{N}$ and $R_1, ..., R_j \in \mathbb{R}[\alpha_1, ..., \alpha_k]$, $H_{0,j}[R_1, ..., R_j]$ is an atomic hypersequent with only one sequent, so $\phi_{H_{0,j}}$ has at least one existential.

Let $j \in \mathbb{N}_{>0}$, $k \in \mathbb{N}$ and $R_1, ..., R_j \in \mathbb{R}[\alpha_1, ..., \alpha_k]$. By applying iteratively the logical rules on $H_{1,j}$, we obtain the basic hypersequent $H^b = [\vdash R_1.\Diamond x, ..., R_j.\Diamond x]^{2^j}$ and ϕ_{H^b} has $2^j + 1$ existentials. So $\phi_{H_{1,j}}(R_1, ..., R_j)$ has $2^j + 1 \geq 2^j$ existentials.

Let us now analyse $\phi_{H_{i+1,j}}(R_1, ..., R_j)$ for $i > 0$. By applying iteratively the logical rules, we obtain only one basic hypersequent : $H^b_{i+1,j}(R_1, ..., R_j) = [\vdash R_1.\Diamond A_i, ..., R_j.\Diamond A_i]^{2^j}$. We notice that one of the subformulas of $\phi_{H^b_{i+1,j}}(R_1, ..., R_j)$ is $\phi_{H_{i,j\times 2^j}}(\vec{R'})$ for some $\vec{R'} \in (\mathbb{R}[\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_{2^j}])^{j\times 2^j}$, which has at least $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{j\times 2^j}}}}}}}_{i \text{ times}} \geq \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^j}}}}}}_{i+1 \text{ times}}$ existentials.

So $\phi_{H_{i+1,j}}(R_1, ..., R_j)$ has at least $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^j}}}}}}_{i+1 \text{ times}}$ existentials. Since $G_i = H_{i,1}(1)$, G_i has at least $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{i \text{ times}}$ existentials. □

Remark 43. It is still an open question whether or not there is an elementary algorithm to decide if a hypersequent is derivable in the system **HMR**.

3.3.3 Archimedean

In this Section, we will show that free modal Riesz spaces are Archimedean, i.e., that for all modal Riesz terms A and B, if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash nA \leq B$ for all n then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq 0$. Note that this result was previously left open in [FMM20, §6.3].

As often, we will reduce the Archimedean property to the system **HMR**. To do so, we first rephrase the Archimedean property. We have

$$\begin{aligned} \forall n, \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash nA \leq B \quad &\Leftrightarrow \quad \forall n, \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq B - nA \\ &\Leftrightarrow \quad \forall n, \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq \frac{1}{n}B - A \end{aligned}$$

and

$$\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq 0 \quad \Leftrightarrow \quad \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -A$$

thus the Archimedean property is equivalent to the following implication

$$\forall n, \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq \frac{1}{n}B - A \Rightarrow \mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash 0 \leq -A$$

or using the soundness, completeness and invertibility of the logical rules of the system **HMR**

$$\forall n, \rhd_{\mathbf{HMR}} \vdash \frac{1}{n}.B, 1.\overline{A} \Rightarrow \rhd_{\mathbf{HMR}} \vdash 1.\overline{A}$$

In order to establish the implication above, we prove a stronger result of independent interest about the hypersequent calculus **HMR**. This states that derivability in **HMR** is continuous in the sense that derivability preserves limits of scalars in hypersequents. More formally, this means that for all hypersequent G parametrized by $\alpha_1, ..., \alpha_l$, sequence of scalars $(s_{1,n}, ..., s_{l,n})_{n\in\mathbb{N}}$ and vector of scalars $s_1, ..., s_l$ such that $\lim_{n\to+\infty} s_{i,n} = s_i$ for all i, we have

$$\forall n, \rhd_{\mathbf{HMR}} G(s_{1,n}, ..., s_{l,n}) \Rightarrow \rhd_{\mathbf{HMR}} G(s_1, ..., s_l)$$

This continuity comes from the continuity of the polynomials used in the algebraic property: if G is basic and $G(s_{1,n}, ..., s_{l,n})$ satisfies the algebraic property for all n, by continuity of the polynomials, so does $G(s_1, ..., s_l)$.

Lemma 106 (Continuity of $\rhd_{\mathbf{HMR}}$)**.** *For all hypersequent G parametrized by $\alpha_1, ..., \alpha_l$, sequence of valid assignments $(s_{1,n}, ..., s_{l,n})_{n\in\mathbb{N}}$ of G and a valid assignment $s_1, ..., s_l$ of G such that $\lim_{n\to+\infty} s_{i,n} = s_i$ for all i, we have*

$$\forall n, \rhd_{\boldsymbol{HMR}} G(s_{1,n}, ..., s_{l,n}) \Rightarrow \rhd_{\boldsymbol{HMR}} G(s_1, ..., s_l)$$

Proof. We will show this result by induction on the complexity of G. We distinguish three cases:

- If G is not basic, then we can use the invertibility of the logical rules to reduce the complexity of G. For instance if
$$G = \ H \mid \vdash \Gamma, \vec{R}.A + B$$
with $\vdash \Gamma, \vec{R}.A + B$. We define
$$G' = \ H \mid \vdash \Gamma, \vec{R}.A, \vec{R}.B$$
Since $\rhd_{\mathbf{HMR}} G(s_{1,n}, ..., s_{l,n})$ for all n, by using the invertibility of the $+$ rule, we have $\rhd_{\mathbf{HMR}} G'(s_{1,n}, ..., s_{l,n})$.
By induction hypothesis, we have $\rhd_{\mathbf{HMR}} G'(s_1, ..., s_l)$ and thus we can conclude by using the $+$ rule.
- If $G = \vdash \vec{R}.1, \vec{S}.\overline{1}$. Since $\rhd_{\mathbf{HMR}} G(s_{1,n}, ..., s_{l,n})$ for all n then by using the algebraic property 2.4.5, we have
$$\forall n, \sum \vec{S}(s_{1,n}, ..., s_{l,n}) \leq \sum \vec{R}(s_{1,n}, ..., s_{l,n})$$
and thus by continuity of the polynomials,
$$\sum \vec{S}(s_1, ..., s_l) \leq \sum \vec{R}(s_1, ..., s_l)$$
and so
$$\cfrac{\cfrac{}{\vdash}\ \text{INIT}}{\vdash \vec{R}(s_1, ..., s_l).1, \vec{S}(s_1, ..., s_l).\overline{1}}\ 1, \sum \vec{S}(s_1, ..., s_l) \leq \sum \vec{R}(s_1, ..., s_l)$$

- Otherwise G has the form

$$\vdash \Gamma_1, \Diamond\Delta_1, \overrightarrow{R'}_1.1, \overrightarrow{S'}_1.\overline{1} \mid \ \dots \mid \vdash \Gamma_m, \Diamond\Delta_m, \overrightarrow{R}_m.1, \overrightarrow{S'}_m.\overline{1}$$

where $\Gamma_i = \overrightarrow{R}_{i,1}.x_1, ..., \overrightarrow{R}_{i,k}.x_k, \overrightarrow{S}_{i,1}.\overline{x_1}, ..., \overrightarrow{S}_{i,k}.\overline{x_{i,k}}$

According to the algebraic property 2.4.5, since $G(s_{1,n}, ..., s_{l,n})$ has a derivation for all n, then for all n there exist numbers $t_{1,n}, ..., t_{m,n} \in \mathbb{R}_{\geq 0}$ such that:

  - there exists $i \in [1..m]$ such that $t_{i,n} \neq 0$ and,
  - for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_{i,n}(\sum \overrightarrow{R}_{i,j}(s_{1,n}, ..., s_{l,n}) - \sum \overrightarrow{S}_{i,j}(s_{1,n}, ..., s_{l,n})) = 0$$

    and,
  - $0 \leq \sum_{i=1}^{n} t_{i,n}(\sum \overrightarrow{R'}_i(s_{1,n}, ..., s_{l,n}) - \sum \overrightarrow{S'}_i(s_{1,n}, ..., s_{l,n}))$ and,
  - $\rhd_{\mathbf{HMR}} H(s_{1,n}, ..., s_{l,n}, t_{1,n}, ..., t_{m,n})$

where

$$H = \vdash \alpha_{l+1}.(\Delta_1, \overrightarrow{R'}_1.1, \overrightarrow{S'}_1.\overline{1}), , ..., \alpha_{l+m}.(\Delta_m, \overrightarrow{R'}_m.1, t_{m,n}\overrightarrow{S'}_m.\overline{1})$$

Notice that for all n, if $t_{1,n}, ..., t_{m,n} \in \mathbb{R}_{\geq 0}$ satisfies the algebraic property for G, then for all $a > 0$, $\frac{t_{1,n}}{a}, ..., \frac{t_{m,n}}{a}$ also satisfies the algebraic property. By taking $a = \max\limits_{i \in [1..m]} t_{i,n}$, we can assume that for all n, $t_{1,n}, ..., t_{m,n} \in [0, 1]$ and that $t_{i_n,n} = 1$ for some i_n.

Since $[0, 1]$ is compact and for all n, there is $i_n \in [1..m]$ such that $t_{i_n,n} = 1$, then there is a subsequence $(\sigma(n))$ such that for all $j \in [1..m]$, $t_{j,\sigma(n)}$ converge toward some $t_j \in [0, 1]$ and there is $i \in [1..m]$ such that $t_{i,\sigma(n)} = 1$ for all n (therefore $t_i = 1$).

If $\Delta_i = \emptyset$ for all i, then $H = \vdash \vec{R}.1, \vec{S}.\overline{1}$ for some $\vec{R}$ and $\vec{S}$ and we conclude as in the previous case to show that $H(s_1, ..., s_{l+m})$ has a derivation.

Otherwise, the complexity of H is strictly lesser that the complexity of G and by induction hypothesis, $H(s_1, ..., s_{l+m})$ has a derivation.

In both case $H(s_1, ..., s_{l+m})$ has a derivation and by continuity

  - for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum \overrightarrow{R}_{i,j}(s_1, ..., s_l) - \sum \overrightarrow{S}_{i,j}(s_1, ..., s_l)) = 0$$

    and,
  - $0 \leq \sum_{i=1}^{n} t_i(\sum \overrightarrow{R'}_i(s_1, ..., s_l) - \sum \overrightarrow{S'}_i(s_1, ..., s_l))$

so according to the algebraic property 2.4.5, $G(s_1, ..., s_l)$ has a derivation.

□

Theorem 3.3.2 (Archimedean property)**.** *For all modal Riesz terms A and B, if $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash nA \leq B$ for all n then $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A \leq 0$.*

Proof. Let's assume that $\mathcal{A}^{\diamond}_{\text{Riesz}} \vdash nA \leq B$ for all n. Then for all $n \geq 1$,

$$\mathcal{A}_{\text{Riesz}} \vdash 0 \leq \frac{1}{n}B - A$$

and thus by the completeness Theorem 2.2.2 of the system **HR** and the invertibility of the logical rules,

$$\forall n \geq 1, \triangleright_{\mathbf{HMR}} \vdash \frac{1}{n}.B, 1.\overline{A}$$

We can now use Lemma 106 on the parametrized hypersequent

$$G(\alpha_1) = \vdash \alpha_1.B, 1.\overline{A}$$

to obtain

$$\triangleright_{\mathbf{HMR}} G(0)$$

and by the soundness Theorem 2.4.1 of the system **HMR**

$$\mathcal{A}^{\diamond}_{\text{Riesz}} \vdash 0 \leq -A$$

and therefore

$$\mathcal{A}^{\diamond}_{\text{Riesz}} \vdash A \leq 0$$

which concludes the proof. □

Conclusion and future work

Recall the four steps for our direction of research announced in the introduction:

1. Find a simple real–valued modal logic for expressing properties of probabilistic systems.
2. Design a good structural proof system for this logic.
3. Extend the logic with the (co)inductive defined operators necessary to obtain the expressiveness of pCTL.
4. Build upon the proof system designed at step 2 to obtain a well–behaved proof system for the logic obtained at step 3, thus having a good proof system for pCTL.

In this thesis, we have taken the work of [MFM17, Mio18, FMM20] as the step one and we have accomplished step two. Indeed, we have built upon the hypersequent calculus **GA** of Metcalfe, Gabbay and Olivetti [MOG05] to develop a hypersequent calculus for the theory of modal Riesz spaces called **HMR**. To do so, we have built hypersequent calculi for different fragments of modal Riesz spaces, namely **HR** for Riesz spaces and **MGA** for modal lattice–ordered Abelian groups, before combining them to obtain **HMR**. For each of those systems, we have shown that they satisfy the basic properties required for well–behaved structural proof systems, i.e., among others, the soundness, completeness and CAN elimination theorems.

We then have used the hypersequent calculus **HMR** to prove new results regarding modal Riesz spaces. Among them, we have shown that the equational theory of modal Riesz spaces is decidable. We also solved a problem left open in [FMM20][§6.3], namely that free modal Riesz spaces are Archimedean.

Future work

The natural continuation of our work is to implement steps three and four mentioned above.

Step three - Mio extended Riesz modal logic with threshold operators (denoted by $\mathbb{T}_{>0}$) to implement a logic sufficiently expressive to interpret the bounded–fragment of pCTL and an axiomatisation for this logic. The threshold operator $\mathbb{T}_{>0}$ is defined using a least fixed–point definition, in the style discussed in Section 1.5. See [Mio18] for further details. Thus, the only remaining operator of pCTL not included in the logic of [Mio18] is the *unbounded Until* operator. Mio suggests that further extending the logic with the following fixed–point defined binary operator U:

$$U(F, G) = \mu X.\mathbb{T}_{>0}(G) \sqcup (\mathbb{T}_{>0}(F) \sqcap \Diamond X)$$

should allow for the encoding of the unbounded Until operator of pCTL.

However, no axiomatisation has yet been found for this extended logic, and this is therefore an interesting line of research.

Step four - For the implementation of a well–behaved proof system, the main difficulty is to extend our hypersequent calculus with rules for (co)inductive operators. One of the most promising approach is the study of *cyclic proofs* originally introduced for logics with (co)inductive definitions such as the modal μ–calculus [DHL06, Stu07, BS11].

A cyclic proof is a derivation that can contain cycles and should satisfy a certain condition in order to ensure soundness. For instance, if we consider the following usual rules for the fixed–points operators:

$$\frac{G \mid \vdash \Gamma, \phi(\nu X.\phi(X))}{G \mid \vdash \Gamma, \nu X.\phi(X)} \qquad\qquad \frac{G \mid \vdash \Gamma, \phi(\mu X.\phi(X))}{G \mid \vdash \Gamma, \mu X.\phi(X)}$$

and the hypersequent $\vdash \nu X.\Diamond(X) \sqcup -1 \sqcap 1$, which is semantically equivalent to the formula P_{NT} defined in Section 1.5, then a possible cyclic proof for this hypersequent would be

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \nu X.\Diamond(X) \sqcup -1 \sqcap 1^{\dagger}}{\vdash \Diamond(\nu X.\Diamond(X) \sqcup -1 \sqcap 1)}\ \Diamond}{\vdash \Diamond(\nu X.\Diamond(X) \sqcup -1 \sqcap 1) \mid \vdash -1}\ \mathrm{W}}{\vdash \Diamond(\nu X.\Diamond(X) \sqcup -1 \sqcap 1) \sqcup -1}\ \sqcup \qquad \dfrac{\dfrac{}{\vdash}\ \mathrm{INIT}}{\vdash 1}\ 1}{\vdash \Diamond(\nu X.\Diamond(X) \sqcup -1 \sqcap 1) \sqcup -1 \sqcap 1}\ \sqcap}{\vdash \nu X.\Diamond(X) \sqcup -1 \sqcap 1^{\dagger}}$$

where † marks the cycle. However, not all cyclic proofs are valid. A similar cyclic proof is possible for the hypersequent $\vdash \mu X.\Diamond(X) \sqcup -1 \sqcap 1$, but this hypersequent should not be derivable since its interpretation is not positive. Indeed, in $(\mathbb{R}, \leq, \Diamond = x \mapsto x)$, the interpretation of $\vdash \mu X.\Diamond(X) \sqcup -1 \sqcap 1$ is the greatest number $r \in \mathbb{R}$ such that $r = \min(\max(r, -1), 1)$, and therefore, it is the real number -1. Since -1 is not positive, the sequent should not be derivable. Thus it is necessary to find adequate conditions for a cyclic proof to be valid. Those conditions must be strong enough for the hypersequent calculus to be sound, but must also be weak enough for the hypersequent calculus to be complete.

The theory of cyclic proofs is an active area of research, and has been mostly developed around the μ-calculus (see, e.g., [DHL06, Stu07, BS11] and the recent [Dou17]). Cyclic proof systems have also been considered for Linear Logic (see [Gir87, Gir95] for an introduction to Linear Logic and, e.g., [Dou17][§2.4.3] for an introduction to the theory of cyclic proofs for several logics including Linear Logic). In this direction, some parallels can be done between the hypersequent calculus **GA** and the sequent calculus for Linear Logic. Firstly, **GA** can be seen as a sequent calculus by removing the S,C and W rules and restricting every other rules to act on hypersequent with only one sequent, e.g., the + rules become

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A + B}\ +$$

and the $\sqcup$ rule becomes the two rules

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \sqcup B}\ \sqcup_1 \qquad\qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \sqcup B}\ \sqcup_2$$

This sequent calculus is no longer complete for the equational theory we consider (the sequent $\vdash x \sqcup \overline{x}$ is no longer derivable), but it is very similar to the sequent calculus of the Multiplicative

Additive fragment of Linear Logic (MALL) [Gir87, Gir95] extended with the usual nullary and binary mix rules (see, among others, [Gir87, FR94, AJ94]):

$$\frac{}{\vdash}\ \text{Mix}_0 \qquad\qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ \text{Mix}_2$$

In fact, we can directly translate a derivation of the sequent version of **GA** into a derivation of MALL, and vice–versa, with the following translations:

Operator of **GA**	Rule of **GA**	Operator of Linear Logic	Rule of Linear Logic
	INIT axiom		Mix_0 axiom
	ID axiom		ID axiom
	M rule		Mix_2 rule
$+$	$+$ rule	$\parr$	$\parr$ rule
	$+$ rule and M rule	$\otimes$	$\otimes$ rule
$\sqcup$	$\sqcup_1$ rule	$\oplus$	$\oplus_1$ rule
	$\sqcup_2$ rule		$\oplus_2$ rule
$\sqcap$	$\sqcap$ rule	$\&$	$\&$ rule
0	0 rule	$\bot$	$\bot$ rule
	0 rule and INIT axiom	1	1 axiom

Note that the translation from MALL to **GA** sends both multiplicative operators $\parr$ and $\otimes$ of MALL to the operator $+$ of **GA** and also sends both multiplicative constants $\bot$ and 1 of MALL to the constant 0 of **GA**. It is well known as folklore that in MALL, the binary MIX rule is equivalent to the derivability of the sequent

$$\vdash 1, 1$$

and the nullary MIX rule is equivalent to the derivability of the sequent

$$\vdash \bot, \bot$$

and both these sequents are trivially derivable when the multiplicative constants are equivalent since $\vdash 1, 1 = \vdash \bot, \bot = \vdash 1, \bot$ and $\vdash 1, \bot$ has the following derivation:

$$\dfrac{\dfrac{}{\vdash 1}\ 1}{\vdash 1, \bot}\ \bot$$

Thus, since the multiplicative constants are the same, the nullary and binary MIX rules are redundant and the sequent version of **GA** is exactly the logic MALL where the multiplicative operators and multiplicative constants collapse.

One possible line of research is then to further study the connections between Linear Logic and the hypersequent calculus **HMR** and how the circular proof techniques for Linear Logic can be exported to hypersequent calculi.

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