TD1: Natural deduction

Christophe Lucas, Colin Riba

Foreword: *Exercises marked with a* $\cdot^\dagger$ *can be done in Coq if you are confident enough to make the correspondences between* NJ *rules and Coq's tactics. Do it at your own risk, the corrections will only be done in* NJ.

Exercise† **1** (Warm up!)**.** Among the different historical presentations of logic, Hilbert's deductive system relied on the following two axioms (for all propositions A, B and C):

$$(K)\quad A \Rightarrow B \Rightarrow A \qquad\qquad (S)\quad (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

Can you build derivation trees in NJ for:

1. (K);
2. (S);
3. $\vdash_{\mathrm{NJ}} A \wedge B \Rightarrow A \vee B$;
4. $\vdash_{\mathrm{NJ}} A \vee B \Rightarrow (\neg A) \Rightarrow B$;
5. $\vdash_{\mathrm{NJ}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$;
6. $\vdash_{\mathrm{NJ}} (A \wedge B) \vee (A \wedge C) \Rightarrow A \wedge (B \vee C)$.

Exercise† **2** (There is no *alternative fact*...)**.** Prove the following judgments:

1. $\vdash_{\mathrm{NJ}} A \Rightarrow \neg\neg A$;
2. $\vdash_{\mathrm{NJ}} \neg(A \wedge \neg A)$;
3. $\vdash_{\mathrm{NJ}} \neg\neg\neg A \Rightarrow \neg A$.

Exercise† **3** (Classical reasoning principles)**.** Peirce's law is defined by the proposition:

$$(\mathrm{PL}) \qquad ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

Together with the excludded-middle and the double-negation elimination principle, it is one of the usual reasoning principles used to obtain classical natural deduction. Can you prove, in NJ, the equivalence:

$$(\mathrm{PL}) \quad \Leftrightarrow \quad (\mathrm{EM}) \quad \Leftrightarrow \quad (\mathrm{DNE})$$

Hint: you may want to think carefully before choosing how to prove this.

Exercise† **4** (Towards Glivenko's theorem)**.** In the same line of exercise 2 prove the following statements involving double negation.

1. $\vdash_{\mathrm{NJ}} \neg\neg A \Rightarrow \neg\neg(A \vee B)$;
2. $\vdash_{\mathrm{NJ}} (A \Rightarrow \neg\neg B) \Rightarrow \neg\neg(A \Rightarrow B)$;
3. $\vdash_{\mathrm{NJ}} (\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg(A \wedge B)$.

Exercise† **5** (De Morgan laws)**.** As usual, equivalence is syntactic sugar: $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$. For the following well-known equivalences, decide for each side if the judgment holds in NJ, or provide an informal argument that refute this judgment:

1. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$
2. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

Exercise 6 (Stability and decidability)**.** We say that a proposition is ***stable*** when $\vdash_{NJ} \neg\neg A \Rightarrow A$; it is ***decidable*** when $\vdash_{NJ} A \vee \neg A$. The following properties amount to analysing the action of $\neg\neg(\cdot)$ as a closure operator on propositions:

1. For $A \in \{\bot, \top\}$, check whether A is stable (resp. decidable).
2. For $\diamond \in \{\Rightarrow, \wedge, \vee\}$, is the proposition $A \diamond B$ stable (resp. decidable) in general?
3. Show that the set of stable propositions is (algebrically) stable under $\Rightarrow$ and $\wedge$.
4. Prove the existence of an operator $F(\cdot)$ such that, for any proposition A, $F(A)$ stable implies A decidable.

Exercise 7 (Toolbox)**.** In NJ, a rule (R) is ***derivable*** if the conclusion can be reached, starting from the premises, using NJ inference rules ; it is ***admissible*** if the conclusion is derivable whenever all the premises are derivable.

1. Prove that any derivable rule is admissible.
2. Let (R) be some admissible rule wrt to NJ. Prove that for any judgment $\Delta \vdash A$,

   $$\Delta \vdash_{\mathrm{NJ}} A \quad \text{iff} \quad \Delta \vdash_{\mathrm{NJ}+R} A$$

   Hint: one way or the other you may need to reason by induction over the proof tree.

   You have just proved that adding an admissible rule to a system does not change its *expressive power*!
3. Prove that the following rules are derivable:

   $$(R_1)\ \frac{\Delta, A \Rightarrow B \vdash A}{\Delta, A \Rightarrow B \vdash B} \qquad (R_2)\ \frac{\Delta, \neg A \vdash A}{\Delta, \neg A \vdash \bot}$$

4. Are the following rules derivable ? admissible ?

   $$(Weakening)\ \frac{\Delta \vdash B}{\Delta, A \vdash B} \qquad (Contraction)\ \frac{\Delta, A, A \vdash B}{\Delta, A \vdash B}$$

5. Prove that the following rules are admissible:

   $$(\mathrm{Cut})\ \frac{\Delta, A \vdash B \quad \Delta \vdash A}{\Delta \vdash B} \qquad (\wedge_L)\ \frac{\Delta, A, B \vdash C}{\Delta, A \wedge B \vdash C} \qquad (\vee_L)\ \frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \vee B \vdash C}$$