ENS DE LYON

TD de Sémantique et Vérification

III– Topology

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In this set of exercises, we will discuss topological characterisations of safety and liveness properties.

Topological Spaces

- A *topological space* is a pair $(X,\mathcal{U})$ of a set X and a subset $\mathcal{U}$ of $\mathcal{P}(X)$, called the *open sets of* X, such that

  1. $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;
  2. for any set I and family $\{U_i \in \mathcal{U}\}_{i\in I}$, also $\bigcup_{i\in I} U_i \in \mathcal{U}$; and
  3. for all $U, V \in \mathcal{U}$, also $U \cap V \in \mathcal{U}$.

  A set $U \in \mathcal{U}$ is called *open* and elements $x \in X$ are called points. If $\mathcal{U}$ is clear from the context, we often refer to X is the topological space.

- Given a point $x \in X$, we say that N is a *neighbourhood* of x if there is an open set U, such that $x \in U$ and $U \subseteq N$. The collection of all neighbourhoods of x is denoted by $\mathcal{N}_x$.

- Given a topological X, we say that $F \in X$ is *closed*, if F^C is open.

- Give a topological space $(X,\mathcal{U})$, a set D is said dense if $D \cap U \neq \emptyset$ for all non-empty $U \in \mathcal{U}$.

Exercise 1.
Show that

1. $\emptyset$ and X are closed
2. for any set I and family $\{F_i \text{ closed}\}_{i\in I}$, also $\bigcap_{i\in I} F_i$ closed; and
3. for all closed F and G, also $F \cup G$ is closed.

For any set $A \subset X$, we define the *closure* $\overline{A}$ of A by

$$\overline{A} = \bigcap\{F \subseteq X \mid F \text{ closed and } A \subseteq F\},$$

which makes sense by the previous exercise.

Exercise 2.
Let $(X,\mathcal{U})$ be a topological space and $A \subseteq X$.

1. Show that $\overline{A} = \{x \in X \mid \forall N \in \mathcal{N}_x.\, N \cap A \neq \emptyset\}$.
2. Show that A is closed iff $\overline{A} = A$.
3. Show A is dense iff $\overline{A} = X$.

Metric Spaces

- A *metric space* is a pair (X, d), where X is a set and d is a map $d\colon X \times X \to \mathbb{R}_{\geq 0}$, such that for all $x, y, z \in X$
  1. $d(x, y) = 0$ iff $x = y$ (positive definiteness);
  2. $d(x, y) = d(y, x)$ (symmetry); and
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

  The map d is then called a *metric*.
- Given a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we define the *ε-ball* $B_\varepsilon(x)$ around x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Exercise 3.
Let (X, d) be a metric space and define $\mathcal{U} \subseteq \mathcal{P}(X)$ by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U. \exists \varepsilon > 0. B_\varepsilon(x) \subseteq U\}.$$

1. Show that the thus defined $(X, \mathcal{U})$ is a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall \varepsilon > 0. B_\varepsilon(x) \cap S \neq \emptyset\}$.

Let AP be a finite set. The set of infinite sequences over (2^{AP}) is denoted by $(2^{\mathrm{AP}})^\omega$ as before. Let $d\colon (2^{\mathrm{AP}})^\omega \times (2^{\mathrm{AP}})^\omega \to \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ.

Exercise 4.

1. Show that $((2^{\mathrm{AP}})^\omega, d)$ is a metric space.
2. Show that the closed sets of $(2^{\mathrm{AP}})^\omega$ are exactly the safety properties.
3. Show that the dense subsets of $(2^{\mathrm{AP}})^\omega$ are exactly the liveness properties.

Decomposition Theorem

Exercise 5.

1. Let $(X, \mathcal{U})$ be a topological space. Show that for all $A \subseteq X$, there is a closed set C and a dense set D such that $A = C \cap D$.
2. Show that for every LT property $P \subseteq (2^{\mathrm{AP}})^\omega$, there is a safety property P_{safe} and a liveness property P_{live} such that $P = P_{safe} \cap P_{live}$.