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Semantics and Verification (M1)

ENS DE LYON

TD de Sémantique et Vérification

IV– Partial Orders and Lattices

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Partial Orders and Lattices

- A *partial order* is a pair $(A, \leq)$ of a set A and binary relation $\leq$ which is
  1. reflexive: $a \leq a$ for all $a \in A$,
  2. transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$,
  3. antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$.
- A *join (or least upper bound)* of $S \subseteq A$ is an upper bound $\bigvee S$ such that $\bigvee S \leq b$ for every upper bound b of S.
- A *meet (or greatest lower bound)* of $S \subseteq A$ is a lower bound $\bigwedge S$ such that $b \leq \bigwedge S$ for every lower bound b of S.
- A *complete lattice* is a partial order $(A, \leq)$ such that every subset $S \subseteq L$ has both a join and a meet.
- Given a topological space $(X,\mathcal{U})$, the *interior* of a set A is $\mathring{A} = \bigcup\{U \in \mathcal{U} \mid U \subseteq A\}$.

Exercise 1.
Show that the following are equivalent for a partial order $(L, \leq)$:
1. $(L, \leq)$ is a complete lattice,
2. every subset $S \subseteq L$ has a least upper bound $\bigvee S \in L$,
3. every subset $S \subseteq L$ has a greatest lower bound $\bigwedge S \in L$.

Exercise 2.
1. Consider a topological space $(X,\mathcal{U})$. Show that $(\mathcal{U}, \subseteq)$ is a complete lattice.
2. Show that the interior of a set $A \subseteq X$ is open.
3. Show that the greatest lower bound of a family of opens $(U_i)_{i\in I}$ is the interior of $\bigcap_i U_i$.

Closure operators

A *closure operator* on a partial order $(L, \leq)$ is a function $c : L \to L$ which is:

- monotone: $c(a) \leq c(b)$ if $a \leq b$,
- expansive: $a \leq c(a)$,
- idempotent: $c(c(a)) = c(a)$.

Exercise 3.

Consider a closure operator c on a complete lattice $(L, \leq)$. Show that $L^c = \{a \in L \mid c(a) = a\}$ is a complete lattice with greatest lower bounds $\bigsqcap S = \bigwedge S$ and least upper bounds $\bigsqcup S = c(\bigvee S)$.

Exercise 4.

Consider a closure operator c on a complete lattice $(L, \leq)$. Show that for all $a \in L$, we have

$$c(a) = \bigwedge\{c(b) \mid a \leq c(b)\}$$

Exercise 5.

A *Kuratowski closure operator* is a closure operator $c : 2^X \to 2^X$ such that $c(\emptyset) = \emptyset$ and $c(A \cup B) = c(A) \cup c(B)$.

1. Consider a topological space $(X, \mathcal{U})$. Show that $\overline{(-)}$ is a Kuratowski closure operator.
2. Given a Kuratowski closure operator $c : 2^X \to 2^X$, show that there is topology $\mathcal{U}$ on X such that the closed sets for $\mathcal{U}$ are exactly the closed sets for c, ie the sets such that $A = c(A)$.

Metric Spaces

- A *metric space* is a pair (X, d), where X is a set and d is a map $d\colon X \times X \to \mathbb{R}_{\geq 0}$, such that for all $x, y, z \in X$
  1. $d(x, y) = 0$ iff $x = y$ (positive definiteness);
  2. $d(x, y) = d(y, x)$ (symmetry); and
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

  The map d is then called a *metric*.
- Given a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we define the *ε-ball* $B_\varepsilon(x)$ around x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Exercise 6.

Let (X, d) be a metric space and define $\mathcal{U} \subseteq \mathcal{P}(X)$ by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U.\, \exists \varepsilon > 0.\, B_\varepsilon(x) \subseteq U\}.$$

1. Show that the thus defined $(X, \mathcal{U})$ is a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall \varepsilon > 0.\, B_\varepsilon(x) \cap S \neq \emptyset\}$.

Let AP be a finite set. The set of infinite sequences over (2^{AP}) is denoted by $(2^{\mathrm{AP}})^\omega$ as before. Let $d\colon (2^{\mathrm{AP}})^\omega \times (2^{\mathrm{AP}})^\omega \to \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ.

Exercise 7.

1. Show that $((2^{\mathrm{AP}})^\omega, d)$ is a metric space.
2. Show that the closed sets of $(2^{\mathrm{AP}})^\omega$ are exactly the safety properties.
3. Show that the dense subsets of $(2^{\mathrm{AP}})^\omega$ are exactly the liveness properties.

Limits and Cauchy Sequences

Given a metric space (X, d), we say that a sequence $(x_n)_{n=0}^{\infty}$ in X with $x_n \in X$ *converges to* $x \in X$, if

$$\forall \varepsilon > 0. \exists N \in \mathbb{N}. \forall n \geq N. d(x_n, x) < \varepsilon.$$

We say that x is the *limit* of $(x_n)_{n=0}^{\infty}$ and write $x = \lim_{n\to\infty} x_n$. It is easily verified that limits are unique, and that $x = \lim_{n\to\infty} x_n$ iff $\forall N \in \mathcal{N}_x. \exists N \in \mathbb{N}. \forall n \geq N. x_n \in N$.

A special type of sequences are Cauchy sequences. We call a sequence $(x_n)_{n=0}^{\infty}$ in X a *Cauchy sequence*, if

$$\forall \varepsilon > 0. \exists N \in \mathbb{N}. \forall n, m \geq N. d(x_n, x_m) < \varepsilon.$$

The metric space X is called *complete*, if every Cauchy sequence in X converges.

Exercise 8.

1. Let (X, d) be a metric space and $S \subseteq X$ Show that $\overline{S}$ consists of all those points that are the limit of a sequence in S.
2. Show that the space (Σ^{ω}, d) is Cauchy complete.