TD de Sémantique et Vérification

V– Galois Connections

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- A *closure operator* on a partial order $(L, \leq)$ is a function $c : L \to L$ which is
  1. *monotone*: if $a \leq b$ then $c(a) \leq c(b)$,
  2. *expansive*: $a \leq c(a)$,
  3. *idempotent*: $c(c(a)) = c(a)$.
- Consider $(L, \leq)$ a partial order and c a closure operator. An element $a \in L$ is said *closed* if $c(a) = a$. We write L^c the set of closed elements of L.
- Given partial orders $(A, \leq_A)$ and $(B, \leq_B)$, a *Galois connection* $g \dashv f : A \to B$ is given by a pair of functions $g : A \to B$ and $f : B \to A$ such that for all $a \in A$ and $b \in B$, we have

$$g(a) \leq_B b \text{ iff } a \leq_A f(b).$$

  g (resp. f) is called the *lower adjoint* (resp. *upper adjoint*).
- Given a non-empty set A, we define

$$\begin{array}{rcccl} \mathrm{Pref} & : & 2^{A^\omega} & \to & 2^{A^*} \\ & & P & \mapsto & \bigcup\{\mathrm{Pref}(\sigma) \mid \sigma \in P\} \end{array}$$

$$\begin{array}{rcccl} \mathrm{cl} & : & 2^{A^*} & \to & 2^{A^\omega} \\ & & W & \mapsto & \{\sigma \in A^\omega \mid \mathrm{Pref}(\sigma) \subseteq W\} \end{array}$$

  Notice that the function cl defined in the second tutorial is actually $\mathrm{cl}(\mathrm{Pref}(P))$ here.

Galois connexion

Exercise 1.
Consider a Galois connection $g \dashv f : A \to B$.

1. Show that both f and g are monotone.
2. Show that $f \circ g$ is a closure operator.

Exercise 2.
Show that $\mathrm{Pref} \dashv \mathrm{cl} : 2^{A^\omega} \to 2^{A^*}$ form a Galois connection.

Exercise 3.
Consider two complete lattices $(A, \leq_A)$ and $(B, \leq_B)$.

1. Show that a function $f : B \to A$ preserves greatest lower bounds iff f has a lower adjoint $g : A \to B$.
2. Show that a function $g : A \to B$ preserves least upper bounds iff g has an upper adjoint $f : B \to A$.

Exercise 4.

Consider a closure operator $c : A \to A$. Since for all $a \in A$, $c(a)$ is closed, c is a function from A to A^c.
Show that $c : A \to A^c$ is part of a Galois connection $c \dashv \iota$ for some function $\iota : A^c \to A$.

Metric Spaces

- A *metric space* is a pair (X, d), where X is a set and d is a map $d\colon X \times X \to \mathbb{R}_{\geq 0}$, such that for all $x, y, z \in X$
  1. $d(x, y) = 0$ iff $x = y$ (positive definiteness);
  2. $d(x, y) = d(y, x)$ (symmetry); and
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

  The map d is then called a *metric*.
- Given a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we define the ε-*ball* $B_\varepsilon(x)$ around x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Exercise 5.

Let (X, d) be a metric space and define $\mathcal{U} \subseteq \mathcal{P}(X)$ by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U. \exists \varepsilon > 0. B_\varepsilon(x) \subseteq U\}.$$

1. Show that the thus defined $(X, \mathcal{U})$ is a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall \varepsilon > 0. B_\varepsilon(x) \cap S \neq \emptyset\}$.

Let AP be a finite set. The set of infinite sequences over (2^{AP}) is denoted by $(2^{\mathrm{AP}})^\omega$ as before. Let $d\colon (2^{\mathrm{AP}})^\omega \times (2^{\mathrm{AP}})^\omega \to \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ.

Exercise 6.

1. Show that $((2^{\mathrm{AP}})^\omega, d)$ is a metric space.
2. Consider $P \subseteq A^\omega$. Show that $\overline{P} = \mathrm{cl}(\mathrm{Pref}(P))$.