ENS DE LYON

TD de Sémantique et Vérification

VIII– LTL and Büchi Automata

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- We denote the words characterised by an LTL formula φ by $[\![\varphi]\!] = \{\sigma \mid \sigma \vDash \varphi\}$.
- For a set of finite words W, we denote by $\overrightarrow{W}$ the set of words having an infinite number of prefixes in W: $\overrightarrow{W} = \{\sigma \in \Sigma^\omega \mid \exists^\infty \hat{\sigma} \in W, \hat{\sigma} \subseteq \sigma\}$

Let A be a set. Recall that

- For $u \in A^*$, $\text{ext}(u) = \{\sigma \in A^\omega \mid u \subseteq \sigma\}$
- We can equip A^ω with the topology $\mathcal{U} = \{\text{ext}(U) \mid U \subseteq A^*\}$

Decomposition of ω-regular Linear Time Properties

Exercise 1.

We want to show the decomposition theorem for ω-regular properties.
Let $U \subseteq \Sigma^*$.

1. Show that if $A \subseteq \Sigma^\omega$ is ω-regular, then $\text{pref}(A) \subseteq \Sigma^*$ is regular. (You may use that $\text{pref}(U)$ is regular if $U \subseteq \Sigma^*$ is regular.)
2. Show that $\text{cl}(U)$ is a safety property induced by $P_{bad} = U^c$.
3. Deduce that if $P \subseteq \mathcal{P}(\text{AP})^\omega$ is an ω-regular safety property, then P is a *regular* safety property.
4. Let P_{bad} be a set of bad prefix of $\text{cl}(U)$ and $\mathcal{A}$ be a complete deterministic automaton recognizing P_{bad} such that for all final state q_F of $\mathcal{A}$ and all $a \in \Sigma, \delta(q_F, a) = q_F$. Show that $\mathcal{L}_\omega(\mathcal{A}) = (2^{\text{AP}})^\omega \backslash \text{cl}(U)$.
5. Show that if U is regular then $\text{cl}(U)$ is ω-regular *(we recall that if* $\text{cl}(U)$ *is a regular safety property, we can always find* P_{bad} *and* $\mathcal{A}$ *satisfying the properties of the previous question. Try to adapt* $\mathcal{A}$ *to recognize* $\text{cl}(U)$*)*.
6. Show that for every ω-regular linear time property P, there is a ω-regular safety property P_{safe} and a ω-regular liveness property P_{live} such that $P = P_{safe} \cap P_{live}$.

From LTL to NBAs

Exercise 2.

Let φ be a the LTL formula over $\text{AP} = \{a, b, c\}$ given by $\varphi = \Box a \wedge (b\, \mathcal{U}\, \neg c)$. Construct an (G)NBA $\mathcal{A}$, such that, $\mathcal{L}_\omega(\mathcal{A}) = [\![\varphi]\!]$.

Characterization of DBAs

Exercise 3.

We want to show that the languages recognized by DBAs are exactly the languages $\overrightarrow{W}$ with W a regular language.

1. Let $\mathcal{A}$ be a NBA and $W = \mathcal{L}(\mathcal{A})$ the set of *finite* words recognized by $\mathcal{A}$ seen as a NFA. Show that $\mathcal{L}_\omega(\mathcal{A}) \subseteq \overrightarrow{W}$.
2. Show that the converse does not necessarily hold, meaning that there is a NBA $\mathcal{A}$ such that $\overrightarrow{\mathcal{L}(\mathcal{A})} \not\subseteq \mathcal{L}_\omega(\mathcal{A})$.
3. Show that if $\mathcal{A}$ is a DBA, then $\overrightarrow{\mathcal{L}(\mathcal{A})} \subseteq \mathcal{L}_\omega(\mathcal{A})$.
4. Show that L is the language of a DBA iff $L = \overrightarrow{W}$ for some regular W.

Exercise 4.

1. Let $\mathrm{AP} = \{a, b\}$. Show that $\{\sigma \mid \exists^\infty t, a \in \sigma(t)\} \subseteq (2^{\mathrm{AP}})^\omega$ is not closed.
2. Show that there is a DBA $\mathcal{A}$ such that $\mathcal{L}_\omega(\mathcal{A})$ is not a safety property.
3. Show that for every regular safety property P, there is a DBA $\mathcal{A}$ such that $P = \mathcal{L}_\omega(\mathcal{A})$.