ENS DE LYON

**TD de Sémantique et Vérification**

# X– Duality and Ultrafilter extension

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## Duality

Recall that:

- for two Boolean algebras $B$ and $B'$ and a function $f : B \to B'$, the dual of $f$ is the function $f^\delta : B \to B'$ defined by $f^\delta(b) = \neg' f(\neg b)$.
- for a set Act and $\alpha \in \text{Act}$, the functions $[\alpha] : \mathfrak{L}(\text{HML}) \to \mathfrak{L}(\text{HML})$ and $\langle\alpha\rangle : \mathfrak{L}(\text{HML}) \to \mathfrak{L}(\text{HML})$ are defined by
  - $[\alpha](\phi) = [\alpha]\phi$
  - $\langle\alpha\rangle(\phi) = \langle\alpha\rangle\phi$
- for $TS = (S, \text{Act}, \to, I, \text{AP}, L)$, the functions $[\![[\alpha]]\!]$ and $[\![\langle\alpha\rangle]\!]$ are defined by
  - $[\![[\alpha]]\!](A) = \{s \in S \mid \forall s' \in \text{Succ}^\alpha(s), s' \in A\}$
  - $[\![\langle\alpha\rangle]\!](A) = \{s \in S \mid \exists s' \in \text{Succ}^\alpha(s), s' \in A\}$

**Exercise 1.**

1. Consider a transition system $TS = (S, \text{Act}, \to, I, \text{AP}, L)$ and $\alpha \in \text{Act}$. Show that:
   - $[\![[\alpha]]\!] = [\![\langle\alpha\rangle]\!]^\delta$
   - $[\![\langle\alpha\rangle]\!] = [\![[\alpha]]\!]^\delta$
2. Let $\alpha \in \text{Act}$. Show that:
   - $[\alpha] = \langle\alpha\rangle^\delta$
   - $\langle\alpha\rangle = [\alpha]^\delta$

**Exercise 2.**

Let $B$ and $B'$ be two Boolean algebras and $f : B \to B'$. Show that:

1. ${f^\delta}^\delta = f$
2. If $f$ is map of join (resp. meet) semilattices, then $f^\delta$ is map of meet (resp. join) semilattices.
3. If $f$ is a map of lattices, then $f^\delta = f$.

## Ultrafilter extension

Recall that:

- for a BAO $B^+ = (B, (f_\alpha)_{\alpha\in\mathrm{Act}})$, the ultrafilter frame $\mathfrak{Uf}(B^+)$ is defined as
  - the states are the ultrafilter over $B$, and
  - given $\mathcal{F}, \mathcal{H}$ two ultrafilters, $\mathcal{F} \xrightarrow{\alpha} \mathcal{H}$ iff $\forall b \in B, b \in \mathcal{H} \Rightarrow f_\alpha(b) \in \mathcal{F}$.
- For a set $X$, we define the function $\pi : X \to \mathfrak{Uf}(X)$ by $\pi(x) = \{A \in \mathcal{P}(X) \mid x \in A\}$. $\pi$ is a bijection between $X$ and $\mathfrak{Uf}(X)$.
- for a $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$, the ultrafilter extension $\mathfrak{Uf}(TS)$ is the transition system where
  - the states are the ultrafilters $\mathfrak{Uf}(S)$ on $S$
  - $\mathcal{F} \xrightarrow{\alpha} \mathcal{H}$ iff $[\![\langle\alpha\rangle]\!](A) \in \mathcal{F}$ whenever $A \in \mathcal{H}$
  - $a \in L(\mathcal{F})$ iff $\{s \in S \mid a \in L(s)\} \in \mathcal{F}$
  - the inital states are $\{\pi(s) \mid s \in I\}$

**Exercise 3.**
Consider a BAO $B^+ = (B, (f_\alpha)_{\alpha\in\mathrm{Act}})$. Sho that in the ultrafilter frame $\mathfrak{Uf}(B^+)$, we have

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{H} \text{ iff } \forall b \in B, f_\alpha^\delta(b) \in \mathcal{F} \Rightarrow b \in \mathcal{H}$$

**Exercise 4.**
Consider a $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$. Show that:

1. Given $s \in S$ and $a \in \mathrm{AP}$, $a \in L(s)$ in $TS$ iff $a \in L(\pi(s))$ in $\mathfrak{Uf}(TS)$.
2. Show that
$$\mathcal{F} \xrightarrow{\alpha} \mathcal{H} \text{ iff } [\![[\alpha]]\!](A) \in \mathcal{F} \Rightarrow A \in \mathcal{H}$$
3. Given $s, s' \in S$ and $\alpha \in \mathrm{Act}$, $s \xrightarrow{\alpha} s'$ in $TS$ iff $\pi(s) \xrightarrow{\alpha} \pi(s')$ in $\mathfrak{Uf}(TS)$.