ENS DE LYON

TD de Sémantique et Vérification

IV– Partial Orders and Lattices

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Partial Orders and Lattices

- A *partial order* is a pair $(A, \leq)$ of a set A and binary relation $\leq$ which is
  1. reflexive: $a \leq a$ for all $a \in A$,
  2. transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$,
  3. antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$.
- A *join (or least upper bound)* of $S \subseteq A$ is an upper bound $\bigvee S$ such that $\bigvee S \leq b$ for every upper bound b of S.
- A *meet (or greatest lower bound)* of $S \subseteq A$ is a lower bound $\bigwedge S$ such that $b \leq \bigwedge S$ for every lower bound b of S.
- A *complete lattice* is a partial order $(A, \leq)$ such that every subset $S \subseteq L$ has both a join and a meet.
- Given a topological space $(X, \mathcal{U})$, the *interior* of a set A is $\mathring{A} = \bigcup\{U \in \mathcal{U} \mid U \subseteq A\}$.

Exercise 1.

Show that the following are equivalent for a partial order $(L, \leq)$:

1. $(L, \leq)$ is a complete lattice,
2. every subset $S \subseteq L$ has a least upper bound $\bigvee S \in L$,
3. every subset $S \subseteq L$ has a greatest lower bound $\bigwedge S \in L$.

Exercise 2.

Consider the space A^ω with $A = \{a, b\}$. Show that $\bigcap_{n \in \mathbb{N}} \mathrm{ext}(a^n)$ is not open.

Closure operators

A *closure operator* on a partial order $(L, \leq)$ is a function $c : L \to L$ which is:

- monotone: $c(a) \leq c(b)$ if $a \leq b$,
- expansive: $a \leq c(a)$,
- idempotent: $c(c(a)) = c(a)$.

Exercise 3.

Consider a closure operator c on a complete lattice $(L, \leq)$. Show that $L^c = \{a \in L \mid c(a) = a\}$ is a complete lattice with greatest lower bounds $\bigsqcap S = \bigwedge S$ and least upper bounds $\bigsqcup S = c(\bigvee S)$.

Exercise 4.

A *Kuratowski closure operator* is a closure operator $c : 2^X \to 2^X$ such that $c(\emptyset) = \emptyset$ and $c(A \cup B) = c(A) \cup c(B)$.

1. Consider a topological space $(X,\mathcal{U})$. Show that $\overline{(-)}$ is a Kuratowski closure operator.
2. Given a Kuratowski closure operator $c : 2^X \to 2^X$, show that there is topology $\mathcal{U}$ on X such that the closed sets for $\mathcal{U}$ are exactly the closed sets for c, ie the sets such that $A = c(A)$.

Galois connexion

- Given partial orders $(A, \leq_A)$ and $(B, \leq_B)$, a *Galois connection* $g \dashv f : A \to B$ is given by a pair of functions $g : A \to B$ and $f : B \to A$ such that for all $a \in A$ and $b \in B$, we have

  $$g(a) \leq_B b \text{ iff } a \leq_A f(b).$$

  g (resp. f) is called the *lower adjoint* (resp. *upper adjoint*).
- Given a non-empty set A, we define

  $$\begin{array}{lccl} \text{pref} & : & 2^{A^\omega} & \to & 2^{A^*} \\ & & P & \mapsto & \bigcup\{\text{pref}(\sigma) \mid \sigma \in P\} \end{array}$$

  $$\begin{array}{lccl} \text{cl} & : & 2^{A^*} & \to & 2^{A^\omega} \\ & & W & \mapsto & \{\sigma \in A^\omega \mid \text{pref}(\sigma) \subseteq W\} \end{array}$$

  Notice that the function cl defined in the second tutorial is actually $\text{cl}(\text{Pref}(P))$ here.

Exercise 5.

Consider a Galois connection $g \dashv f : A \to B$.

1. Show that both f and g are monotone.
2. Show that $f \circ g$ is a closure operator.

Exercise 6.

Consider two complete lattices $(A, \leq_A)$ and $(B, \leq_B)$.

1. Show that a function $f : B \to A$ preserves greatest lower bounds iff f has a lower adjoint $g : A \to B$.
2. Show that a function $g : A \to B$ preserves least upper bounds iff g has an upper adjoint $f : B \to A$.

Exercise 7.

Show that $\text{Pref} \dashv \text{cl} : 2^{A^\omega} \to 2^{A^*}$ form a Galois connection.

Exercise 8.

Given $P \subseteq A^\omega$, show that $\overline{P} = \text{cl}(\text{pref}(P))$.

Continous Functions

Consider topological spaces $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$. A function $f : X \to Y$ is continous if $f^{-1}(V)$ is open in X whenever V is open in Y.

Exercise 9.

Show that $f : A^\omega \to B^\omega$ is continuous iff

$$\forall n \in \mathbb{N},\ \forall \alpha \in A^\omega,\ \exists k \in \mathbb{N},\ \forall \beta \in A^\omega \Big(\beta(0) \cdots \beta(k) = \alpha(0) \cdots \alpha(k) \quad \Rightarrow \\ f(\beta)(0) \cdots f(\beta)(n) = f(\alpha)(0) \cdots f(\alpha)(n)\Big)$$