ENS DE LYON

TD de Sémantique et Vérification

V– Observable properties and LML

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Christophe Lucas
christophe.lucas@ens-lyon.fr

We recall that

- Given sets X and Y and a function $f : X \to Y$, we define $f^{-1}(B) = \{x \mid f(x) \in B\}$ for $B \subseteq Y$.
- Given topological spaces X and Y, a function $X \to Y$ is continuous iff $f^{-1}(B)$ is open whenever B is open.
- A clopen is a set that is both open and closed.
- For a set A, a property $P \subseteq (2^A)^\omega$ is said obversable if P is a clopen.
- For a topolocial space $(X,\mathcal{U})$, a set $A \subseteq X$ is compact if every open cover of A contains a finite cover of A. If X is compact, $(X,\mathcal{U})$ is called a compact space.
- A topological space $(X,\mathcal{U})$ is Hausdorff if for any distinct points $x, y \in X$, there are disjoint opens U, V such that $x \in U$ and $y \in V$.

Exercise 1.
Show that $f : A^\omega \to B^\omega$ is continuous iff

$$\forall n \in \mathbb{N},\ \forall \alpha \in A^\omega,\ \exists k \in \mathbb{N},\ \forall \beta \in A^\omega \Big(\beta(0)\cdots\beta(k) = \alpha(0)\cdots\alpha(k) \quad \Rightarrow \\ f(\beta)(0)\cdots f(\beta)(n) = f(\alpha)(0)\cdots f(\alpha)(n)\Big)$$

Exercise 2.
Let A be a set.

1. Show that $\mathrm{ext}(u)$ is a clopen for every $u \in A^*$.
2. Show that for every finite subset $U \subseteq A^*, \mathrm{ext}(U)$ is a clopen.
3. Let $A = \mathbb{N}$. Show that there is no finite $U \subseteq \mathbb{N}^*$ such that $\mathrm{ext}(U) = \mathrm{ext}(\mathbb{N}_{>0})$.

Exercise 3.
Show that a closed subset of a compact space is compact.

Exercise 4.
Show that if A is infinite, then A^ω is not compact.

Exercise 5.
Let AP be a finite set. Show that $P \subseteq (2^{\mathrm{AP}})^\omega$ is observable iff there is a finite $W \subseteq (2^{\mathrm{AP}})^*$ such that $P = \mathrm{ext}(W)$.

Exercise 6.
Let A be a set. Show that A^ω is Hausdorff.

Exercise 7.
Let AP be a set.

1. Let ϕ be a closed LML-formula. Show that $[\![\phi]\!]$ is a clopen subset of $(2^{\mathrm{AP}})^\omega$.

2. Show that if AP is finite, then for every clopen $P \subseteq (2^{\mathrm{AP}})^\omega$, there exists a closed LML-formula ϕ such that $[\![\phi]\!] = P$.
3. Let $\mathrm{AP} = \mathbb{N}$ and $2\mathbb{N} \subseteq \mathrm{AP}$ be the set of even numbers. Show that there is no closed LML-formula ϕ such that $[\![\phi]\!] = \mathrm{ext}(2\mathbb{N})$.