

TD de Sémantique et Vérification

VI– LTL

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Recall that:

- $\llbracket \Diamond \varphi \rrbracket_\rho = \{\sigma \in (2^{\text{AP}})^\omega \mid \exists i, \sigma \upharpoonright i \in \llbracket \varphi \rrbracket_\rho\}$
- $\llbracket \Box \varphi \rrbracket_\rho = \{\sigma \in (2^{\text{AP}})^\omega \mid \forall i, \sigma \upharpoonright i \in \llbracket \varphi \rrbracket_\rho\}$
- $\llbracket \varphi \text{ U } \psi \rrbracket_\rho = \{\sigma \in (2^{\text{AP}})^\omega \mid \exists i, \sigma \upharpoonright i \in \llbracket \psi \rrbracket_\rho, \forall j < i, \sigma \upharpoonright j \in \llbracket \varphi \rrbracket_\rho\}$
- $\varphi \text{ W } \psi := \neg(\neg\psi \text{ U } \neg(\varphi \vee \psi))$

Moreover, a state  $s$  of a transition system satisfies a LTL formula  $\varphi$  with parameter  $\rho$  if and only if  $\text{Traces}(s) \subseteq \llbracket \varphi \rrbracket_\rho$ .

**Exercise 1.**

Let  $L$  be a complete lattice and let  $f : L \rightarrow L$  be a monotone function. Show that  $\mu(f) = \bigwedge \{a \in L \mid f(a) \leq a\}$  (resp.  $\nu(f) = \bigvee \{a \in L \mid a \leq f(a)\}$ ) is the least fixpoint (resp. greatest fixpoint) of  $f$ .

**Exercise 2.**

Show that:

1.  $\neg(\varphi \text{ W } \psi) \equiv \neg\psi \text{ U } (\neg\varphi \wedge \neg\psi)$
2.  $\neg(\varphi \text{ U } \psi) \equiv \neg\psi \text{ W } (\neg\varphi \wedge \neg\psi)$
3.  $\bigcirc(\varphi \text{ U } \psi) \equiv \bigcirc\varphi \text{ U } \bigcirc\psi$
4.  $\Diamond\phi \equiv \neg\Box\neg\phi$
5.  $\Diamond\phi \equiv \phi \vee \bigcirc\Diamond\phi$

**Exercise 3.**

Let  $\phi$  be a formula with parameters  $\rho$ . We define  $\phi_\Box(X) = \phi \wedge \bigcirc X$ . Show that for every valuation  $\rho$ ,  $\llbracket \Box\phi \rrbracket_\rho$  is the greatest fixpoint of  $\llbracket \phi_\Box \rrbracket_\rho$ .

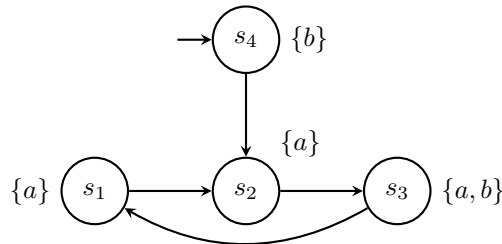
**Exercise 4.**

Show that:

1.  $\llbracket \top \text{ U } \varphi \rrbracket_\rho = \{\sigma \in (2^{\text{AP}})^\omega \mid \exists i, \sigma \upharpoonright i \in \llbracket \varphi \rrbracket_\rho\}$
2.  $\llbracket \varphi \text{ W } \perp \rrbracket_\rho = \{\sigma \in (2^{\text{AP}})^\omega \mid \forall i, \sigma \upharpoonright i \in \llbracket \varphi \rrbracket_\rho\}$

**Exercise 5.**

Consider the following transition system over the set of atomic propositions  $\{a, b\}$ :



Indicate for each of the following LTL formulae the set of states for which these formulae are fulfilled:

A.  $\bigcirc a$

C.  $\Box b$

E.  $\Box(b \text{ U } a)$

B.  $\bigcirc \bigcirc \bigcirc b$

D.  $\Box \Diamond a$

F.  $\Diamond(a \text{ U } b)$

**Exercise 6.**

A formula  $\phi$  is said to be in positive normal form if the negations is only applied to atoms or variables (for instance, with  $a, b \in AP$ ,  $a \wedge \neg b$  is in positive normal form but  $\neg(a \vee b)$  is not).

1. Show that every formula is equivalent to a formula in positive normal form.
2. We replace the operator  $W$  by the release operator  $R$  satisfying  $\neg(\phi R \psi) \equiv (\neg\phi) \text{ U } (\neg\psi)$ .  
Show that, with this operator, every formula is still equivalent to a formula in positive normal form.
3. Compare the two methods regarding the size of the formulae.