An optimal algorithm to generate tilings

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1 The result

A lot of progress has been made in tiling theory in the last ten years after Thurston ([Thu90]), building on previous work by Conway and Lagarias ([CL90]), introduced height functions as a tool to encode and study tilings.

This allowed the authors of this paper, in previous work ([Rém99], [Des01]), to prove that the set of lozenge (or domino) tilings of a hole-free, general-shape domain in the plane can be endowed with a distributive lattice structure.

In this paper, we see that this structure allows us in turn to construct an algorithm that is optimal with respect to both space and execution time to generate all the tilings of a domain D. We first recall some results about tilings, and then we describe the algorithm.

2 Properties of tilings

Lozenge tilings We assume that the cells of the triangular lattice of the plane are colored in black or white, in such a way that two cells sharing an edge have different colors. This coloration induces a direction on the edges of the lattice: they are directed clockwise around black cells and (consequently) counterclockwise around white cells. A *domain* is a finite and simply connected union of cells of the lattice. A *lozenge* is a union of two cells sharing an edge, which is called the *central axis* of the lozenge. A *lozenge tiling* of a domain is a set of lozenges that cover the whole area with neither gap nor overlap.

Height functions The *height functions*, introduced by W. P. Thurston ([Thu90]) and independently in the statistical physics litterature (see [BH97] for a review) and precisely studied and generalized by several authors ([Cha96], [Pro01], [Rém99], [Des01]) are a very powerful tool to study lozenge tilings. A lozenge tiling T of a domain D can be encoded by a height function h_T, defined as follows: fix an origin vertex O on the boundary of D and set $h_T(O) = 0$; if (v, v') is a directed edge such that $[v, v']$ is the central axis of a lozenge of T, then $h_T(v') = h_T(v) - 2$; otherwise, $h_T(v') = h_T(v) + 1$. This definition is coherent since it is coherent for each triangle and D is simply connected.

Order Let (T, T') be a pair of tilings of D. We say that $T \leq T'$ if for each vertex v of D, $h_T(v) \leq h_{T'}(v)$. The functions $h_{\inf(T,T')} = \min(h_T, h_{T'})$ and $h_{\sup(T,T')} = \max(h_T, h_{T'})$ are themselves height functions that encode tilings, which implies that the set of the tilings of D has a structure of distributive lattice (see for example [DP90] for an introduction to lattice theory).

Flips Let v be an interior vertex of D such that all the directed edges ending in v are central axes of lozenges in a tiling T. A flip is the replacement of these three lozenges by three lozenges whose central axis are edges starting in v. A new tiling T_{flip} is thus obtained; T and T_{flip} are comparable for the order defined above. More generally, $T \leq T'$ if and only if there exists an increasing sequence $(T = T_0, T_1, \ldots, T_p = T')$ of tilings such that $\forall\ 0 \leq i < p$, T_{i+1} is deduced from T_i by a flip. As a corollary we get the *flip connectivity*: given any pair (T, T') of tilings of D, one can pass from T to T' by a sequence of flips and, more precisely, the min-

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imal number of flips to pass from T to T' is $\sum_v |h_T(v) - h_{T'}(v)|/3$.

Construction (Thurston's algorithm) There exists a minimal tiling whose corresponding height function has no local maximum except on the boundary of D; indeed, a flip could otherwise be done around the local maximum, yielding a new minimal tiling. From this property we deduce a linear algorithm which constructs the minimal tiling if D can be tiled, or proves that D is not tileable.

3 Algorithm

In order to generate all the tilings of a domain, a natural idea is to encode tilings by words and then to provide a way to find the successor of an element in the lexicographic order. We now describe this two ingredients. There are many ways to encode tilings by words. The difficult part is to select one that nicely allows a successor function.

Total order Since height functions encode tilings, it suffices to fix an arbitrary order on the vertices of D (any order will do) to obtain an encoding of the tilings by words, which are ordered lexicographically. We see that this encoding follows closely the height functions and is therefore very natural. This total order is a linear extension of the order defining the lattice structure.

Computation of the successor Given a word that encodes a tiling, we now need to find its successor in the lexicographic order. Or rather, the next word that encodes a tiling. We know that the set of the tilings of D is connected by flips. Our first step to find the successor is to identify the vertices that can be submitted to a flip and then to select the right-most one in the encoding by a word. Then performing the flip yields a new tiling, greater than the former one in the lexicographic order, but maybe not the successor. To construct the latter, we use a generalized version of Thurston's algorithm: instead of assigning values on the boundary of D and then computing the minimal tiling, let us suppose that values are assigned to the vertices on the boundary of D *and* on any number of vertices inside D, such that at least one tiling exists under these conditions. Our generalized version constructs the tiling which has the minimal height function with respect to the values already assigned, and this provides the successor.

Analysis Let us sketch the analysis of this algorithm. We suppose that tilings are discarded as soon as they are exhibited. All we need to find a successor is the knowledge of one tiling, so the space required is the one needed to encode a tiling by a height function, that is $O(|D| \ln |D|)$. Since the generalized version of Thurston's algorithm is linear in the number of vertices of D, the execution time is $O(|D|)$ times the number of tilings.

We have thus provided an algorithm that makes a non-trivial use of the lattice structure of the tilings, is linear in the number of tilings and requires a fixed (small) space. Our method can also be applied to tilings by dominoes. Furthermore, it should be rather straightforward to generalize it to domains with holes.

References

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