Computing matrix perturbations with minimal leading Jordan structure.

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Abstract. This article aims to briefly outline a new algorithm for computing some eigenvalue first-order terms of an arbitrary matrix perturbation, i.e. a formal power series matrix $A(\epsilon) = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots$ with arbitrary n by n leading matrix A_0. The eigenvalue first-order terms we recover are all those whose inverse is a positive integer. Here no assumption is made on neither the nature of the eigenvalues of $A(\epsilon)$ nor the form of A_0. This improves previous results [1] where it is required that the leading exponent of any eigenvalue of $A(\epsilon)$ be the inverse of a positive integer and that A_0 be a nilpotent matrix in Jordan canonical form.

Key words. Eigenvalues of matrix perturbations, nilpotent Jordan form, similarity transforms, Newton diagram.

Matrix perturbations with minimal leading Jordan structure. Assuming that the leading matrix A_0 is a nilpotent Jordan canonical form, Lidskii [2] provided sufficient conditions on some entries of A_1 for $A(\epsilon)$ to admit rm eigenvalues with order $\mathrm{O}(\epsilon^{1/m})$ when A_0 has r Jordan blocks with size m; he further showed how to recover the associated leading coefficients from these particular entries exclusively. However, though generically satisfied, Lidskii's conditions are not necessary and the same eigenvalue splitting may occur for some other leading Jordan structures. Consider for example the two 4×4 formally similar matrices

$$A(\epsilon) = \left[\begin{array}{ccc|c} 0 & 1 & & \\ \epsilon & 0 & 1 & \\ & & 0 & \epsilon \\ \hline \epsilon & & & 0 \end{array}\right] \quad \sim \quad \tilde{A}(\epsilon) = \left[\begin{array}{cc|cc} 0 & 1 & & \\ \epsilon & 0 & \epsilon & \\ \hline & & 0 & 1 \\ \epsilon & & & 0 \end{array}\right].$$

One could verify that both matrix perturbations have four eigenvalues with order $\mathrm{O}(\epsilon^{1/2})$ $(r = m = 2)$. Actually, the Jordan structure A_0 is "misleading" in the sense that it does not consist of two 2×2 nilpotent Jordan blocks. On the other hand, $A(\epsilon)$ is similar for all $\epsilon \neq 0$ to $\tilde{A}(\epsilon)$ whose leading structure matches the eigenvalue splitting. The structure $\tilde{A}_0$ is thus "better" than A_0. Viewing nilpotent Jordan structures as integer partitions ordered with lexicographic ordering $<$, we will see that "better" actually means "smaller with respect to $<$". More precisely, any matrix perturbation $\tilde{A}(\epsilon)$ similar to $A(\epsilon)$ and having minimal leading Jordan structure has the following property : all the eigenvalue first-order terms of $A(\epsilon)$ whose exponent is the inverse of a positive integer can be recovered from $\tilde{A}_0$ and $\tilde{A}_1$ by applying Lidskii's perturbation theory.

A minimization algorithm. In order to design an algorithm that computes a similar form of $A(\epsilon)$ with minimal leading Jordan structure, we notice first that the leading rank of such a form must be minimal. This leads to the following two stage algorithmic scheme

1. compute a matrix perturbation $A^{(1)}(\epsilon)$ similar to $A(\epsilon)$ with minimal leading rank;

2. minimize the leading Jordan structure of $A^{(1)}(\epsilon)$ without any leading rank change.

Step 1 can be done by removing from A_0 as many columns with valuation zero as possible by means of appropriate similarity transforms whose entries are polynomials in ϵ. The leading coefficient of the resulting matrix perturbation has the structure shown by $A_0^{(1)}$ in the example below. Step 2 progressively builds the minimal leading Jordan structure by removing nonzero columns of $A_0^{(1)}$ in order to create as many nilpotent Jordan blocks as possible with increasing sizes : roughly, every time a nonzero column is removed, an additional 1 appears on the first upper diagonal of the leading matrix, thus increasing by one the size of an existing block while keeping the leading rank minimal. Again, the similarity transforms used are polynomial matrices : for $i \geq 1$, stepping from $A^{(i)}(\epsilon)$ to $A^{(i+1)}(\epsilon)$ is achieved by means of a similarity transform $P^{(i)}(\epsilon) = C_1^{(i)} D^{(i)}(\epsilon) C_2^{(i)}$ where $C_1^{(i)}$ and $C_2^{(i)}$ are invertible constant matrices and $D^{(i)}(\epsilon)$ is a diagonal matrix having only 1's and ϵ's on the diagonal. Evidence of the correctness of the algorithm is provided by the Newton diagram analysis of Moro, Burke and Overton [3, §3] where a $p \times p$ nilpotent Jordan block is identified with a segment with slope $1/p$ and whose projection on the horizontal axis has length p : step 1 actually defines the best lower approximation of the Newton diagram $\mathcal{N}$ of $A(\epsilon)$ by means of segments whose slope is either 0 or 1; then step 2 first allows to use segments with slope 1/2, then segments with slope 1/3 etc. We stop when the leading Jordan structure of the current matrix defines the best approximation of $\mathcal{N}$ with segments whose slope is either 0 or the inverse of a positive integer, for this means that such a structure is minimal. Notice that the smallest slope of this approximation is then the inverse of the size of the biggest Jordan block(s) in the minimal leading structure we are looking for.

It should also be emphasized that this approach - contrary to the one proposed in [1] - does not require that the Jordan canonical form of A_0 be known from the outset nor that $A(\epsilon)$ admit $\mathrm{O}(\epsilon^{1/m_i})$-eigenvalues only, with $m_1, m_2, \ldots$ some positive integers. Additionally, the algorithm can still be used when A_0 is not nilpotent. The output will then have a leading matrix that can be block partitioned as

$$\begin{bmatrix} * & 0 \\ * & J \end{bmatrix}$$

where J is a nilpotent Jordan structure of size $p < n$ so that $n - p$ is the number of $\mathrm{O}(1)$-eigenvalues of $A(\epsilon)$. ($p = n$ if and only if A_0 is nilpotent.)

Example. As an example, let us consider a 7 by 7 matrix perturbation $A(\epsilon)$ with three eigenvalues with order $\mathrm{O}(\epsilon^{1/3})$, three eigenvalues with order $\mathrm{O}(\epsilon^{2/3})$ and one eigenvalue with order $\mathrm{O}(\epsilon)$. Our claim is that $A(\epsilon)$ has minimal leading rank 3 and that our minimization algorithm successively computes similar matrices $A^{(1)}(\epsilon)$, $A^{(2)}(\epsilon)$, $A^{(3)}(\epsilon)$ and $A^{(4)}(\epsilon)$ with leading coefficients all having rank 3 but the following different shapes. (The

$*$'s may of course take different values at each different step.)

$$A_0^{(1)} = \left[\begin{array}{ccc|cccc} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ \hline * & * & * & 0 & & & \\ * & * & * & & 0 & & \\ * & * & * & & & 0 & \\ * & * & * & & & & 0 \end{array}\right] \longrightarrow A_0^{(2)} = \left[\begin{array}{cc|cc|ccc} * & * & & & & & \\ * & * & & & & & \\ \hline * & * & 0 & 1 & & & \\ * & * & & 0 & & & \\ \hline * & * & & & 0 & & \\ * & * & & & & 0 & \\ * & * & & & & & 0 \end{array}\right]$$

$$\longrightarrow A_0^{(3)} = \left[\begin{array}{c|cc|cc|cc} * & & & & & & \\ \hline * & 0 & 1 & & & & \\ * & & 0 & & & & \\ \hline * & & & 0 & 1 & & \\ * & & & & 0 & & \\ \hline * & & & & & 0 & \\ * & & & & & & 0 \end{array}\right] \longrightarrow A_0^{(4)} = \left[\begin{array}{ccc|cc|cc} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ \hline & & & 0 & 1 & & \\ & & & & 0 & & \\ \hline & & & & & 0 & \\ & & & & & & 0 \end{array}\right].$$

The last leading matrix $A_4^{(0)}$ is the minimal leading Jordan structure associated to the initial matrix perturbation $A(\epsilon)$ and Lidskii's theorem [3, Theorem 2.1 page 798] then allows to recover all its $\mathrm{O}(\epsilon^{1/3})$- and $\mathrm{O}(\epsilon)$-eigenvalue first-order terms. The corresponding Newton diagram interpretation is as follows, where the envelope terminology has been borrowed from [3] : when i increases, the leading matrix $A_0^{(i)}$ defines an envelope $\mathcal{E}^{(i)}$ that "better" approximates the Newton diagram $\mathcal{N}$ of $A(\epsilon)$ (see Figure 1). Envelope $\mathcal{E}^{(4)}$ is the best approximation of $\mathcal{N}$ that is possible by using segments with slope either 0 or $1/s$ with s a positive integer. Notice that here $s \leq 3$ since the minimal Jordan structure $A_0^{(4)}$ has Jordan blocks of size at most 3.

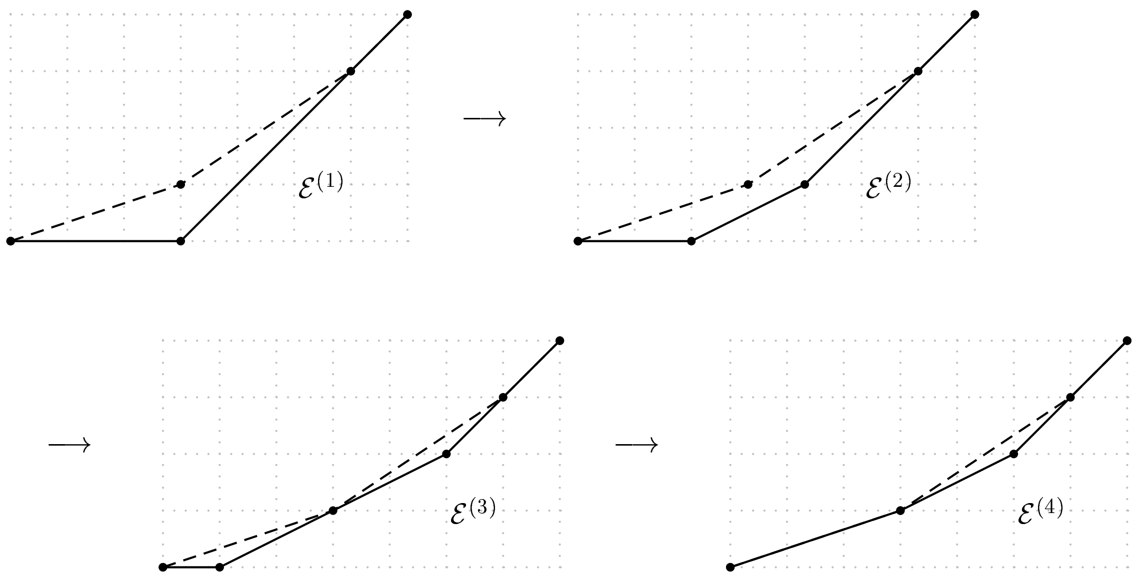

Figure 1: The Newton diagram $\mathcal{N}$ is shown as a dashed line and its four successive approximations $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, $\mathcal{E}^{(3)}$, $\mathcal{E}^{(4)}$ as solid lines. Notice the best approximation of the segment of $\mathcal{N}$ with slope $2/3$ by means of two segments with slopes $1/2$ and 1.

References

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