An Algorithm for the Eigenvalue Perturbation Problem: Reduction of a κ-Matrix to a Lidskii Matrix.

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ABSTRACT

In this article, we present an algorithmic approach to the eigenvalue perturbation problem. We show that any matrix perturbation $A(\epsilon)$ of an arbitrary nilpotent Jordan canonical form J with all eigenvalues having an order of the form $\mathrm{O}(\epsilon^{1/(\text{a positive integer})})$ is similar to a matrix perturbation $\tilde{A}(\epsilon)$ in Arnold normal form that can be seen as generic. Calling $A(\epsilon)$ a κ-matrix and $\tilde{A}(\epsilon)$ a Lidskii-Arnold matrix, we also provide a reduction algorithm for the computation of the Lidskii-Arnold form of a κ-matrix. It is based on the minimization of the leading Jordan structure J and on Lidskii's genericity conditions for perturbed eigenvalues.

1. INTRODUCTION

Let $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ be an n by n matrix perturbation where J is a nilpotent Jordan canonical form (JCF), B a complex matrix and ϵ a small parameter. The entries of $A(\epsilon)$ are assumed to be holomorphic at $\epsilon = 0$. We study the link between the first two coefficients J, B of $A(\epsilon)$ considered as the formal power series $\sum_{i\geq 0} A_i\epsilon^i$, and the leading terms $\mu\epsilon^\beta$ of the Puiseux expansions $\lambda(\epsilon) = \mu\epsilon^\beta + o(\epsilon^\beta)$ of the eigenvalues of $A(\epsilon)$. (See [2, 8] for the existence of such Puiseux expansions at $\epsilon = 0$.) This is traditionally referred to as the perturbation theory for the JCF [11]. In particular, it has been used to explain the generic and also some nongeneric behaviors that may occur when computing numerical eigenvalues [11, 10, 4].

Using Arnold's compact notation to denote by 0^k a nilpotent Jordan block of size k and by $(0^k)^r$ the block-diagonal matrix that consists of r such blocks, let $J = (0^{n_1})^{r_1} \dots (0^{n_q})^{r_q}$, and assume that $n_1 > \dots > n_q$. (The r_i are positive integers, and clearly $n = \sum_{i=1}^q r_i n_i$.) Lidskii [9, 11] has shown that generically $A(\epsilon)$ admits $r_j n_j$ eigenvalues with order $\mathrm{O}(\epsilon^{1/n_j})$. Furthermore, denoting by t_j the partial sum $r_1 + \dots + r_j$, the leading coefficients μ of these eigenvalues depend exclusively on t_j^2 particular entries of the principal submatrix of B of order $r_1n_1 + \dots + r_jn_j$. In practice, a generic B will mean a matrix B that is chosen dense, at random.

In this paper, we address the "reciprocal". Though it may be easily verified that for a holomorphic matrix $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ with all eigenvalues having orders of the form $\mathrm{O}(\epsilon^{1/(\text{a positive integer})})$, the sizes of the blocks of J differ from these positive integers, we may wonder whether there exists a matrix $\tilde{A}(\epsilon)$ similar to $A(\epsilon)$ whose first two coefficients $\tilde{J}$, $\tilde{B}$ contain the generic behavior described by Lidskii.

The characterization we give for Lidskii matrices allows for a positive answer to this question. Furthermore, we provide a constructive proof that leads to a reduction algorithm transforming $A(\epsilon)$ into a normalization of $\tilde{A}(\epsilon)$. The central idea is the following. While $A(\epsilon)$ is not a Lidskii matrix, its leading JCF J can be "decreased", in a sense to be precised, by means of a similarity transform.

In the sequel, the matrices $A(\epsilon)$ and $\tilde{A}(\epsilon)$ will be respectively called **κ-matrix** and **Lidskii matrix**. (We prefer to say 'Lidskii matrix' for $\tilde{A}(\epsilon)$ rather than 'generic matrix' in order to avoid confusion with the usual genericity of the first-order perturbation $A_1 = B$.)

The main motivation for this study is an algorithmic use of Lidskii's eigenvalue perturbation theory, which we believe to be new. As far as we know, this is indeed the first time a matricial algorithm is proposed for perturbed eigenvalues whose order is *not* an integer. (The integer case was studied in [7]). Furthermore, it should be considered as a first step towards the more general goal of recovering *any* leading term (e.g. $\epsilon^{2/5}$) from a few, low order, coefficients of a matrix similar to $A(\epsilon)$.

Applications arise in quantum physics [6] and when computing the formal solutions of linear differential systems [12, 3, 13].

Let $\mathsf{K} = \mathbb{C}$ and $\mathsf{F} = \mathsf{K}[[\epsilon]]$ be the ring of formal power series in ϵ, and denote by $\mathsf{F}[\epsilon^{-1}]$ its quotient field. Viewing $A(\epsilon)$ as an element of $\mathsf{F}[\epsilon^{-1}]^{n\times n}$, and denoting by $GL(n, \mathsf{F}[\epsilon^{-1}])$ the group of units of $\mathsf{F}[\epsilon^{-1}]^{n\times n}$, the orbit of matrices **similar** to $A(\epsilon)$ is $\mathcal{O}_A \equiv \{P^{-1}AP : \det P(\epsilon) \neq 0\}$.

The **valuation** of $A(\epsilon)$ is defined as the smallest power of

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ϵ that appears in the entries of the matrix, and is denoted by $\operatorname{val} A(\epsilon)$. The valuation β of an eigenvalue of $A(\epsilon)$ is merely called **leading exponent**, and, together with the associated **leading coefficient** (or ring constant) μ, it forms the leading term of the eigenvalue. In the sequel, $A(\epsilon)$ will be chosen with no loss of generality of the form $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ with J a nilpotent JCF, and identified with $\sum_{i\geq 0} A_i \epsilon^i$.

In Section 2.1, we recall representations of nilpotent Jordan structures. Sections 2.2 and 2.3 are devoted to Lidskii matrices and Lidkii's eigenvalue perturbation result, and to its Newton diagram-based interpretation. Section 3 presents the first step of our algorithm: the minimization of the rank of the leading matrix J. Then, a further improvement of J is detailed in Sections 4.2 and 4.3, using the properties given in Section 4.1. A characterization of Lidskii matrices follows in Section 4.4. The algorithm that results from both Sections 3 and 4 is given in Section 5. The Lidskii matrix we obtain may be normalized as shown in Section 6. We conclude in Section 7.

2. PRELIMINARIES AND PREVIOUS RESULTS

2.1 Ordering nilpotent Jordan structures

The block sizes of a nilpotent Jordan form J are known as the **Segré characteristics** of J [4, §2.2]. When counted with their multiplicity, they define J completely. Consider the n by n nilpotent Jordan structure $J = (0^{n_1})^{r_1} \cdots (0^{n_q})^{r_q}$ with $n_1 > \ldots > n_q$, and let $t_j = r_1 + \ldots + r_j$, $s_j = r_1 n_1 + \ldots + r_j n_j$, $t = t_q$ and $s = s_q = n$. The Segré characteristics of J may be represented by the partition of n

$$\kappa(J) = (\underbrace{n_1, \ldots, n_1}_{r_1}, \ldots, \underbrace{n_q, \ldots, n_q}_{r_q})$$

which we may also write $\kappa(J) = ((r_j, n_j))_{1\leq j\leq q}$. The order used is the lexicographic ordering. This allows for the comparison of any two nilpotent Jordan matrices J and $\tilde{J}$. In the case when J and $\tilde{J}$ have the same rank, such an ordering is easily illustrated by two dual representations, Ferrer diagrams [4] (or Young tableaux [5]) and Newton envelopes [11].

The **Ferrer diagram** $\mathcal{F}(J)$ of J represents each Jordan block 0^k as a stack of k coins. The stacks are ordered by nonincreasing heights, from the left to the right. Clearly, there are exactly n coins and the total number of stacks is equal to $n - \operatorname{rank} J$. The move of a coin from a stack to another, from the left to the right, without modifying the monotonicity of the partition is called a rightward coin move. (See Figure 1.)

In their Newton diagram-based study of the Lidskii eigenvalue perturbation theory, Moro, Burke and Overton [11] use another diagram to represent a nilpotent Jordan structure. Set $s_0 = t_0 = 0$ and $\mathcal{P}_0 \equiv (0,0)$, and define for $j = 1, \ldots, q$, $k = 1, \ldots, r_j$

$$\mathcal{P}_j^k \equiv (s_{j-1} + k n_j, t_{j-1} + k) \quad \text{and} \quad \mathcal{P}_j \equiv \mathcal{P}_j^{r_j} \equiv (s_j, t_j)\ ,$$

Following [11, p.808], the lower convex hull $\mathcal{E}(J)$ of the set $\{\mathcal{P}_j\}_{0\leq j\leq q}$ will be referred to as the **Newton envelope** associated with J. This is the lowest possible Newton diagram that can be obtained by a perturbation of J. We say that $\mathcal{E}(\tilde{J})$ lies above $\mathcal{E}(J)$ when all the $\tilde{\mathcal{P}}_j^k$ lie on $\mathcal{E}(J)$ with at least one of them lying strictly above $\mathcal{E}(J)$. (See Figure 2.)

Since the entries of the integer partitions we consider are arranged in nonincreasing order and by definition of the Ferrer diagram and of the Newton envelope, we have the following

FACT 1. *Assume that* $\operatorname{rank} J = \operatorname{rank} \tilde{J}$. *Then,*

(i) $\kappa(\tilde{J}) < \kappa(J)$ *if and only if* $\mathcal{F}(\tilde{J})$ *can be derived from* $\mathcal{F}(J)$ *by a rightward coin move.*

(ii) $\kappa(\tilde{J}) < \kappa(J)$ *if and only if* $\mathcal{E}(\tilde{J})$ *lies above* $\mathcal{E}(J)$.

EXAMPLE 1. *Consider* $J = (0^3)^2(0^2)^3 0$ *and* $\tilde{J} = 0^3(0^2)^5$. *Then* $\kappa(J) = (3,3,2,2,2,1) > \kappa(\tilde{J}) = (3,2,2,2,2,2)$ *and "moving" from* J *to* $\tilde{J}$ *is illustrated in Figures 1 and 2.*

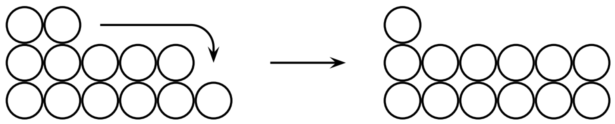

Figure 1: Ferrer diagrams for $J = (0^3)^2(0^2)^3 0$ and $\tilde{J} = 0^3(0^2)^5$ and rightward coin move indicating that $\kappa(J) > \kappa(\tilde{J})$.

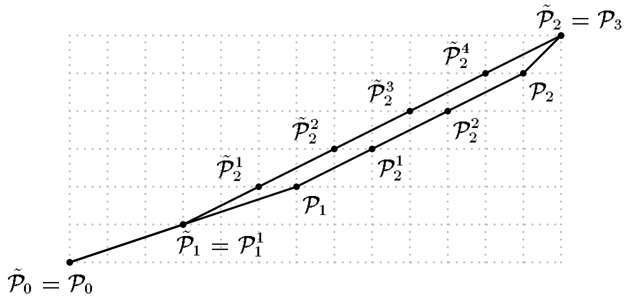

Figure 2: Newton envelopes for $J = (0^3)^2(0^2)^3 0$ and $\tilde{J} = 0^3(0^2)^5$ also indicating that $\kappa(\tilde{J}) < \kappa(J)$.

We end this paragraph with the following definition, which will be used throughout the rest of the paper.

DEFINITION 1. *Let* $A(\epsilon) \in \mathsf{F}[\epsilon^{-1}]^{n\times n}$. *When* $\operatorname{val} A(\epsilon) \geq 0$, *write* $A(\epsilon) = \sum_{i\geq 0} A_i \epsilon^i$ *and let* $\kappa(A) = \kappa(J)$ *where* J *is the JCF of* A_0. *If* $\operatorname{val} A(\epsilon) < 0$, *we set* $\kappa(A) = (+\infty)$.

This will allow for the minimization of $\kappa(A)$ over $\mathcal{O}_A$, and thus of $\kappa(J)$ over the set of matrix perturbations similar to $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$.

2.2 Lidskii matrices

Recall first the definition of a ring of eigenvalues.

Definition 2 ([10]). *Let $w = e^{\frac{2\pi i}{k}}$. A set of k eigenvalues of $A(\epsilon)$ that satisfy*

$$\lambda_l(\epsilon) = \mu w^l \epsilon^{1/k} + o(\epsilon^{1/k}), \quad l = 0, \ldots, k-1, \quad 0 \neq \mu \in \mathbb{C},$$

*is called a k-**ring**. The number μ is the **ring constant**.*

The subclass of matrix perturbations we will consider in the sequel is given as follows.

Definition 3. *Let $\kappa = ((r_j, n_j))_{1 \leq j \leq q}$ be a partition of n. A κ-**matrix** is a matrix $A(\epsilon)$ whose n eigenvalues split for $j = 1, \ldots, q$ into r_j n_j-rings.*

Note that in practice the value of κ is not known from the outset. This prevents from setting $z = \epsilon^{1/n_j}$ into $A(\epsilon)$, and applying to $A(z^{n_j})$ the reduction algorithm of [7, p.123]. Indeed, it handles integer leading exponents only [7, Proposition 1].

Definition 4. *A **Lidskii matrix** is a κ-matrix of the form $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ where $\kappa(A) = \kappa$.*

Hence, the eigenvalue splitting of a Lidskii matrix can be identified with its leading Jordan structure.

Since the valuation of a κ-matrix is at most 1, the valuation of a Lidskii matrix is either 0 or 1.

An arbitrary nilpotent Jordan structure J being given, Lidskii [9, 11] provided some sufficient conditions on B for $A(\epsilon)$ to be a κ-matrix with $\kappa = \kappa(A)$. Recalling that $t = r_1 + \ldots + r_q$ is the number of blocks of J, partition B conformally with the block structure of J into t^2 blocks B_{kl}. Denote by b_{kl} the lower left corner of each block B_{kl} and form for $j = 1, \ldots, q$ the t_j by t_j constant matrices $L_j = [b_{kl}]_{1 \leq k, l \leq t_j}$.

Definition 5. *We call j**th Lidskii submatrix** associated with $A(\epsilon)$ the matrix L_j. Let also $\Delta_j \equiv \det L_j$, and, for $k = 1, \ldots, r_j$, denote by Δ_j^k the sum of the principal minors of L_j with order $t_{j-1} + k$ that contain L_{j-1}.*

Theorem 1 ([9],[11]). *A κ-matrix is a Lidskii matrix if and only if $\Delta_j \neq 0$ for $j = 1, \ldots, q$. Additionally, the n_jth power of the ring constants for each j are the roots of the polynomial $\det(L_j - \lambda E_j)$ of degree r_j where $E_j = diag(0_{t_{j-1}}, I_{r_j})$.*

When B is chosen dense, at random, such conditions are satisfied in general. In other words, a generic perturbation of an arbitrary nilpotent Jordan structure is a Lidskii matrix.

Example 2. *Let $J = (0^3)^2 (0^2)^3 0$ as in Example 1, and let B a 13 by 13 matrix partitioned conformally with J.*

$$B = \left[\begin{array}{ccc|ccc|cc|cc|cc|c}
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
\heartsuit & * & * & \heartsuit & * & * & \spadesuit & * & \spadesuit & * & \spadesuit & * & \diamondsuit \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
\heartsuit & * & * & \heartsuit & * & * & \spadesuit & * & \spadesuit & * & \spadesuit & * & \diamondsuit \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
\spadesuit & * & * & \spadesuit & * & * & \spadesuit & * & \spadesuit & * & \spadesuit & * & \diamondsuit \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
\spadesuit & * & * & \spadesuit & * & * & \spadesuit & * & \spadesuit & * & \spadesuit & * & \diamondsuit \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
\spadesuit & * & * & \spadesuit & * & * & \spadesuit & * & \spadesuit & * & \spadesuit & * & \diamondsuit \\
\hline
\diamondsuit & * & * & \diamondsuit & * & * & \diamondsuit & * & \diamondsuit & * & \diamondsuit & * & \diamondsuit
\end{array}\right]$$

Then

$$L_1 = \begin{bmatrix} \heartsuit & \heartsuit \\ \heartsuit & \heartsuit \end{bmatrix}, \quad L_2 = \begin{bmatrix} \heartsuit & \heartsuit & \spadesuit & \spadesuit & \spadesuit \\ \heartsuit & \heartsuit & \spadesuit & \spadesuit & \spadesuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit \end{bmatrix},$$

$$L_3 = \begin{bmatrix} \heartsuit & \heartsuit & \spadesuit & \spadesuit & \spadesuit & \diamondsuit \\ \heartsuit & \heartsuit & \spadesuit & \spadesuit & \spadesuit & \diamondsuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \diamondsuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \diamondsuit \\ \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \diamondsuit \\ \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit \end{bmatrix}$$

and

$$E_1(\lambda) = diag(\lambda, \lambda), \quad E_2(\lambda) = diag(0, 0, \lambda, \lambda, \lambda),$$

$$E_3(\lambda) = diag(0, 0, 0, 0, 0, \lambda).$$

If B is generic, a matrix of the form $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ is a Lidskii matrix according to Theorem 1. However, if, for example, B has all its entries equal to zero, except for $B_{3,1}$, $B_{6,7}$, $B_{8,9}$, $B_{10,11}$, $B_{12,13}$ and $B_{13,4}$ which are set to one, $A(\epsilon) = J + \epsilon B$ is only a κ-matrix with $\kappa = (3, 2, 2, 2, 2, 2) < \kappa(A)$. In the next examples, we keep considering this non Lidskii B.

2.3 A Newton diagram characterization

In [11], Moro, Burke and Overton provide a characterization of Lidskii's sufficient conditions "$\Delta_j \neq 0$" in terms of the Newton diagram of the matrix $A(\epsilon)$. (This geometric construction is recalled in [11, p.805-806].) One has $\Delta_j \neq 0$ for $j = 1, \ldots, q$ if and only if the segments of both the Newton diagram $\mathcal{N}$ of $A(\epsilon) = J + \epsilon B + \mathrm{O}(\epsilon^2)$ and the envelope $\mathcal{E}(J)$ of J coincide. More precisely, Theorem 3.1 in [11, p.807] implies that $\Delta_k^j \neq 0$ if and only if the point $\mathcal{P}_j^k$ is a point of $\mathcal{N}$. (See Figure 3 below.)

Such a characterization allows for the following

Proposition 1. *If $A(\epsilon)$ is a Lidskii matrix then $\kappa(A)$ is minimum over $\mathcal{O}_A$.*

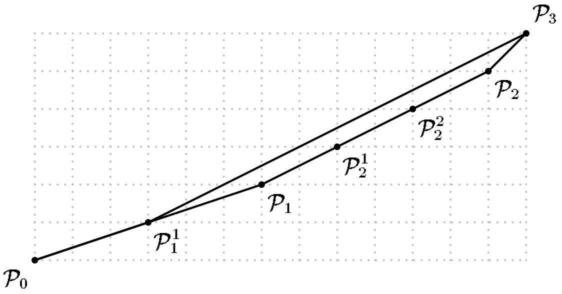

Figure 3: Newton diagram and Newton envelope for Example 2. Here $\Delta_1^1 \neq 0$ and $\Delta_3 \neq 0$, whereas $\Delta_1 = \Delta_2^1 = \Delta_2^2 = \Delta_2 = 0$.

PROOF. Recall that for a given J, the envelope is the lowest possible Newton diagram associated with a perturbation of J. Hence, conversely, for a prescribed Newton diagram, a J whose associated envelope coincides with that Newton diagram is necessarily the highest envelope. According to Definition 1, and by definition of the Newton envelope, this means that $\kappa(A)$ is minimum over $\mathcal{O}_A$. $\square$

Fact 1 and Proposition 1 suggest two natural necessary steps: minimize first $\operatorname{rank} J$ over $\mathcal{O}_A$, then minimize $\kappa(A)$ over the set of matrices similar to $A(\epsilon)$ having the same minimum leading rank. They will prove to be sufficient in the case when $A(\epsilon)$ is a κ-matrix. (See Section 5.)

3. MINIMIZATION OF $\operatorname{rank} J$

Let $r(A)$ be the rank of A_0 when $\operatorname{val} A(\epsilon) \geq 0$. If $\operatorname{val} A(\epsilon) < 0$ we set $r(A) = +\infty$. The proposition below provides a necessary condition for a matrix perturbation to be a Lidskii matrix in terms of the rank of the leading matrix. Then we explain how to reduce $A(\epsilon)$ for this condition to be satisfied. The reciprocal holds true in the case of κ-matrix.

PROPOSITION 2. *If $\Delta_q \neq 0$ (and a fortiori if $A(\epsilon)$ is a Lidskii matrix), then $r(A) = \operatorname{rank} J$ is minimum over $\mathcal{O}_A$.*

PROOF. By definition of the Newton envelope, the ordinate h of the highest point of $\mathcal{E}(J)$ satisfies $h = n - r(A)$. Hence, minimizing $r(A)$ is equivalent to maximizing h. But $\Delta_q \neq 0$ implies that h is also the ordinate of the highest point of $\mathcal{N}$. Hence h is maximum over $\mathcal{O}_A$. $\square$

Assume that $\operatorname{val} A(\epsilon) \geq 0$. A criteria for $r(A)$ to be minimum over $\mathcal{O}_A$ is (see [12])

$$\theta_A(\lambda) = \lim_{\epsilon \to 0} \epsilon^{\operatorname{rank} A_0} \det(\frac{A_0}{\epsilon} + A_1 - \lambda I) \not\equiv 0.$$

If $\theta_A(\lambda) \equiv 0$, we compute an invertible polynomial matrix $P(\epsilon)$ with the method of [7] to obtain a matrix $\tilde{A}(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$ so that $\theta_{\tilde{A}}(\lambda) \not\equiv 0$.

Reciprocally, we have the following

PROPOSITION 3. *If $A(\epsilon)$ is a κ-matrix with $r(A)$ minimum over $\mathcal{O}_A$, then $\Delta_q \neq 0$.*

PROOF. Because $A_0 = J$ is in JCF, it follows from the definition of the polynomial $\theta_A(\lambda)$ that $\theta_A(\lambda) = \pm\Delta_q$ if $n_q > 1$, and $\theta_A(\lambda) = \pm\det(L_q - \lambda E_q)$ if $n_q = 1$. Hence, if J has blocks with size > 1 only, $\Delta_q = \theta_A(\lambda) \neq 0$. On the other hand, if $n_q = 1$, then $\theta_A(\lambda)$ has no zero root; otherwise $A(\epsilon)$ has at least one eigenvalue with leading exponent > 1, which is impossible if $A(\epsilon)$ is a κ-matrix. Hence $\Delta_q = \theta_A(0) \neq 0$. $\square$

However, the minimality condition on $\operatorname{rank} J$ is clearly not sufficient for $A(\epsilon)$ to be a Lidskii matrix. Although $\theta_A(\lambda) \not\equiv 0$ for the matrix $A(\epsilon)$ in Example 2, this matrix is not a Lidskii matrix. This indicates that the "ones" lying on the first upper diagonal of J should be moved along this upper diagonal. This is the purpose of the next Section.

4. MINIMIZATION OF $\kappa(A)$

This section is devoted to the computation of a similarity $P(\epsilon) \in GL(n, \mathsf{F}[\epsilon^{-1}])$ that transforms a κ-matrix $A(\epsilon)$ which is not a Lidskii matrix, into $\tilde{A}(\epsilon) = \tilde{J} + \epsilon\tilde{B} + \mathrm{O}(\epsilon^2)$ such that $\kappa(\tilde{A}) = \kappa(\tilde{J}) < \kappa(J) = \kappa(A)$. Since it may be assumed from Section 3 that $r(A)$ is minimum over $\mathcal{O}_A$, we are actually looking for a matrix $\tilde{J}$ with the same (minimum) rank as J.

Denote by j the smallest integer of $\{2, \ldots, q\}$ such that $\Delta_{j-1} = 0$. (We know from Proposition 3 that $\Delta_q \neq 0$, hence $n_{j-1} > 1$.)

4.1 Technical lemmata

We give here four results that will be used throughout Sections 4.2 and 4.3.

LEMMA 1. *Let $(\alpha_1, \ldots, \alpha_n) \in \mathsf{K}^n$. There exists a similarity $P \in GL(n+1, \mathsf{K})$ that transforms*

$$M = \left[\begin{array}{cccc|c} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & 0 & 1 & \\ & & & 0 & 0 \\ \hline \alpha_1 & \alpha_2 & \cdots & \alpha_n & 0 \end{array}\right] \quad \text{into} \quad \tilde{M} = \left[\begin{array}{cccc|c} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & 0 & 1 & \\ & & & 0 & 0 \\ \hline \tilde{\alpha}_1 & 0 & \cdots & 0 & 0 \end{array}\right].$$

where $\tilde{\alpha}_1 \in \mathsf{K}$.

PROOF. Notice that C_{n+1}, the last column of M, is identically zero, and apply to M the elementary operations

$$R_{n+1} \leftarrow R_{n+1} - \alpha_{i+1}R_i\,, \quad C_i \leftarrow C_i + \alpha_{i+1}C_{n+1}$$

for $i = 1, \ldots, n-1$. $\square$

LEMMA 2. *Let $\alpha \in \mathsf{K}$, $\alpha \neq 0$, and let*

$$M_{ij}(\alpha) = \left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & \alpha & & \\ & & 0 & 1 & & & & \\ & & & 0 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{array}\right] \in \mathsf{K}^{(m+n)\times(m+n)}$$

where $i \in \{1, \ldots, m-1\}$, $j \in \{m+2, \ldots, m+n\}$, $m \geq n$, and α is the (i,j) entry of $M_{ij}(\alpha)$. Then there exists a similarity $P \in GL(m+n, \mathsf{K})$ that transforms $M_{ij}(\alpha)$ into $M_{ij}(0) = 0^m 0^n$.

PROOF. Apply to $M_{ij}(\alpha)$ the following elementary operation

$$C_j \leftarrow C_j - \alpha C_{i+1}, \quad R_{i+1} \leftarrow R_{i+1} + \alpha R_j$$

It transforms $M_{ij}(\alpha)$ into $M_{i+1,j+1}(\alpha)$. Repeat the process until $j = m+n$. Then $R_{m+n} = 0$. □

LEMMA 3. *Let*

$$M = \left[\begin{array}{c|c} 0 & u \\ \hline 0 & \\ \vdots & J \\ 0 & \end{array}\right] \in \mathsf{K}^{(n+1)\times(n+1)}$$

where $J = 0^{n_1} \ldots 0^{n_q} \in \mathsf{K}^{n\times n}$, $n_1 \geq \ldots \geq n_q$, *and* $u \in \mathsf{K}^n$. *Assume that* $u = [u_1, \ldots, u_q] \neq 0$ *where, for* $j = 1, \ldots, q$, $u_j = [*, 0, \ldots, 0] \in \mathsf{K}^{n_j}$. *If* j *corresponds to the first subvector* u_j *of* u *whose "$*$" entry is nonzero, then the JCF of* M *is equal to*

$$0^{n_1} \ldots 0^{n_{j-1}} 0^{n_j+1} 0^{n_{j+1}} \ldots 0^{n_q}$$

up to block permutations.

PROOF. All the remaining nonzero "$*$" entries can be eliminated by using the nonzero one of u_j, and by applying Lemma 2 at each intermediate step. Then, it is possible to swap the blocks of diagonal blocks of the resulting matrix to obtain the similar matrix $0^{n_1} \ldots 0^{n_{j-1}} \tilde{M}\, 0^{n_{j+1}} \ldots 0^{n_q}$, where

$$\tilde{M} = \left[\begin{array}{c|cccc} 0 & * & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & 0^{n_j} & & \\ 0 & & & & \end{array}\right] \in \mathsf{K}^{(n_j+1)\times(n_j+1)}.$$

Since $* \neq 0$, the JCF of $\tilde{M}$ is clearly 0^{n_j+1}. □

LEMMA 4. *One can compute* $P \in GL(n, \mathsf{K})$ *such that* $\tilde{J} = P^{-1}JP = J$ *and the* $(j-1)$*th Lidskii submatrix* $\tilde{L}_{j-1}$ *of* $\tilde{B} = P^{-1}BP$ *has its last row identically zero.*

PROOF. Since $\Delta_{j-2} \neq 0$ and $\Delta_{j-1} = 0$, there exists $t_{j-2} < k \leq t_{j-1}$ such that the kth row of L_{j-1} depends linearly on the other rows. Computing a vector in the (left) kernel of L_{j-1} provides the coefficients for such a linear dependency:

$$L_{j-1}[k, .] = \sum_{l=1, l\neq k}^{t_{j-1}} \alpha_l L_{j-1}[l, .].$$

Let $i_k = s_{j-2} + kn_{j-1}$. The kth row of L_{j-1} corresponds to entries of the i_kth row of B. Applying to $A(\epsilon)$ the following elementary row operations

$$R_{i_k} \leftarrow R_{i_k} - \alpha_l R_{i_l} \tag{1}$$

for $l = 1, \ldots, t_{j-1}$, $l \neq k$ clearly sets the kth row of L_{j-1} to zero. Furthermore, it does not modify J since the rows of J with index i_k or i_l are identically zero.

To obtain a similarity transformation, each row operation of type (1) must be followed by its associated elementary column operation

$$C_{i_l} \leftarrow C_{i_l} + \alpha_l C_{i_k} \tag{2}$$

Let us now carefully examine the effect of transformations (2) on both the $(j-1)$th Lidskii submatrix and J. First, (2) does not modify the coefficients α_l. Hence the zeroing process of the kth row of L_{j-1} is not perturbed by (2). However, each operation (2) for a given l modifies J by adding $-\alpha_l$ to its (i_k, i_l+1) entry (which is initially zero). Denote by $\tilde{J}_l$ this new JCF. Applying Lemma 2, using the "ones" of the block of $\tilde{J}_l$ that contains its (i_l+1, i_l+1) diagonal entry reduces $\tilde{J}_l$ back to J. Furthermore, the elementary operations involved here do not modify the $(j-1)$th Lidskii submatrix, for its entries stem from rows and columns which are not those used for such operations.

Hence, applying (1), (2) and Lemma 2 for $l = 1, \ldots, t_{j-1}$, $l \neq k$, zeroes the kth row of L_{j-1} without modifying J.

Finally, an appropriate swap of the r_{j-1} blocks of J of size n_{j-1} allows for the kth row of L_{j-1} we have just zeroed to become its last row. □

4.2 Case $n_j < n_{j-1} - 1$

In this situation, all we know is that $\Delta_{j-1} = 0$. However, the simplification of L_{j-1} provided by Lemma 4 suffices to decrease $\kappa(A)$ strictly.

PROPOSITION 4. *With* j *given as above, assume that the last row of* L_{j-1} *is zero. Let* $i = s_{j-1}$ *and let*

$$D(\epsilon) = diag(I_{i-1}, \epsilon, I_{n-i}). \tag{3}$$

Then $\hat{A} = D^{-1}AD$ *satisfies* $\kappa(\hat{A}) < \kappa(A)$.

PROOF. The valuation of $\hat{A}(\epsilon)$ is still nonnegative, and the new leading matrix $\hat{A}_0$ is

$$\hat{A}_0 = \left[\begin{array}{c|c|c} & 0 & \\ \hat{J} & \vdots & 0 \\ & 0 & \\ \hline \hat{u} & 0 & \hat{v} \\ \hline & 0 & \\ 0 & \vdots & \hat{K} \\ & 0 & \end{array}\right]$$

where $\hat{J} = (0^{n_1})^{r_1} \ldots (0^{n_{j-2}})^{r_{j-2}} (0^{n_{j-1}})^{r_{j-1}-1} 0^{n_{j-1}-1}$ has size $i-1$, and $\hat{K} = (0^{n_j})^{r_j} \ldots (0^{n_q})^{r_q}$ has size $n-i$. Furthermore, the ith row $[\hat{u}, 0, \hat{v}]$ of $\hat{A}_0$ is equal to the ith row of B except for its ith entry, which is zero. Hence,

$$\hat{u} = [\hat{u}_1^1, \ldots, \hat{u}_1^{r_1}, \ldots, \hat{u}_{j-2}^1, \ldots, \hat{u}_{j-2}^{r_{j-2}}, \hat{u}_{j-1}^1, \ldots, \hat{u}_{j-1}^{r_{j-1}-1}, \hat{u}']$$

where

$$\hat{u}_{j_1}^{j_2} = [0, \underbrace{*, \ldots, *}_{n_{j_1}-1}], \quad \text{and} \quad \hat{u}' = [0, \underbrace{*, \ldots, *}_{n_{j-1}-2}]$$

for $j_1 = 1, \ldots, j-1$ and for $j_2 = 1, \ldots, r_{j_1}$ if $j_1 < j-1$ and $j_2 = 1, \ldots, r_{j-1}-1$ if $j_1 = j-1$.

Here the "$*$" terms denote the corresponding entries in the ith row of B, and the "0" terms stem from the zero row of L_{j-1}.

Since the ith column of $\hat{A}_0$ is identically zero, apply Lemma 1 to eliminate the entries of $[\hat{u}, 0, \hat{v}]$ that lie on the same column as a "one" of the Jordan blocks of $\hat{J}$ and $\hat{K}$. This results in a matrix $\bar{A}_0$ similar to $\hat{A}_0$ and having the same structure. More precisely, $\bar{J} = \hat{J}$, $\bar{K} = \hat{K}$, and $\bar{u} = 0$. Furthermore, since $\Delta_q \neq 0$ (Proposition 3),

$$\bar{v} = [\bar{v}_j^1, \ldots, \bar{v}_j^{r_j}, \ldots, \bar{v}_q^1, \ldots, \bar{v}_q^{r_q}] \neq 0 \tag{4}$$

where, for $j_1 = j, \dots, q$ and $j_2 = 1, \dots, r_{j_1}$,

$$\bar{v}_{j_1}^{j_2} = [\,*, \underbrace{0, \dots, 0}_{n_{j_1}-1}].$$

According to Lemma 3, the submatrix

$$\left[\begin{array}{c|c} 0 & \bar{v} \\ \hline 0 & \\ \vdots & \hat{K} \\ 0 & \end{array}\right]$$

of $\bar{A}_0$ has a JCF with block sizes at most $n_j + 1$. Since it was assumed that $n_j + 1 < n_{j-1}$, the JCF $\hat{J}$ of $\hat{A}_0$ satisfies $\kappa(\hat{J}) < \kappa(J)$, and $\kappa(\hat{A}) < \kappa(A)$. □

Corollary 1. *If $n_j < n_{j-1} - 1$ one can compute $P(\epsilon) = P_1 D(\epsilon) P_2$, where $D(\epsilon)$ is given by (3) and P_1 and P_2 are in $GL(n, \mathsf{K})$, such that $\tilde{A} = P^{-1}AP$ is a matrix of the form $\tilde{J} + \epsilon\tilde{B} + \mathrm{O}(\epsilon^2)$ and $\kappa(\tilde{A}) = \kappa(\tilde{J}) < \kappa(J) = \kappa(A)$.*

The Ferrer diagram interpretation of this case is illustrated in Figure 4.

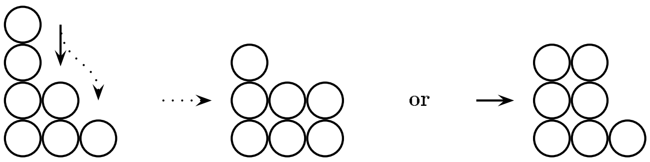

Figure 4: Coin moves from $0^4 0^2 0$ to $(0^3)^2 0$ and to $0^3 (0^2)^2$.

Notice that there no guarantee that the $P(\epsilon)$ of Corollary 1 defines the shortest coin move.

4.3 Case $n_j = n_{j-1} - 1$

In this case, one has more than the single condition "$\Delta_{j-1} = 0$", as shown below.

Lemma 5. *If $n_j = n_{j-1} - 1$ then $\Delta_j^k = 0$ for $k = 1, \dots, r_j$.*

Proof. Let $k \in \{1, \dots, r_j\}$ and assume that $\Delta_j^k \neq 0$. Recall that $\Delta_{j-2} \neq 0$ and that $\Delta_{j-1} = \Delta_{j-1}^{r_{j-1}} = 0$, and let k' be the smallest index of $\{1, \dots, r_{j-1}\}$ such that $\Delta_{j-1}^{r_{j-1}-k'} \neq 0$. (We note $\Delta_{j-1}^0 = \Delta_{j-2}$ and $\mathcal{P}_{j-1}^0 = \mathcal{P}_{j-2}$ for simplicity.) Then both points $\mathcal{P}_{j-1}^{r_{j-1}-k'}$ and $\mathcal{P}_j^k$ lie on $\mathcal{N}$, and the segment of $\mathcal{N}$ they define has length $k'n_{j-1} + kn_j$. Furthermore there exists $p \in \mathbb{N}^*$ such that $1/p$ is the slope of this segment, for $A(\epsilon)$ is a κ-matrix. Hence, $1/p = (k'+k)/(k'n_{j-1} + kn_j)$. If $n_j = n_{j-1} - 1$, $p = k'/(k'+k) + n_j \notin \mathbb{N}^*$. Contradiction. □

In particular, Lemma 5 implies that $\Delta_j = \Delta_j^{r_j} = 0$. However, using the intermediate conditions on the Δ_j^k's allows for a further simplification of the jth Lidskii submatrix.

Lemma 6. *One can compute $P \in GL(n, \mathsf{K})$ such that $\tilde{J} = P^{-1}JP = J$ and the jth Lidskii submatrix $\tilde{L}_j$ of $\tilde{B} = P^{-1}BP$ has the block structure*

$$\tilde{L}_j = \begin{bmatrix} \tilde{L}_{j-1} & * \\ 0 & \tilde{X} \end{bmatrix} \tag{5}$$

where $\tilde{X}$ is strictly upper triangular and the last row of $\tilde{L}_{j-1}$ is zero.

Proof. Since $\Delta_j = 0$, zero the last row of L_j as in Lemma 4, and denote by $L_j^{(1)}$ the resulting matrix and by $M_j^{(1)}$ the square submatrix of $L_j^{(1)}$ with order $t_j - 1$ that contains $L_{j-1}^{(1)}$. Therefore, $\Delta_j^{r_j-1}$ now reduces to a single matrix determinant, namely $\det M_j^{(1)}$. From Lemma 5, $\det M_j^{(1)} = 0$ and we may zero its last row, thus also reducing $\Delta_j^{r_j-2}$ to a single matrix determinant. Repeating this argument for $M_j^{(2)}, \dots, M_j^{(r_j-1)}$ yields the strictly upper triangular matrix $\tilde{X}$. Finally, zero the last row of the current L_{j-1} using Lemma 4. □

Proposition 5. *With j given as above, assume that L_j has the form (5). Define*

$$D(\epsilon) = diag(I_{s_{j-1}-1}, \epsilon, \underbrace{I_{n_j-1}, \epsilon, \dots, I_{n_j-1}, \epsilon}_{r_j n_j}, I_{n-s_j}). \tag{6}$$

Then $\hat{A} = D^{-1}AD$ satisfies $\kappa(\hat{A}) < \kappa(A)$.

Proof. Applying Lemma 1 and Lemma 3 as in the proof of Proposition 4 yields a simplified leading matrix

$$\bar{A}_0 = \begin{bmatrix} \bar{A}_0^{11} & \bar{A}_0^{12} & \bar{A}_0^{13} \\ 0 & \bar{A}_0^{22} & \bar{A}_0^{23} \\ 0 & 0 & \bar{A}_0^{33} \end{bmatrix}.$$

Here,

$$\bar{A}_0^{11} = \left[\begin{array}{ccc|c} & & & 0 \\ & \hat{J} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right]$$

is the same as before. Additionally, all the entries of $[\,\bar{A}_0^{12}\bar{A}_0^{13}\,]$ are zero, except for the last row, which has the form $\bar{v}$ given in (4). The matrix $[\,\bar{A}_0^{22}\bar{A}_0^{23}\,]$ is defined in the same way but $\bar{A}_0^{22}$ is strictly upper triangular. Finally,

$$\bar{A}_0^{33} = (0^{n_{j+1}})^{r_{j+1}} \cdots (0^{n_q})^{r_q}.$$

From this form, it follows that the rank of $\bar{A}_0$ is still minimum, and that the first Jordan block to be modified has size $n_{j-1} - 1$. Furthermore, the other blocks that may arise have size at most n_j. □

Corollary 2. *If $n_j = n_{j-1} - 1$, one can compute $P(\epsilon) = P_1 D(\epsilon) P_2$ where $D(\epsilon)$ is given by (3) and P_1 and P_2 are in $GL(n, \mathsf{K})$, such that $\tilde{A} = P^{-1}AP$ is a matrix of the form $\tilde{J} = \epsilon\tilde{B} + \mathrm{O}(\epsilon^2)$ and $\kappa(\tilde{A}) = \kappa(\tilde{J}) < \kappa(J) = \kappa(A)$.*

The Ferrer diagram illustration of this case is Figure 1. The coin has first to roll before falling.

4.4 Characterization of Lidskii matrices

From Proposition 1 and Corollaries 1 and 2, we obtain the following criteria for a κ-matrix to be a Lidskii matrix.

THEOREM 2. *A κ-matrix $A(\epsilon)$ is a Lidskii matrix if and only if $\kappa(A)$ is minimum over $\mathcal{O}_A$.*

Since when $A(\epsilon)$ is not a Lidskii matrix, it is possible from the two previous subsections to minimize its leading Jordan structure by similarity, we also obtain the following

COROLLARY 3. *Any κ-matrix is similar to a Lidskii matrix.*

5. THE κL ALGORITHM

The results of Sections 3 and 4 lead immediately to an algorithm that computes for an arbitrary κ-matrix $A(\epsilon)$ a similarity $P(\epsilon)$ such that $P^{-1}AP$ is a Lidskii matrix.

It consists of two steps: 1) minimize $r(A)$, 2) minimize $\kappa(A)$ by a succession of strict decreases.

Notice that the entries of $P(\epsilon)$ are actually polynomials w.r.t. the variable ϵ, and not series.

Algorithm κL

Input: A, an $n \times n$ κ-matrix of the form $J + \epsilon B + \mathrm{O}(\epsilon^2)$.

$\tilde{A} \leftarrow$ a matrix similar to A with minimum $r(\tilde{A})$.
Reduce $\tilde{A}$ to the form $\tilde{J} + \epsilon\tilde{B} + \mathrm{O}(\epsilon^2)$.
while $\tilde{A}$ is not a Lidskii matrix
compute $\hat{A}$ a matrix similar to $\tilde{A}$ such that $\kappa(\hat{A}) < \kappa(\tilde{A})$.
$\hat{A}$ has the form $\hat{J} + \epsilon\hat{B} + \mathrm{O}(\epsilon^2)$.
$\tilde{A} \leftarrow \hat{A}$.
if no such matrix is found
print "The input matrix is not a κ-matrix".
return $\tilde{A}$.
endif.
endwhile.

Output: $\tilde{A}$.

The algorithm stops when either all Lidskii's conditions "$\Delta_j \neq 0$" are satisfied, or $\kappa(A)$ cannot be decreased strictly. In the first case, $\tilde{A}$ is a Lidskii matrix.

The second situation means that the input matrix was not a κ-matrix. The returned $\tilde{A}$ has a leading Jordan form whose block sizes do not differ by more than one. (Otherwise, one more coin could be moved rightward, which means that $\kappa(\tilde{A})$ could be decreased strictly.)

6. NORMALIZATION WITH THE ARNOLD NORMAL FORM

The leading JCF J of the Lidskii matrix $A(\epsilon)$ we have computed is unique. Hence, $A(\epsilon)$ may be normalized by being transformed into its Arnold normal form [1] by means of a holomorphic similarity $P(\epsilon)$. We may use the algorithm described [3] to compute such a $P(\epsilon)$. Here $P(0) = I_n$ and the leading JCF of $P^{-1}AP$ is still J. Higher order terms are set to zero, except for some particular entries, as shown on the example below: let $A(\epsilon) = 0^3 (0^2)^2 + \epsilon B + \mathrm{O}(\epsilon^2)$. Then, the Arnold normal form of $A(\epsilon)$ has the form

$$\tilde{A}(\epsilon) = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ * & * & * & * & * & * & * \\ * & & & 0 & 1 & & \\ * & & & * & * & * & * \\ * & & & * & & 0 & 1 \\ * & & & * & & * & * \end{bmatrix}$$

where each blank entry is equal to zero, and each "$*$" entry is a function of ϵ, holomorphic at $\epsilon = 0$, and vanishing at $\epsilon = 0$.

Furthermore, if $A(\epsilon)$ is a Lidskii matrix, its Arnold normal form is also a Lidskii matrix for such a normalization does not modify the leading JCF.

THEOREM 3. *Any κ-matrix $A(\epsilon)$ is similar to a Lidskii matrix in Arnold normal form. Algorithm κL followed by an Arnold normal form computation provides a similarity $P(\epsilon)$ that reduces $A(\epsilon)$ to this form.*

7. FUTURE WORK

Assume that the κL algorithm is given as input the first of the two matrices

$$A_1(\epsilon) = \left[\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \epsilon & \\ \hline & & & 0 & 1 \\ \epsilon & & & & 0 \end{array}\right] \qquad A_2(\epsilon) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ \epsilon^2 & & & & 0 \end{bmatrix}$$

which are similar for all $\epsilon \neq 0$, and whose eigenvalues are $1^{1/5}\epsilon^{2/5}$. Clearly, $A_1(\epsilon)$ is neither a κ-matrix, nor a Lidskii matrix. But $\kappa(A_1)$ is minimum over $\mathcal{O}_{A_1}$, and the algorithm detects that $A_1(\epsilon)$ is *not* a κ-matrix and stops.

Now looking at $A_2(\epsilon)$, we may observe that, although

$$\operatorname{rank} A_2(0) > \operatorname{rank} A_1(0)\,,$$

$A_2(z^{1/2})$ is actually a Lidskii matrix w.r.t. variable z. In other words, the leading exponent $2/5$ can be viewed on $A_2(\epsilon)$ as

$$2/5 = (\text{valuation of the perturbation})/(\text{size of the block}).$$

This strongly suggests the extension of both Lidskii's criteria (Theorem 1) and algorithm κL to matrix perturbations with rows or columns of valuation greater than one. This is left for further research.

8. REFERENCES

[1] V. Arnold. On matrices depending on parameters. *Russian Math. Surveys*, 26:29–43, 1971.

[2] H. Baumgärtel. *Analytic perturbation theory for matrices and operators.* Birkhäuser Verlag, Basel, 1985.

[3] G. Chen. Solutions formelles de systèmes d'équations différentielles linéaires ordinaires homogènes. Ph.D. Thesis, Université Joseph Fourier, Grenoble, France, 1990.

[4] A. Edelman, E. Elmroth, and B. Kagstrom. A geometric approach to perturbation theory of matrices and matrix pencils: Part 2: stratification-enhanced staircase algorithm. *SIAM J. Mat. Anal. Appl.*, 20(3):667–699, 1999.

[5] E. Gansner. Acyclic digraphs, Young tableaux, and nilpotent matrices. *SIAM J. Alg. Disc. Meth.*, 2:429–440, 1981.

[6] C. P. Jeannerod, N. Maillard, and E. Pflügel. An algorithmic approach for the symmetric perturbed eigenvalue problem. Application to the solution of a Schrödinger equation by a kp-perturbation method. *Proceedings IMACS ACA'98*, 1998.

[7] C.-P. Jeannerod and E. Pflügel. A reduction algorithm for matrices depending on a parameter. *Proceedings of ISSAC 99*, pages 121–128, Vancouver, Canada, July 29-31, 1999.

[8] T. Kato. *Perturbation theory for linear operators.* Springer, Berlin, 1980.

[9] V. Lidskii. Perturbation theory of non-conjugate operators. *U.S.S.R. Comput. Math. and Math. Phys.*, 1:73–85, 1965.

[10] Y. Ma and A. Edelman. Nongeneric perturbations of Jordan blocks. *Linear Algebra Appl.*, 273:45–63, 1998.

[11] J. Moro, J. Burke, and M. Overton. On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure. *SIAM J. Matrix Anal. Appl.*, 18:793–817, 1997.

[12] J. Moser. The order of a singularity in Fuchs' theory. *Math. Z.*, pages 379–398, 1960.

[13] E. Pflügel. Résolution symbolique des systèmes différentiels linéaires. Ph.D. Thesis, Université Joseph Fourier, oct. 1998.