Eigenvalues of matrices depending on a parameter
and the **Linset** Maple library.

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The goal of this talk is twofold. First, explain how the matrix perturbation theory for eigenvalues leads to the definition and the computation of a **valuation reduced form** of a matrix depending on a parameter. Second, present a Maple V.5 package called **Linset**. This library aims to provide tools and algorithms to handle polynomial matrix fractions and perturbed matrices. In particular, it implements some valuation reduced form-based methods in order to solve the eigenvalue perturbation problem.

Valuation reduced matrices and the eigenvalue perturbation problem. Let $A(\epsilon) = A_0 + A_1\epsilon + A_2\epsilon^2 + \cdots$ be an n by n matrix depending on a small parameter ϵ. Each matrix coefficient A_i has its entries in a field K. Typically, the field is $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. An algebraic closure of K is denoted by $\bar{K}$. Clearly, $A(\epsilon)$ may be seen as a perturbation of the constant matrix A_0 by order $O(\epsilon)$. What is considered here is the so-called **eigenvalue perturbation problem** (see [6]). It is well known [1, 4] that the solutions $\lambda(\epsilon)$ of $\det(A(\epsilon) - \lambda I) = 0$ can be expanded into Puiseux series $\lambda(\epsilon) = \mu\epsilon^{\beta} + \cdots$ ($\mu \in \bar{K}\backslash\{0\}$, $\beta \in \mathbb{Q}$) for ϵ in some neighborhood of zero. Such eigenvalues are called $O(\epsilon^{\beta})$ eigenvalues. For us, solving the eigenvalue perturbation problem will mean determining the first-order terms $\mu\epsilon^{\beta}$ of the perturbed eigenvalues $\lambda(\epsilon)$ from some particular entries of either $A(\epsilon)$, or a matrix similar to $A(\epsilon)$.

When A_0 is a single Jordan block, then typically the zero eigenvalue splits up into n $\mathrm{O}(\epsilon^{1/\mathrm{n}})$ eigenvalues. For an arbitrary (nilpotent) Jordan structure A_0, perturbed eigenvalues generically split according to the sizes and the numbers of the Jordan blocks of A_0. Assuming that the Jordan structure of A_0 is known, Lidskii (see [7]) has shown that A_1 generically provides the first-order terms of the eigenvalues of $A(\epsilon)$. Here, "generically" means that A_1 is dense, chosen at random. Using Arnold's compact notation to denote by 0^k a nilpotent Jordan block of size k and by $(0^k)^r$ the block-diagonal matrix that consists of r such blocks, we recall Lidskii's perturbation result for eigenvalues. It says that, if $A_0 = (0^{n_1})^{r_1} \ldots (0^{n_q})^{r_q}$ then $A(\epsilon) = A_0 + A_1\epsilon + A_2\epsilon^2 + \cdots$ generically admits $r_j n_j$ $O(\epsilon^{1/n_j})$ eigenvalues. Furthermore, the first-order coefficients "μ" depend on some particular entries of A_1 with position given by the Jordan structure of A_0.

Hence, the question is [5]: how to deal with *nongeneric* situations ? We provide answers in two directions, motivated by the two matrices

$$A(\epsilon) = A_0 + A_1\epsilon = \left[\begin{array}{cc|c} 0 & 1 & \epsilon \\ \epsilon & 0 & \\ \hline & \epsilon & 0 \end{array}\right], \quad \text{and} \quad B(\epsilon) = B_0 + B_1\epsilon = \left[\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \epsilon & \\ \hline \epsilon & & & 0 & 1 \\ & & & & \epsilon \end{array}\right].$$

Both matrices $A(\epsilon)$ and $B(\epsilon)$ are nongeneric perturbations of order $O(\epsilon)$ of, respectively, $0^2 0$ and $0^3 0^2$. Indeed, when applying Lidskii's theory, all we can say is the following. $A(\epsilon)$ admits two eigenvalues of order $O(\epsilon^{1/2})$ and one of order $O(\epsilon)$. The actual eigenvalues of $A(\epsilon)$ have the form $\pm\epsilon^{1/2} + \cdots$ and $-\epsilon^2 + \cdots$. The eigenvalue $-\epsilon^2 + \cdots$ is seen as an $\mathrm{O}(\epsilon)$ eigenvalue. The matrix $B(\epsilon)$ admits three eigenvalues of order $O(\epsilon^{1/3})$ and two eigenvalues of order $O(\epsilon^{1/2})$. However the actual eigenvalues are ϵ and $1^{1/4}\epsilon^{1/2}$.

To explain such failures, we propose to focus on the **valuation** and the **column valuations** of the perturbed matrix $A(\epsilon)$ and of its perturbative part $A(\epsilon) - A_0$. (As a square matrix is similar to its transpose, we could also consider row valuations.) Define the valuation ν_{ij} of the (i,j) entry of $A(\epsilon)$ as the smallest power of ϵ that appears in this entry. (An entry identically zero is assumed to have valuation $+\infty$.) Hence the two definitions: the valuation ν of $A(\epsilon)$ is $\nu = \min_{1\le i,j\le n} \nu_{ij}$ and the valuation ν_j of the jth column of $A(\epsilon)$ is $\nu_j = \min_{1\le i\le n} \nu_{ij}$.

Additionally, let the **valuation profile** of $A(\epsilon) - A_0$ be the n-tuple $(\nu_1, \ldots, \nu_n)$ of the column valuations of $A(\epsilon) - A_0$. It is said **conforming to the Jordan structure** $A_0 = (0^{n_1})^{r_1} \ldots (0^{n_q})^{r_q}$ if it can be partitioned into n_j-tuples ordered as the blocks of A_0, and such that for each $j = 1, \ldots, q$, the n_j components are equal to either the same finite value or $+\infty$. Hence a conforming valuation profile may be identified with the q-tuple $(v_1, \ldots, v_q)$. This means that all the nonzero columns of $A(\epsilon) - A_0$ corresponding to the same Jordan block 0^{n_j} of A_0 have the same valuation v_j.

This allows for a presentation of the following independent points.

1. **The integer leading exponent case.** One can compute a polynomial matrix invertible for all $\epsilon \neq 0$ that transforms $A(\epsilon)$ with similarity into a so-called **valuation reduced form**. By definition, this form has a maximal valuation, and minimal numbers of columns of each valuation, among all similar matrices. It is a generalization of the CRV form used in [2]. The main property of such a form is to allow to recover all the eigenvalue first-order terms whose leading exponent β is an integer. Consequently, to handle the $\mathrm{O}(\epsilon^2)$ eigenvalue of $A(\epsilon)$ in the example above, it suffices to transform it into valuation reduced form. Furthermore, it appears that the invariants displayed by valuation reduced matrices, which are the Newton polynomials associated with integer leading exponents β, can be computed using a minimal number of terms of the power series $A(\epsilon)$. This allows for the computation of eigenvalues of polynomial matrix fractions or analytic matrices. Both standard and lazy algorithms for the computation of a valuation reduced form will be presented.

2. **Valuation-based extension of Lidskii's eigenvalue perturbation theorem.** The result due to Lidskii we mentioned above can be extended to a class of nongeneric perturbations that clearly explain why the Jordan structure of A_0 can be misleading and why it "creates" nongenericity [3]. Consider a perturbation $A(\epsilon)$ of $A_0 = 0^{n_{11}} \ldots 0^{n_{1r_1}} \ldots 0^{n_{q1}} \ldots 0^{n_{qr_q}}$ such that the valuation profile of $A(\epsilon) - A_0$ is conforming to A_0. Denote it by $(v_{11}, \ldots, v_{1r_1}, \ldots, v_{q1}, \ldots, v_{qr_q})$. Assuming with no

loss of generality that the blocks of A_0 are ordered so that

$$\beta_1 = v_{11}/n_{11} = \ldots = v_{1r_1}/n_{1r_1} < \ldots < \beta_q = v_{q1}/n_{q1} = \ldots = v_{qr_q}/n_{qr_q},$$

our claim is the following: $A(\epsilon)$ generically admits $\underline{v}_j/\beta_j$ $O(\epsilon^{\beta_j})$ eigenvalues where $\underline{v}_j = v_{j1} + \ldots + v_{jr_j}$. Furthermore, the first-order coefficients "μ" are the roots of polynomials defined by some particular entries of $A(\epsilon) - A_0$ whose position depend both on the Jordan structure of A_0 and on the valuation profile of $A(\epsilon) - A_0$.

• The first consequence of this result is that Lidskii's theorem follows as a particular case with all column valuations v_{jk} $(j = 1, \ldots, q,\ k = 1, \ldots, r_j)$ equal to 1.

• Second, it leads to a very natural definition of the "best Jordan structure" of the leading matrix A_0 of a matrix depending on a parameter $A(\epsilon)$. Indeed, consider the matrix $B(\epsilon)$ of the example above. Forall $\epsilon \neq 0$, it is similar to

$$C(\epsilon) = \left[\begin{array}{cc|cc|c} 0 & 1 & & & \\ & 0 & \epsilon & & \\ \hline & & 0 & 1 & \\ \epsilon & & & 0 & \\ \hline & & & & \epsilon \end{array}\right].$$

whose leading matrix is $C_0 = (0^2)^2 0$. Furthermore, one could verify that $C(\epsilon)$ is generic in Lidskii's sense [7]. Hence the leading Jordan structure C_0 is the best we can find for the eigenvalue perturbation problem. On the contrary, the leading Jordan structure B_0 of $B(\epsilon)$ is somehow misleading since we have seen that $B(\epsilon)$ is nongeneric. Notice also that rank $C_0 = 2 <$ rank $B_0 = 3$, which indicates that $B(\epsilon)$ is *not* in valuation reduced form. (We will see why $C(\epsilon)$ is.)

The fact that in the extension of Lidskii's result valuations greater than 1 are considered (*i.e.* the v_{jk} ar not necessarily equal to 1) is fundamental, for it allows for the study of eigenvalue first-order terms with an *arbitrary* exponent β. (In particular, β need not be of the form 1/(size of a Jordan block) anymore, as required by Lidskii.) Consider for example the matrix $D(\epsilon)$ below [8], and the matrix $E(\epsilon)$ which is similar to $D(\epsilon)$ for all $\epsilon \neq 0$.

$$D(\epsilon) = \left[\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \epsilon & \\ \hline & & & 0 & 1 \\ \epsilon & & & & 0 \end{array}\right], \quad E(\epsilon) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ \epsilon^2 & & & & 0 \end{bmatrix}.$$

The matrix $D(\epsilon)$ is a well-known example of nongeneric perturbation of $0^3 0^2$ which has 5 $O(\epsilon^{2/5})$ eigenvalues. However, we will see that $E(\epsilon)$ is generic in the sense of the extension of Lidskii's result and it involves a perturbation of valuation 2. Consequently, the leading Jordan structure of $E(\epsilon)$ is the best we can find and exponent 2/5 is displayed on $E(\epsilon)$ through the simple formula

$$2/5 = \text{(valuation of the perturbation)/(size of the Jordan block)}.$$

Presentation of Linset: a package for matrices depending on a parameter. A Maple V.5 implementation of the approach described above is available in the **Linset** package. This library implements both basic and sophisticated tools for handling matrices depending on a parameter, that is polynomial matrices, polynomial matrix fractions, matrix pencils and analytic matrices. The main topics are

- canonical and reduced forms (Popov, KCF, Arnold, valuation reduced,...),
- eigenvalues and eigenvectors of perturbed matrices,
- generalized eigenvalues of perturbed polynomial matrices,
- random matrix perturbations,
- matrix visualization.

Various equivalence relations are used, depending on the problem. In particular, valuation reduced form-based algorithms rely on polynomial matrix similarity. Note also that the algorithms combining symbolic and numerical computations call some Matlab routines such as `eig` (numerical eigenvalues of a constant matrix). Finally, some graphical tools (Newton diagram, bipartite graph, random eigenvalue and matrix plots) are also available.

What we propose is an illustration of the main functionalities of this library with the algorithms described in the first part of the talk.

References

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