A Dialectica-Like Approach to Tree Automata

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Abstract

We propose a fibred monoidal closed category of alternating tree automata. Our notion is based on Dialectica-like categories, suggested by the specific logical form of the transitions of alternating automata. The basic monoidal closed structure gives a realizability interpretation of proofs of a first-order multiplicative linear logic as winning strategies in corresponding acceptance games.

Moreover, we show that the usual powerset operation translating an alternating automaton to an equivalent non-deterministic one satisfies the deduction rules of the '!' modality of linear logic. We thus get a deduction system for intuitionistic linear logic, which in particular gives deduction for minimal intuitionistic predicate logic via the Girard translation. We also get a weak form of completeness of our realizers wrt language inclusion, based on the '?' modality.

1. Introduction

We propose fibred monoidal closed categories for tree automata. We extend the approach of [35], based on the slogan: *Automata as objects, Executions as morphisms*.

We consider a variation of alternating automata on infinite trees. Alternating tree automata (see e.g. [10, 29, 30, 42]) are equivalent in expressive power to the Monadic Second-Order Logic (MSO), which subsumes most of the logics used in verification.

The models presented here provide a computational interpretation of a deduction system for tree automata (see §6). It encompass the constructions on automata reflecting the connectives of MSO, which are used in the translation of MSO-formulas to tree automata (see e.g. [10, 40, 42]) and give the decidability of MSO [34] (*via* decidability of emptiness checking).

Tree automata and MSO are traditionally viewed as positive objects: one is primarily interested in satisfaction or satisfiability, the Boolean connectives are disjunction and negation and the primitive notion of quantification is existential. In contrast, Curry-Howard approaches tend to favor proof-theoretic oriented and negative approaches, *i.e.* approaches in which the predominant logical connective is the implication, and where the predominant form of quantification is universal. We build on [35] which proposed fibred categories of tree automata with a monoidal product and existential quantifications.

Wrt the decomposition of MSO formulas in tree automata, the switch from a disjunction to a conjunction implemented by a monoidal product is unproblematic. The main problems concern the interplay between negation and existential quantification. While alternating automata are linearly closed under complement, they have no correct primitive notion of existential quantification. On the other hand, non-deterministic automata have existential quantification but no linear notion of complement. Alternating automata can be translated to non-deterministic one (the *Simulation Theorem* [30]) at an exponential cost. It follows that quantifier alternations in MSO formulas reflect the non-elementary complexity of the translation to tree automata.

This paper shows that this decomposition corresponds to some extent to the decomposition of intuitionistic logic in linear logic [9]. We propose a monoidal closed structure for tree automata, which encompass operations on alternating tree automata (including complement). Moreover, we show that the usual powerset operation translating an alternating automaton to an equivalent non-deterministic one satisfies the deduction rules of the '!' modality of linear logic. We thus get a deduction system for intuitionistic linear logic, which in particular gives deduction for minimal intuitionistic predicate logic via the Girard translation.

Most modern approaches to MSO and tree automata use games (see e.g. [10, 32, 40]), because game determinacy provides a convenient approach to the complementation of alternating tree automata. Following [35], our models are based on game semantics. The notion of morphisms is based on a synchronous restriction of the linear arrow of *simple games* (see e.g. [1, 14]). This restriction allows to internalize homsets in tree automata (as required by the *closed* structure), so that a realizer in our computational interpretation can always be described as an accepting run of some tree automata (with decidable emptiness checking in the case of *regular* automata, equivalent to MSO).

Our main contributions (wrt [35]) are the *closed* structure on automata and a primitive notion of universal quantification (see §3). We also explicit a deduction system (§6), as well as the fact that the simulation of alternating automata by non-deterministic ones satisfies the deduction rules (but unfortunately not cut-elimination) of the ! modality of intuitionistic linear logic (§7).

We use Gödel's *Dialectica* interpretation (see e.g. [4, 23]) in two related ways. First, Dialectica can be seen as a constructive notion of prenex $\exists\forall$-formulas, on which we base the transitions of the internal implication of tree automata. This leads to our notion of automata presented in §2. Second, our notion of morphism (issued from [35]) is based on *zig-zag* strategies, which can be represented using Dialectica-like categories (see e.g. [8, 13, 15]). As a by-product, the fibred structure of [35], based on codomain fibrations, is simplified to variants of simple fibrations (see e.g. [18]).

The paper is organized as follows. We begin in §2 by an overview of the Curry-Howard-like approach to tree automata

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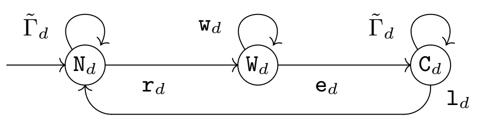
of [35], and §3 presents the main aspects of our Dialectica-like approach. We then give in §4 a presentation of a fibred monoidal closed structure on zig-zag games, on top of which we present our fibrations of automata in §5. Finally, §6 presents our deduction system and §7 the interpretation of exponential rules. Appendix A gives some definitions and basic facts on simple and zig-zag games.

2. A Curry-Howard Approach to Tree Automata

Fix a non-empty finite set D of *tree directions*. Alphabets (denoted Σ, Γ, etc) are finite non-empty sets. Fix also a singleton set $\mathbf{1} := \{\bullet\}$, and two-elements sets $\mathbf{2} := \{0, 1\}$ and $\mathbb{B} := \{\mathbb{t}, \mathbb{f}\}$.

Definition 2.1. *A Σ-labeled D-ary tree is a map $T : D^* \to \Sigma$.*

Example 2.2.
(i) A $\mathbf{2}$-labelled tree $T : D^* \to \mathbf{2}$ is the characteristic function of the set $S \subseteq D^*$ such that $p \in S$ iff $T(p) = 1$.
(ii) Given $T : D^* \to \Sigma_1 \times \cdots \times \Sigma_n$ and a projection $\mathsf{p}_{\Sigma_i} : \Sigma_1 \times \cdots \times \Sigma_n \to \Sigma_i$, we have $\mathsf{p}_{\Sigma_i} T = \mathsf{p}_{\Sigma_i} \circ T : D^* \to \Sigma_i$.
(iii) Take $D := \mathbf{2}$. A finite execution of the interleaving $(P_0 \,|||\, P_1)$ of two processes P_0 and P_1 can be described as a finite word $p \in D^*$, with $p_n = d$ iff an atomic instruction of P_d is executed at step n. An infinite execution of $(P_0 \,|||\, P_1)$ can then be seen as an infinite path χ in the full binary tree D^*, that is an ω-word $\chi \in D^\omega$.
(iv) Continuing (iii), consider, for each $d \in D$, the following transition system $\mathcal{P}_d$, where $\tilde{\Gamma}_d$ is a finite non-empty set:



$(\mathcal{P}_d)$

We see $\mathcal{P}_d$ as representing the possible executions of a process P_d with states $\Sigma_d = \{\mathtt{N}_d, \mathtt{W}_d, \mathtt{C}_d\}$ and actions $\Gamma_d = \tilde{\Gamma}_d + \{\mathtt{r}_d, \mathtt{w}_d, \mathtt{e}_d, \mathtt{l}_d\}$. Process P_d can perform actions in its non-critical section $\mathtt{N}_d$, but also by action $\mathtt{r}_d$ request access to its critical section $\mathtt{C}_d$. Accesses to $\mathtt{C}_d$ are guarded by the waiting state $\mathtt{W}_d$.
A binary tree $I : D^* \to \Delta_0 \times \Delta_1$ where $\Delta_d := \mathbf{1} + \Sigma_d \times \Gamma_d$ can be seen as a particular implementation of $(P_0 \,|||\, P_1)$ if it is *correct* in the sense that $I(\varepsilon) = (\bullet, \bullet)$ (standing for $(\mathtt{N}_0, \mathtt{N}_1)$), $\mathsf{p}_{\Delta_d} I(p.(1-d)) = \mathsf{p}_{\Delta_d} I(p)$, and moreover the transitions of $\mathcal{P}_d$ are respected from $\mathtt{N}_d$ by each sequence $(\mathsf{p}_{\Delta_d} I(p_k))_k$ s.t. $p_{k+1} = p_k.(1-d)^{n_k}.d$ for some $(n_k)_k$.

2.1 Tree Automata and Games

Our notion of tree automaton is a variation of the usual notion of alternating automaton. It allows to naturally see tree automata as Dialectica-like $\exists\forall$-forms and to build a linear implication automaton $\mathcal{A} \multimap \mathcal{B}$ from automata $\mathcal{A}$ and $\mathcal{B}$ (see §3 below).

Definition 2.3 (Tree Automata)**.** *A* tree automaton $\mathcal{A}$ *over alphabet Σ (notation $\mathcal{A} : \Sigma$) has the form*

$$\mathcal{A} \quad = \quad (Q_\mathcal{A}\,,\, q^\imath_\mathcal{A}\,,\, U\,,\, X\,,\, \delta_\mathcal{A}\,,\, \Omega_\mathcal{A}) \tag{1}$$

where $Q_\mathcal{A}$ is the finite set of states, *$q^\imath_\mathcal{A} \in Q_\mathcal{A}$ is the* initial state, *U and X are finite sets of resp.* P *and* O*-moves, $\Omega_\mathcal{A} \subseteq Q^\omega$ is the* acceptance condition, *and the* transition function *$\delta_\mathcal{A}$ has the form*

$$\delta_\mathcal{A} \quad : \quad Q_\mathcal{A} \times \Sigma \quad\longrightarrow\quad U \times X \quad\longrightarrow\quad (D \longrightarrow Q_\mathcal{A})$$

We suppose for simplicity that automata are *complete*, in the sense that U and X are always non-empty. An automaton $\mathcal{A}$ as in (1) is *non-deterministic* if $X = \mathbf{1}$ and *deterministic* if moreover $U = \mathbf{1}$, in which case we see its transition function as being of the form

$$\delta_\mathcal{A} \quad : \quad Q_\mathcal{A} \times \Sigma \quad\longrightarrow\quad D \quad\longrightarrow\quad Q_\mathcal{A}$$

$\mathbf{1}$	$\mathcal{A}(T)$
	$(p, q_\mathcal{A})$
P	u
O	(x, d)
	$(p.d, q'_\mathcal{A})$
P	u'
O	(x', d')
	$(p.d.d', q''_\mathcal{A})$

$\mathbf{1}$	$\mathcal{A}(M)$	$\multimap$	$\mathcal{B}(N)$	
	$(p, q_\mathcal{A})$		$(p, q_\mathcal{B})$	
O	u			
			v	P
			(y, d)	O
P	(x, d)			
	$(p.d, q'_\mathcal{A})$		$(p.d, q'_\mathcal{B})$	

Figure 1. Plays in Games over $\mathbf{1}$

Example 2.4.
(i) The *unit automaton* $\mathbf{I}_\Sigma : \Sigma$ is the unique deterministic automaton over Σ with state set $\mathbf{1}$ (so that $\bullet$ is initial) and acceptance given by $\Omega_\mathbf{I} := \mathbf{1}^\omega$.
(ii) Usually (see e.g. [29, 30], and also [42]), an alternating tree automaton $\mathcal{A}$ over Σ with state set $Q_\mathcal{A}$ has transitions given by a map $\delta_\mathcal{A}$ taking a state q and an input letter $\mathtt{a}$ to an irredundant disjunctive normal form[1] over $Q_\mathcal{A} \times D$, so that we can assume $\delta_\mathcal{A}(q, \mathtt{a}) \in \mathcal{P}(\mathcal{P}(Q_\mathcal{A} \times D))$ which is read as the $\vee\wedge$-form

$$\bigvee_{\gamma \in \delta_\mathcal{A}(q,\mathtt{a})} \; \bigwedge_{(q',d) \in \gamma} (q', d)$$

This leads to an automaton $\widehat{\mathcal{A}}$ in the sense of Def. 2.3 with states $Q_\mathcal{A} + \mathbb{B}$, P-moves $U := \mathcal{P}(Q_\mathcal{A} \times D)$ and O-moves $X := Q_\mathcal{A}$, with transitions given by $\delta_{\widehat{\mathcal{A}}}(\mathbb{b}, \mathtt{a}, \_, \_, \_) := \mathbb{b}$ if $\mathbb{b} \in \mathbb{B}$ and for $q \in Q_\mathcal{A}$:

$$\delta_{\widehat{\mathcal{A}}}(q, \mathtt{a}, \gamma, q', d) := \begin{cases} q' & \text{if } \gamma \in \delta_{\widehat{\mathcal{A}}}(q, \mathtt{a}) \text{ and } (q', d) \in \gamma \\ \mathbb{t} & \text{if } \gamma \in \delta_{\widehat{\mathcal{A}}}(q, \mathtt{a}) \text{ and } (q', d) \notin \gamma \\ \mathbb{f} & \text{if } \gamma \notin \delta_{\widehat{\mathcal{A}}}(q, \mathtt{a}) \end{cases}$$

and with acceptance condition $\Omega_{\widehat{\mathcal{A}}} := \Omega_\mathcal{A} + Q^*_\mathcal{A}.\mathbb{t}^\omega$.

Similarly to the usual setting (see e.g. [10, 32, 40]), acceptance of a tree $T : D^* \to \Sigma$ by an automaton $\mathcal{A} : \Sigma$ as in (1) can be defined *via* a two-player *acceptance game*. We read transitions of the form

$$\delta_\mathcal{A}(q_\mathcal{A}, \mathtt{a}) \quad : \quad U \times X \quad\longrightarrow\quad (D \longrightarrow Q_\mathcal{A})$$

as $\vee\wedge$-forms

$$\bigvee_{u \in U} \; \bigwedge_{x \in X} \; \bigwedge_{d \in D} \delta_\mathcal{A}(q_\mathcal{A}, \mathtt{a}, u, x, d)$$

generating the game $\mathcal{A}(T)$ depicted on Fig. 1 (left), where the *Proponent* P (also called *Automaton* or $\exists$loïse) plays from the $\bigvee$'s by choosing the $u, u' \in U$, while its *Opponent* O ($\forall$bélard) plays from the $\bigwedge$'s and tries to find a failing path by choosing the $d, d' \in D$ together with the $x, x' \in X$. The state $q'_\mathcal{A}$ is computed from $(p, q_\mathcal{A})$ as $q'_\mathcal{A} := \delta_\mathcal{A}(q_\mathcal{A}, T(p), u, x, d)$ (and similarly for $q''_\mathcal{A}$ from $(p.d, q'_\mathcal{A})$). An infinite play in $\mathcal{A}(T)$ from $(\varepsilon, q^\imath_\mathcal{A})$ generates an infinite sequence of states $(q_n)_n$, and P wins that play iff $(q_n)_n \in \Omega_\mathcal{A}$ (note that there is no finite maximal play). Then $\mathcal{A}$ accepts T iff P has a (total)[2] winning strategy in this game.

Definition 2.5. *Given $\mathcal{A} : \Sigma$, we let $\mathcal{L}(\mathcal{A})$ be the set of $T : D^* \to \Sigma$ such that $\mathcal{A}$ accepts T.*

An automaton $\mathcal{A}$ is *regular* if $\Omega_\mathcal{A}$ is an ω-regular set (see e.g. [10, 32, 40]). *Parity* automata are regular automata $\mathcal{A}$ such that $\Omega_\mathcal{A}$ is generated from a map $c_\mathcal{A} : Q_\mathcal{A} \to \mathbb{N}$ as the set of sequences $(q_n)_n$ such that the maximal k occurring infinitely often in $(c_\mathcal{A}(q_n))_n$

[1] That is, an element of the free distributive lattice over $Q_\mathcal{A} \times D$ [19, §4.8].

[2] All winning strategies considered in this paper are assumed to be also *total* (see App. A).

is even. Regular automata are equivalent in expressive power with MSO (see e.g. [10, 40]).

Definition 2.6. *A set of trees* $\mathcal{L} \subseteq (D^* \to \Sigma)$ *is* regular *if there is a regular automaton* $\mathcal{A} : \Sigma$ *such that* $\mathcal{L} = \mathcal{L}(\mathcal{A})$.

Example 2.7. (i) Continuing Ex. 2.2.(iv), the *mutual exclusion property* (ME) states that in an interleaved execution, P_0 and P_1 can not be both simultaneously in their critical section.
Then a correct I as in Ex. 2.2.(iv) satisfies (ME) iff it is accepted by the deterministic automaton $\mathcal{ME} : \Delta_0 \times \Delta_1$, whose states are $Q_{\mathcal{ME}} := \mathbb{B} = \{\mathbb{t}, \mathbb{f}\}$ with $\mathbb{t}$ initial and $\Omega_{\mathcal{ME}} := \mathbb{t}^\omega$, and whose transition function takes a state $q \in Q_{\mathcal{ME}}$ and $\mathtt{a} \in \Delta := \Delta_0 \times \Delta_1$ to the map

$$\delta_{\mathcal{ME}}(q, \mathtt{a}) \quad : \quad d \in D \mapsto \begin{cases} \mathbb{f} & \text{if } \mathtt{a} = ((\mathtt{C}_0, -), (\mathtt{C}_1, -)) \\ q & \text{otherwise} \end{cases}$$

(ii) Given an alphabet Σ and a set $A \subseteq \Sigma$, we define an automaton $\mathcal{A}^\Sigma_A$ which accepts the trees $T : D^* \to \Sigma$ such that every infinite path in T meets A infinitely often. The state set of $\mathcal{A}^\Sigma_A$ is $\mathbb{B}$ with $\mathbb{t}$ initial, its transitions are given by:

$$\delta_{\mathcal{A}_A}(q, \mathtt{a}) \quad : \quad d \in D \longmapsto \begin{cases} \mathbb{t} & \text{if } \mathtt{a} \in A \\ \mathbb{f} & \text{otherwise} \end{cases}$$

and $(q_n)_n \in \mathbb{B}^\omega$ is accepting iff it contains infinitely many occurrences of $\mathbb{t}$.

(iii) More generally, any LTL property over all paths of a finite state system can be described by a regular deterministic automaton.

2.2 Linear Synchronous Arrow Games

This paper builds from [35] which introduces fibred categories of automata and *substituted acceptance games*, whose objects are automata (possibly instantiated with trees) and whose morphisms are strategies in a suitable restriction of the linear arrow of *simple games* (see e.g. [1, 14]) between corresponding (extended) acceptance games. We now present an adaptation of the *substituted acceptance games* [35] to our context.

Assume that we have a category $\mathbf{T}$ of *trees* (see Def. 4.8 below), whose objects are alphabets and such that the homset $\mathbf{T}[\Sigma, \Gamma]$ contains all functions $M : D^* \to (\Sigma \to \Gamma)$ and such that $(D^* \to \Sigma) \simeq \mathbf{T}[\mathbf{1}, \Sigma]$.

Given automata $\mathcal{A} : \Gamma$ as in (1) and $\mathcal{B} : \Delta$ with

$$\mathcal{B} \quad = \quad (Q_\mathcal{B} \,,\, q^\imath_\mathcal{B} \,,\, V \,,\, Y \,,\, \delta_\mathcal{B} \,,\, \Omega_\mathcal{B}) \tag{2}$$

and tree maps $M : D^* \to (\Sigma \to \Gamma)$ and $N : D^* \to (\Sigma \to \Delta)$, the *substituted acceptance game* [35]

$$\mathcal{A}(M) \quad \multimap \quad \mathcal{B}(N) \qquad \text{(over } \Sigma\text{)}$$

is depicted Fig. 2, where $\mathtt{a} \in \Sigma$, $d \in D$ and $q'_\mathcal{A}$ is computed from $(p, q_\mathcal{A})$ and M by $q'_\mathcal{A} := \delta_\mathcal{A}(q_\mathcal{A}, M(p)(\mathtt{a}), u, x, d)$ (and similarly for $q'_\mathcal{B}$ from $(p, q_\mathcal{B})$ and N). An infinite play ϖ in $\mathcal{A}(M) \multimap \mathcal{B}(N)$ thus determines an infinite sequence $(q_n, q'_n)_n \in (Q_\mathcal{A} \times Q_\mathcal{B})^\omega$. Then ϖ is winning (for P) iff $(q'_n)_n \in \Omega_\mathcal{B}$ whenever $(q_n)_n \in \Omega_\mathcal{A}$.

Note that there is no finite maximal play in $\mathcal{A}(M) \multimap \mathcal{B}(N)$. Moreover, P is forced to play the input character $\mathtt{a} \in \Sigma$ and the tree direction $d \in D$ imposed by O. In particular, the game is D-*synchronous*, in the sense that one infinite play explores exactly one path of the input.

Remark 2.8. A P-strategy σ on $\mathcal{A}(M) \multimap \mathcal{B}(N)$ is a strategy on the linear arrow of simple games with components $\mathcal{A}(M)$ and $\mathcal{B}(N)$, such that the projections to components $\mathcal{A}(M)$ and $\mathcal{B}(N)$ of every play s of σ have the same *trace* in $(D + \Sigma)^*$ (see [35]).

Example 2.9 (Games over $\mathbf{1}$). Figure 1 (right) depicts the particular case of a game over alphabet $\mathbf{1}$ obtained by specializing the game of Fig. 2 to trees in $(D^* \to \Sigma) \simeq \mathbf{T}[\mathbf{1}, \Sigma]$.

Σ	$\mathcal{A}(M)$	$\multimap$	$\mathcal{B}(N)$	
	$(p, q_\mathcal{A})$		$(p, q_\mathcal{B})$	
O	$(\mathtt{a}, u)$			
			$(\mathtt{a}, v)$	P
			(y, d)	O
P	(x, d)			
	$(p.d, q'_\mathcal{A})$		$(p.d, q'_\mathcal{B})$	

Figure 2. Plays in Linear Synchronous Arrow Games

(i) Consider an arbitrary $\mathcal{A} : \Sigma$ and some tree $T : D^* \to \Sigma$. Then P has winning strategy in $\mathcal{A}(T)$ (*i.e.* $T \in \mathcal{L}(\mathcal{A})$) iff P has a winning strategy in $\mathbf{I_1} \multimap \mathcal{A}(T)$.

(ii) Continuing Ex. 2.7.(i), it is possible that only some specific executions of a correct I satisfy (ME). Such specific executions can be described as the infinite paths of a *partial scheduler*[3] $S \subseteq D^*$, and we can check whether I satisfies (ME) along all infinite paths of S by using an automaton $\mathcal{S}$ such that $\mathcal{S}(S) \multimap \mathcal{ME}(I)$ forces O to explore a path in S.
The deterministic automaton $\mathcal{S} : \mathbf{2}$ runs on subsets of D^* represented by their characteristic maps $S : D^* \to \mathbf{2}$ (as in Ex. 2.2.(i) above). Its states are $Q_\mathcal{S} := \mathbb{B}$, with $\mathbb{t}$ initial and $\Omega_\mathcal{S} := \mathbb{t}^\omega$, and its transitions are given by

$$\delta_\mathcal{S}(q, \mathtt{i}) \quad : \quad d \in D \longmapsto \begin{cases} q & \text{if } \mathtt{i} = 1 \\ \mathbb{f} & \text{otherwise} \end{cases}$$

Then P has a winning strategy in $\mathcal{S}(S) \multimap \mathcal{ME}(I)$ iff I satisfies (ME) along all infinite paths of S, since by D-synchronicity P is forced to play the same tree direction in the component $\mathcal{S}(S)$ as imposed by O in $\mathcal{ME}(I)$.

Example 2.10. (i) Consider a game as in Fig. 2 where $\mathcal{A}$ and $\mathcal{B}$ are both over the same alphabet Σ and where M and N are both the identity $\mathbf{T}$-map $\mathrm{Id}_\Sigma := \boldsymbol{\lambda} p.\boldsymbol{\lambda} \mathtt{a}.\mathtt{a}$.
We write $\mathcal{A} \multimap \mathcal{B}$ for the game $\mathcal{A}(\mathrm{Id}_\Sigma) \multimap \mathcal{B}(\mathrm{Id}_\Sigma)$. Note that transitions do not depend anymore from the tree positions $p \in D^*$ since $q'_\mathcal{A} = \delta_\mathcal{A}(q_\mathcal{A}, \mathtt{a}, u, x, d)$ (and similarly for $q'_\mathcal{B}$).

(ii) Continuing (i), it is well-known (see e.g. [10, 32, 40]) that every regular language L of infinite words can be recognized by a deterministic ω-word parity automaton $(Q_L, q^\imath_L, \delta_L, c_L)$. Following [42], given a regular tree automata $\mathcal{A}$ over Σ, let

$$\mathcal{A}^\dagger \quad := \quad (Q_\mathcal{A} \times Q_L \,,\, (q^\imath_\mathcal{A}, q^\imath_L) \,,\, U \,,\, X \,,\, \delta_{\mathcal{A}^\dagger} \,,\, \Omega_{\mathcal{A}^\dagger})$$

where $L = \Omega_\mathcal{A}$, $\Omega_{\mathcal{A}^\dagger}$ is generated from c_L (*via* second projection) and the transition function $\delta_{\mathcal{A}^\dagger}$ is defined as

$$\delta_{\mathcal{A}^\dagger}((q_\mathcal{A}, q_L), \mathtt{a}, u, x, d) \quad := \quad (q'_\mathcal{A} \,,\, \delta_L(q_L, q'_\mathcal{A}))$$

with $q'_\mathcal{A} := \delta_\mathcal{A}(q_\mathcal{A}, \mathtt{a}, u, x, d)$. Note that $\mathcal{A}$ and $\mathcal{A}^\dagger$ have the same sets of P and O-moves, so that the identity ("copy-cat") strategies provide an isomorphism $\mathcal{A} \simeq \mathcal{A}^\dagger$.

(iii) Also continuing (i), assume that $\mathcal{A}$ and $\mathcal{B}$ are moreover both non-deterministic (so that $x = y = \bullet$). Then a map $Q_\mathcal{A} \times U \to V$ (as in [7, Def. 1]) determines a P-strategy on $\mathcal{A} \multimap \mathcal{B}$. It follows from [7, Thm. 1] that for any regular tree language $\mathcal{L}$ there is a non-deterministic automaton $\mathcal{B}$ such that $\mathcal{L}(\mathcal{B}) = \mathcal{L}$ and such that for every parity non-deterministic $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ there is a winning P-strategy on $\mathcal{A} \multimap \mathcal{B}$.

[3] It would make sense to also require S to be prefix-closed, but this is unnecessary if we focus on infinite paths.

2.2.1 Deterministic Linear Implications

Anticipating on §3.1, we indicate how the linear synchronous arrow between *deterministic* automata can be internalized into deterministic automata.

Given deterministic $\mathcal{A}$ and $\mathcal{B}$, both over Σ, the deterministic automaton $\mathcal{A} \multimap \mathcal{B}$ has states $Q_\mathcal{A} \times Q_\mathcal{B}$ (with $(q^\imath_\mathcal{A}, q^\imath_\mathcal{B})$ initial) and transitions given by

$$\delta_{\mathcal{A}\multimap\mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), \mathtt{a}) \quad := \quad (\delta_\mathcal{A}(q_\mathcal{A}, \mathtt{a})\,,\, \delta_\mathcal{B}(q_\mathcal{B}, \mathtt{a}))$$

The acceptance condition of $\mathcal{A} \multimap \mathcal{B}$ consists of those $(q_n, q'_n)_n \in (Q_\mathcal{A} \times Q_\mathcal{B})^\omega$ such that $(q'_n)_n \in \Omega_\mathcal{B}$ whenever $(q_n)_n \in \Omega_\mathcal{A}$. Note that $\mathcal{A} \multimap \mathcal{B}$ is regular as soon as both $\mathcal{A}$ and $\mathcal{B}$ are regular.

Example 2.11. Continuing Ex. 2.9.(ii), we define an automaton $\mathcal{CS} : \mathbf{2} \times \Delta$ such that P wins the game $\mathbf{I_1} \multimap \mathcal{CS}(\langle S, I\rangle)$ iff P wins the game $\mathcal{S}(S) \multimap \mathcal{ME}(I)$. First note that $\mathcal{S}$ and $\mathcal{ME}$ have different input alphabets (resp. $\mathbf{2}$ and Δ). Given the **FinSet**-projection $\mathsf{p}_\mathbf{2} : \mathbf{2} \times \Delta \to \mathbf{2}$, the automaton $\mathcal{S}[\mathsf{p}_\mathbf{2}] : \mathbf{2} \times \Delta$ has the same states as $\mathcal{S}$ and transitions given by

$$\delta_{\mathcal{S}[\mathsf{p}_\mathbf{2}]}(q, (\mathtt{i}, \mathtt{a}), d) \quad := \quad \delta_\mathcal{S}(q, \mathsf{p}_\mathbf{2}(\mathtt{i}, \mathtt{a}), d) \quad = \quad \delta_\mathcal{S}(q, \mathtt{i}, d)$$

Automaton $\mathcal{ME}[\mathsf{p}_\Delta] : \mathbf{2}\times\Delta$ (for the projection $\mathsf{p}_\Delta : \mathbf{2}\times\Delta \to \Delta$) is obtained similarly. We then let $\mathcal{CS} := (\mathcal{S}[\mathsf{p}_\mathbf{2}] \multimap \mathcal{ME}[\mathsf{p}_\Delta])$.

2.3 Indexed Structure

The categories of [35] based on the games in Fig. 2 are actually fibred over the lluf subcategory of $\mathbf{T}$ with homsets restricted to maps of the form $D^* \to (\Sigma \to \Delta)$. We will present in §5 a fibration $\mathsf{DialAut}$ generalizing [35].

It is sufficient to mention here the fact that a tree map $L : D^* \to (\Xi \to \Sigma)$ defines a functor L^*, which maps a game $\mathcal{A}(M) \multimap \mathcal{B}(N)$ over Σ as in Fig. 2, to the game $\mathcal{A}(M \circ L) \multimap \mathcal{B}(N \circ L)$ over Ξ (defined similarly but with $(M \circ L) := \boldsymbol{\lambda} p.\boldsymbol{\lambda}\mathtt{x}.M(p)(L(p)(\mathtt{x}))$ and $N \circ L$, in place of M and N), and maps a (winning) P-strategy σ on $\mathcal{A}(M) \multimap \mathcal{B}(N)$ to a (winning) P-strategy $L^*(\sigma)$ on $\mathcal{A}(M \circ L) \multimap \mathcal{B}(N \circ L)$.

Assume now that P has a winning strategy in the game $\mathcal{A} \multimap \mathcal{B}$ over (say) Σ. Then for every tree $T \in D^* \to \Sigma \simeq \mathbf{T}[\mathbf{1}, \Sigma]$, we have a winning strategy $T^*(\sigma)$ on the game $\mathcal{A}(T) \multimap \mathcal{B}(T)$ over $\mathbf{1}$. Now if $T \in \mathcal{L}(\mathcal{A})$, by Ex. 2.9.(i) there is a winning P-strategy τ on $\mathbf{I_1} \multimap \mathcal{A}(T)$, and since (total) winning P-strategies compose (see App. A), we obtain a winning P-strategy $T^*(\sigma)\circ\tau$ on $\mathbf{I_1} \multimap \mathcal{B}(T)$, so that $T \in \mathcal{L}(\mathcal{B})$. We therefore have:

Proposition 2.12. *If* P *has a winning strategy on* $\mathcal{A} \multimap \mathcal{B}$*, then* $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.

Remark 2.13. In case the map L above is issued from a **FinSet**-map $\mathtt{f} : \Xi \to \Sigma$, then a game of the form $\mathcal{A} \multimap \mathcal{B}$ (hence with $\mathcal{A}, \mathcal{B} : \Sigma$) is mapped to $\mathcal{A}[\mathtt{f}] \multimap \mathcal{B}[\mathtt{f}]$ (where $\mathcal{A}[\mathtt{f}]$ and $\mathcal{B}[\mathtt{f}]$ are defined by internalizing $\mathtt{f}$ into the transition functions, similarly as in Ex. 2.11). It follows that there is a category Aut of automata fibred over (non-empty) finite sets (see §5.2).

2.4 Projection and Existential Quantification

The role of input characters $\mathtt{a} \in \Sigma$ in the play depicted in Fig. 2 is actually issued from the usual operation of *projection* on automata, which implements a form of existential quantification over labeled input trees. In our context, the (almost usual) projection operation of [35] can be adapted as follows:

Definition 2.14. *Given automaton* $\mathcal{A}$ *over* $\Sigma \times \Gamma$ *as in (1), let*

$$\exists_\Gamma \mathcal{A} \quad := \quad (Q_\mathcal{A}\,,\, q^\imath_\mathcal{A}\,,\, \Gamma \times U\,,\, X\,,\, \delta_{\exists_\Gamma \mathcal{A}}\,,\, \Omega_\mathcal{A})$$

where $\delta_{\exists_\Gamma \mathcal{A}}(q, \mathtt{a}, (\mathtt{b}, u), x, d) \;:=\; \delta_\mathcal{A}(q, (\mathtt{a}, \mathtt{b}), u, x, d)$.

	$\mathbf{1}$	$(\exists_\mathbf{2}\mathcal{CS})(I)$
		(p, q)
P		$(\mathtt{i}, \bullet)$
O		d
		$(p.d, q')$
P		$(\mathtt{j}, \bullet)$
O		d'
		$(p.d.d', q'')$

	$\mathbf{1}$	$(\exists_\Delta\mathcal{C}_0)(S)$	$\multimap$	$(\exists_\Delta\mathcal{E}_1)(S)$
		(p, q_0)		(p, q_1)
O		$(\mathtt{a}, \bullet)$		
P				$(\mathtt{b}, \bullet)$
O				d
P		d		
		$(p.d, q'_0)$		$(p.d, q'_1)$

Figure 3. Projection and Existential Quantifications

It is easy to see that if $\mathcal{A}$ accepts $T : D^* \to (\Sigma \times \Gamma)$, then $\exists_\Gamma \mathcal{A}$ accepts $\mathsf{p}_\Sigma T$, so that $\mathsf{p}_\Sigma(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\exists_\Gamma \mathcal{A})$. The converse only holds for *non-deterministic* automata.

Proposition 2.15. *If* $\mathcal{A} : \Sigma\times\Gamma$ *is* non-deterministic *then* $\mathcal{L}(\exists_\Gamma \mathcal{A}) = \mathsf{p}_\Sigma(\mathcal{L}(\mathcal{A}))$.

Example 2.16. (i) Continuing Ex. 2.11, fix a correct $I : D^* \to \Delta$. A play of $\exists_\mathbf{2}\mathcal{CS}(I)$ is depicted on Fig. 3 (left). Winning P-strategies in that game are in 1-1 with partial schedulers $S \subseteq D^*$ such that I satisfies (ME) along any path of S.

(ii) Reversing the perspective of (i), assume now that we want, for a fixed partial scheduler $S \subseteq D^*$, to ensure the existence of a particular behavior of say P_1, assuming the existence of a particular behavior of P_0, e.g. for every infinite path χ of S,

$$\begin{aligned} &\exists P_0 (P_0 \text{ is infinitely often in state } \mathsf{C}_0 \text{ in } \chi) \quad \Longrightarrow \\ &\qquad \exists P_1 (P_1 \text{ infinitely often executes } \mathtt{e}_1 \text{ in } \chi) \end{aligned} \tag{3}$$

Consider the automata $\mathcal{A}^\triangle_{C_0}$ and $\mathcal{A}^\triangle_{E_1}$ defined following Ex. 2.7.(ii), where $\mathtt{a} \in C_0$ iff $\mathsf{p}_{\Delta_0}(\mathtt{a}) = (\mathsf{C}_0, -)$ and similarly for E_1. Following §2.2.1, let $\mathcal{C}_0 := (\mathcal{S}[\mathsf{p}_\mathbf{2}] \multimap \mathcal{A}^\triangle_{C_0}[\mathsf{p}_\Delta])$ and $\mathcal{E}_1 := (\mathcal{S}[\mathsf{p}_\mathbf{2}] \multimap \mathcal{A}^\triangle_{E_1}[\mathsf{p}_\Delta])$. Then a winning P strategy in the game $(\exists_\Delta\mathcal{C}_0)(S) \multimap (\exists_\Delta\mathcal{E}_1)(S)$ (see Fig. 3, right) provides an implementation of (3).

Remark 2.17. Anticipating on §5.3, the existential quantification are usual fibred existential quantifications (called *simple coproducts* in e.g. [18, Def. 1.9.1]) in the fibred category Aut (see Rem. 2.13 and §5). So in particular $\exists_\Gamma$ is left-adjoint to the weakening functor p^*_Σ. More generally, given automata $\mathcal{A} : \Delta \times \Gamma, \mathcal{B} : \Xi$, and tree maps $M : D^* \to (\Sigma \to \Delta)$ and $N : D^* \to (\Sigma \to \Xi)$, there is an isomorphism

$$(\exists_\Gamma\mathcal{A})(M) \multimap \mathcal{B}(N) \;\simeq\; \mathcal{A}(M \times \mathrm{Id}_\Gamma) \multimap \mathcal{B}(N \circ \mathsf{p}_\Sigma) \tag{4}$$

In particular, if $\Sigma = \Delta = \mathbf{1}$ then (modulo $\mathbf{1} \times \Gamma \simeq \Gamma$), the plays of $\mathcal{A} \multimap \mathcal{B}(N \circ \mathbf{1}_\Gamma)$ over Γ are in bijection with plays of $\exists_\Gamma\mathcal{A} \multimap \mathcal{B}(N)$:

	$\mathbf{1}$	$\exists_\Gamma\mathcal{A}$	$\multimap$	$\mathcal{B}(N)$	
		$(p, q_\mathcal{A})$		$(p, q_\mathcal{B})$	
O		$(\mathtt{b}, u)$			
				v	P
				(y, d)	O
P		(x, d)			
		$(p.d, q'_\mathcal{A})$		$(p.d, q'_\mathcal{B})$	

2.5 Monoidal Structure

Tree automata are naturally equipped with a synchronous (direct) product, which gives a symmetric monoidal structure.

Definition 2.18. *Given automata* $\mathcal{A}$ *and* $\mathcal{B}$ *over* Σ *as in (1) and (2),*

$$\mathcal{A} \otimes \mathcal{B} \quad := \quad (Q_\mathcal{A} \times Q_\mathcal{B}, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}), U \times V, X \times Y, \delta_{\mathcal{A}\otimes\mathcal{B}}, \Omega_{\mathcal{A}\otimes\mathcal{B}})$$

with $\delta_{\mathcal{A}\otimes\mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), \mathsf{a}, (u, v), (x, y), d) := (q'_\mathcal{A}, q'_\mathcal{B})$ *where* $q'_\mathcal{A} = \delta_\mathcal{A}(q_\mathcal{A}, \mathsf{a}, u, x, d)$ *and* $q'_\mathcal{B} = \delta_\mathcal{B}(q_\mathcal{B}, \mathsf{a}, v, y, d)$, *and moreover* $((q_n, q'_n))_n \in \Omega_{\mathcal{A}\otimes\mathcal{B}}$ *iff both* $(q_n)_n \in \Omega_\mathcal{A}$ *and* $(q'_n)_n \in \Omega_\mathcal{B}$.

Note that $\mathcal{A} \otimes \mathcal{B}$ is non-deterministic if $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic.

Example 2.19.
(i) Given $\mathcal{A}$ as in (1) and $\mathcal{B}$ as in (2), there is a winning P-strategy on $\mathcal{A} \otimes \mathcal{B} \multimap \mathcal{A}$. It maps $(u, v) \in U \times V$ to $u \in U$ and $x \in X$ to $(x, y) \in X \times Y$, where $y \in Y$ is arbitrary (recall that Y is assumed to be non-empty).
(ii) If $\mathcal{A}$ is non-deterministic, then there is a P-winning strategy on $\mathcal{A} \multimap \mathcal{A} \otimes \mathcal{A}$. Its maps $u \in U$ to $(u, u) \in U \times U$. Note that such strategy may not exist when $X \neq \mathbf{1}$, since O can play two different $(x, x') \in X \times X$ in the component $\mathcal{A} \otimes \mathcal{A}$.

Proposition 2.20. *Given* $\mathcal{A} : \Sigma$ *and* $\mathcal{B} : \Sigma$, *we have* $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.

3. A Dialectica-Like Approach to Automata

This section presents the two main innovations of this paper: the monoidal *closed* structure on tree automata, and a primitive notion of universal quantification.

3.1 Monoidal Closed Structure

The main contribution of this paper w.r.t. [35] is that we obtain a monoidal *closed* structure. The main consequence is the introduction of a *linear implication* connective on automata, satisfying

$$\mathcal{A} \otimes \mathcal{B} \longrightarrow\!\!\circ\; \mathcal{C} \qquad \simeq \qquad \mathcal{B} \longrightarrow\!\!\circ\; (\mathcal{A} \multimap \mathcal{C}) \tag{5}$$

and which is compatible with cut-elimination (see Rem. 6.2).

We now discuss how to build to a linear implication automaton $(\mathcal{A} \multimap \mathcal{B}) : \Sigma$ from automata $\mathcal{A}$ as in (1) and $\mathcal{B}$ as in (2), both over Σ. The isomorphism (5) imposes that $\mathcal{A} \multimap \mathcal{B}$ runs $\mathcal{A}$ and $\mathcal{B}$ in parallel, so that we let $Q_{\mathcal{A}\multimap\mathcal{B}} := Q_\mathcal{A} \times Q_\mathcal{B}$ (with $(q^\imath_\mathcal{A}, q^\imath_\mathcal{B})$ initial).

Recall from §2.1, that in acceptance games (as in Fig. 1, left), we see the transition $\delta_\mathcal{A}(\mathsf{a}, q_\mathcal{A})$ of $\mathcal{A}$ as an $\vee\wedge$-form

$$\bigvee_{u\in U} \bigwedge_{x\in X} \bigwedge_{d\in D} q'_{u,x,d}$$

so that the transition $\delta_{\mathcal{A}\multimap\mathcal{B}}(\mathsf{a}, q_\mathcal{A}, q_\mathcal{B})$ should by a form of linear implication:

$$\bigvee_{u\in U} \bigwedge_{x\in X} \bigwedge_{d\in D} q'_{u,x,d} \qquad \longrightarrow\!\!\circ \qquad \bigvee_{v\in V} \bigwedge_{y\in Y} \bigwedge_{d\in D} q''_{v,y,d}$$

We now follow the pattern of Gödel's Dialectica interpretation (see e.g. [4, 23]). It consists in Skolemization for a suitable (constructive) prenex form:

$$\begin{array}{c} \bigvee_{u\in U} \bigwedge_{x\in X} \bigwedge_{d\in D} q'_{u,x,d} \quad \multimap \quad \bigvee_{v\in V} \bigwedge_{y\in Y} \bigwedge_{d\in D} q''_{v,y,d} \\ \equiv \\ \bigwedge_{u\in U} \bigvee_{v\in V} \bigwedge_{y\in Y} \bigwedge_{d\in D} \bigvee_{x\in X} \bigvee_{d'\in D} (q'_{u,x,d'} \multimap q''_{v,y,d}) \end{array}$$

Since we must have $d = d'$, the above amounts to:

$$\bigwedge_{u\in U} \bigvee_{v\in V} \bigwedge_{y\in Y} \bigwedge_{d\in D} \bigvee_{x\in X} (q'_{u,x,d} \multimap q''_{v,y,d})$$

We now skolemize the $\wedge\vee$, replacing $\wedge_U \vee_V (-)$ by $\vee_{V^U} \wedge_U (-)$ and similarly for the resulting $\wedge_U \wedge_Y \wedge_D \vee_X (-)$. This leads to

$$\begin{array}{c} \bigvee_{f\in V^U} \bigwedge_{u\in U} \bigwedge_{y\in Y} \bigwedge_{d\in D} \bigvee_{x\in X} (q'_{u,x,d} \multimap q''_{f(v),y,d}) \\ \equiv \\ \bigvee_f \bigvee_F \bigwedge_u \bigwedge_y \bigwedge_{d\in D} (q'_{u,F(u,y,d),d} \multimap q''_{f(v),y,d}) \end{array}$$

We therefore put:

Definition 3.1. *Given automata* $\mathcal{A}$ *and* $\mathcal{B}$ *over* Σ *as in (1) and (2), let* $\mathcal{A} \multimap \mathcal{B}$ *be the automaton*

$$(Q_\mathcal{A} \times Q_\mathcal{B}, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}), V^U \times X^{U\times Y\times D}, U \times Y, \delta_{\mathcal{A}\multimap\mathcal{B}}, \Omega_{\mathcal{A}\multimap\mathcal{B}})$$

with $\delta_{\mathcal{A}\multimap\mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), \mathsf{a}, (f, F), (u, y), d) := (q'_\mathcal{A}, q'_\mathcal{B})$ *where* $q'_\mathcal{A} = \delta_\mathcal{A}(q_\mathcal{A}, \mathsf{a}, u, F(u, y, d), d)$ *and* $q'_\mathcal{B} = \delta_\mathcal{B}(q_\mathcal{B}, \mathsf{a}, f(v), y, d)$, *and moreover* $((q_n, q'_n))_n \in \Omega_{\mathcal{A}\multimap\mathcal{B}}$ *iff* $(q'_n)_n \in \Omega_\mathcal{B}$ *whenever* $(q_n)_n \in \Omega_\mathcal{A}$.

We now see how to define a strategy $\mathbf{\Lambda}(\sigma) : \mathbf{I} \multimap (\mathcal{A} \multimap \mathcal{B})$ assuming a given $\sigma : \mathcal{A} \multimap \mathcal{B}$, as in

Σ	$\mathcal{A}$	$\overset{\sigma}{\multimap}$	$\mathcal{B}$	
	$(p, q_\mathcal{A})$		$(p, q_\mathcal{B})$	
O	(a, u)			
			(a, v)	P

Note that σ locally gives, for each fixed $\mathsf{a} \in \Sigma$, a function $f_\mathsf{a} : u \in U \mapsto v \in V$. Similarly, the next O-move against σ must contain some $y \in Y$ and $d \in D$, to which σ answers some $x \in X$. This also defines a function $F_\mathsf{a} : (u, y, d) \mapsto x$. We can then let $\mathbf{\Lambda}(\sigma)$ play $(f_\mathsf{a}, F_\mathsf{a})$. Moreover the next O-move against $\mathbf{\Lambda}(\sigma)$ must be some (u, y, d), as in

Σ	$\mathbf{I}$	$\overset{\mathbf{\Lambda}(\sigma)}{\multimap}$	$(\mathcal{A} \multimap \mathcal{B})$	
	p		$(p, (q_\mathcal{A}, q_\mathcal{B}))$	
O	a			
			$(\mathsf{a}, (f_\mathsf{a}, F_\mathsf{a}))$	P
			(u, y, d)	O
P	d			
	$p.d$		$(p.d, (q'_\mathcal{A}, q'_\mathcal{B}))$	

These (u, y, d) can be transmitted back to σ, which by definition of f_a and F_a, played $v = f_\mathsf{a}(u)$ and $x = F_\mathsf{a}(u, y, d)$. Continuing this way, the two strategies explore the same path of the tree, with the same input in Σ, and produce the same states in $Q_\mathcal{A}$ and $Q_\mathcal{B}$.

Example 3.2. Given $\mathcal{A}$ and $\mathcal{B}$ on the same alphabet, since P has a winning strategy on $\mathcal{A} \otimes \mathcal{B} \multimap \mathcal{A}$, he has a winning strategy on $\mathcal{A} \longrightarrow\!\!\circ\; (\mathcal{B} \multimap \mathcal{A})$.

3.2 Complementation and Falsity

Alternating automata enjoy a complementation construction linear in the number of states (see e.g. [29]). Using the monoidal closed structure, a similar construction can be done with our automata.

In this paragraph we consider *Borel automata*, *i.e.* automata whose acceptance condition is a Borel set (regular sets are Borel).

The *falsity automaton* $\bot$ (over Σ) is $(\mathbb{B}, \mathbb{f}, D, \mathbf{1}, \delta_\bot, \Omega_\bot)$ where $\Omega_\bot := \mathbb{B}^*.\mathbb{t}^\omega$ and the transition function $\delta_\bot$ is defined as follows: let $\delta_\bot(\mathbb{t}, \_, d', \bullet, d) := \mathbb{t}$, and

$$\delta_\bot(\mathbb{f}, \_, d', \bullet, d) \quad := \quad \begin{cases} \mathbb{f} & \text{if} \quad d = d' \\ \mathbb{t} & \text{otherwise} \end{cases}$$

Note that $\bot$ accepts no tree since in an acceptance game, O can always play the same d as P. Given an automaton $\mathcal{A}$ on Σ, let $\mathcal{A}^\bot := \mathcal{A} \multimap \bot$. The automaton $\mathcal{A}^\bot$ can be seen as

$$(Q_\mathcal{A} \times \mathbb{B}, (q^\imath_\mathcal{A}, \mathbb{f}), D^U \times X^{U\times D}, U, \delta_{\mathcal{A}^\bot}, \Omega_{\mathcal{A}^\bot})$$

with $\delta_{\mathcal{A}^\bot}(a, (q_\mathcal{A}, \mathbb{f}), (f, F), u, d) = (q'_\mathcal{A}, \mathbb{b})$ where $\mathbb{b} = \mathbb{f}$ iff $f(u) = d$, and $\delta_{\mathcal{A}^\bot}(a, (q_\mathcal{A}, \mathbb{t}), (f, F), u, d) = (q'_\mathcal{A}, \mathbb{t})$, where $q'_\mathcal{A} := \delta_\mathcal{A}(a, q_\mathcal{A}, u, F(u, d), d)$. Hence O looses as soon as he does not follow the direction proposed by P *via* f. Thanks to the determinacy of Borel games [26], we get:

Proposition 3.3. *Given a Borel automaton* $\mathcal{A} : \Sigma$ *and a tree* $T : D^* \to \Sigma$, $T \in \mathcal{L}(\mathcal{A}^\bot)$ *iff* $T \notin \mathcal{L}(\mathcal{A})$.

Example 3.4. Given non-deterministic Borel automata $\mathcal{A}$ and $\mathcal{B}$ s.t. $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, it follows from (4), and Props. 2.15, 2.20 and 3.3 that P has a winning strategy in $\mathcal{A} \otimes \mathcal{B} \multimap \bot$, and thus in $\mathcal{A} \multimap \mathcal{B}^\bot$. We will show in §6 (Ex.6.1.(ii)) that in this case P has a winning strategy on $(\mathcal{B} \multimap \mathcal{A}) \multimap \mathcal{B}^{\perp}$. It then follows from Ex. 3.2 and Prop. 2.12 that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B} \multimap \mathcal{A}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$.

3.3 Universal Quantifications and the $\exists\forall$-Normal Form

We have seen the projection operation $\exists_{\Sigma}(-)$ in §2.4. We devise now a coprojection operation $\forall_{\Sigma}(-)$, which follows Gödel's Dialectica interpretation of universal quantifications, and also allows to see automata as $\exists\forall$-forms: For an automaton $\mathcal{A}$ with

$$\delta_{\mathcal{A}} \quad : \quad Q_{\mathcal{A}} \times \Sigma \quad \longrightarrow \quad U \times X \quad \longrightarrow \quad (D \longrightarrow Q_{\mathcal{A}})$$

we will have

$$\mathcal{A} \quad \simeq \quad \exists_U \forall_X \mathcal{D} \tag{6}$$

where $\mathcal{D}$ is the *deterministic* automaton whose transition function

$$\delta_{\mathcal{D}} \quad : \quad Q_{\mathcal{A}} \times (\Sigma \times U \times X) \quad \longrightarrow \quad D \quad \longrightarrow \quad Q_{\mathcal{A}}$$

is obtained from $\delta_{\mathcal{A}}$ in the obvious way.

Let us look at what should happen when (say) $\forall_{\Gamma}(-)$ is applied to an automaton $\mathcal{A} = \exists_U \forall_X \mathcal{D}$ over $\Sigma \times \Gamma$, so that $\forall_{\Gamma}\mathcal{A} = \forall_{\Gamma}\exists_U\forall_X\mathcal{D}$. In order to recover an $\exists\forall$-form from $\forall_{\Gamma}\exists_U\forall_X\mathcal{D}$, we apply here the same trick as in Gödel's Dialectica interpretation, replacing $\forall_{\Gamma}\exists_U(-)$ by $\exists_{U^{\Gamma}}\forall_{\Gamma}(-)$, so that

$$\forall_{\Gamma}\mathcal{A} \quad = \quad \forall_{\Gamma}\exists_U\forall_X\mathcal{D} \quad = \quad \exists_{U^{\Gamma}}\forall_{\Gamma\times X}\mathcal{D}'$$

where $\delta_{\mathcal{D}'}(q_{\mathcal{A}}, (\mathtt{a}, f, (\mathtt{b}, x)), d) \;:=\; \delta_{\mathcal{A}}(q_{\mathcal{A}}, (\mathtt{a}, \mathtt{b}), \mathtt{f}(\mathtt{b}), x, d)$. This gives the general definition of coprojection for automata:

Definition 3.5. *Given automaton $\mathcal{A}$ over $\Sigma \times \Gamma$ as in (1), let*

$$\forall_{\Gamma}\mathcal{A} \quad := \quad (Q_{\mathcal{A}}\,,\, q^{\imath}_{\mathcal{A}}\,,\, U^{\Gamma}\,,\, \Gamma \times X\,,\, \delta_{\forall_{\Gamma}\mathcal{A}}\,,\, \Omega_{\mathcal{A}})$$

where $\delta_{\forall_{\Gamma}\mathcal{A}}(q, \mathtt{a}, f, (\mathtt{b}, x), d) \;:=\; \delta_{\mathcal{A}}(q, (\mathtt{a}, \mathtt{b}), f(\mathtt{b}), x, d)$.

Note that if $\mathcal{D}$ (over say $\Sigma \times U \times X$) is deterministic, then $\forall_X\mathcal{D}$ has $\mathbf{1} \simeq \mathbf{1}^{\Gamma}$ as P-moves and $\Gamma \simeq \Gamma \times \mathbf{1}$ as O-moves. It follows that for $\mathcal{A}$ as in (1) we indeed have $\mathcal{A} \simeq \exists_U\forall_X\mathcal{D}$ as in (6).

The inversion of quantifiers from $\forall_{\Gamma}\exists_U(-)$ to $\exists_{U^{\Gamma}}\forall_U(-)$ corresponds, in games, to the usual (constructive) Skolemization performed in Gödel's Dialectica interpretation.

Remark 3.6. Similarly as existential quantifications (in Rem. 2.17), the operation $\forall(-)$ gives usual fibred universal quantifications in Aut (see Rem. 2.13 and §5). It is right adjoint to weakening, and satisfies the dual law of (4):

$$\mathcal{B}(N) \;\multimap\; (\forall_{\Gamma}\mathcal{A})(M) \;\simeq\; \mathcal{B}(N \circ \mathsf{p}_{\Sigma}) \;\multimap\; \mathcal{A}(M \times \mathrm{Id}_{\Gamma}) \tag{7}$$

Example 3.7.
(i) As usual, (7) gives for any $\mathcal{A} : \Sigma$ a winning P-strategy $\epsilon_{\mathcal{A}}$ on $(\forall_{\Sigma}\mathcal{A})[\mathbf{1}_{\Sigma}] \multimap \mathcal{A}$ (modulo $\Sigma \simeq \mathbf{1} \times \Sigma$).
(ii) If $\mathcal{A}$ and $\mathcal{B}$ (both over Σ) are regular, then the game $\mathcal{A} \multimap \mathcal{B}$ is equivalent to a finite regular game. Indeed, by (7), P has a winning strategy on $\mathcal{A} \multimap \mathcal{B}$ iff he has a winning strategy on $\mathbf{I_1} \multimap \forall_{\Sigma}(\mathcal{A} \multimap \mathcal{B})$. But since in that game O can only play $\bullet$ in the component $\mathbf{I_1}$, similarly as in Ex. 2.9.(i), it is equivalent to the acceptance game of the automaton $\forall_{\Sigma}(\mathcal{A} \multimap \mathcal{B}) : \mathbf{1}$ on the unique tree $\mathbf{1} : D^* \to \mathbf{1}$ (in the sense of Fig. 1 left). Since $\forall_{\Sigma}(\mathcal{A} \multimap \mathcal{B}) : \mathbf{1}$ is regular, it is then well-known (see e.g. [40, Ex. 6.12 & Thm. 6.18]) that its acceptance game is equivalent to a finite regular game, and that the winner always effectively has a finite state winning strategy.
(iii) Our internalized linear arrow can handle a construction for the separation property of [36, Thm. 2.7]. Assume $\mathcal{A}$ and $\mathcal{B}$ as in Ex. 3.4 are regular. It follows from (ii) that P has a *finite-state* winning strategy τ on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$, described (say) by the automaton $(Q_{\tau}, q^{\imath}_{\tau}, \delta_{\tau}, \mathsf{o}_{\tau})$ where $\langle \delta_{\tau}, \mathsf{o}_{\tau} \rangle : Q_{\tau} \times (\Sigma \times U \times V) \to Q_{\tau} \times D$. We can then restrict $\mathcal{B} \multimap \mathcal{A}$ along τ. Define $\mathcal{C} : \Sigma$ as follows:

$$\mathcal{C} \;:=\; ((Q_{\mathcal{B}\multimap\mathcal{A}} \times Q_{\tau}) + \{\mathbb{t}\}, (q^{\imath}_{\mathcal{B}\multimap\mathcal{A}}, q^{\imath}_{\tau}), U^V, V, \delta_{\mathcal{C}}, \Omega_{\mathcal{C}})$$

where $\delta_{\mathcal{C}}(\mathbb{t}, \_, \_, \_, \_) := \mathbb{t}$, and $\delta_{\mathcal{C}}((q, q_{\tau}), \mathtt{a}, f, v, d) := \mathbb{t}$ if $\mathsf{o}(q_{\tau}, \mathtt{a}, f(v), v) \neq d$, and otherwise, $\delta_{\mathcal{C}}((q, q_{\tau}), \mathtt{a}, f, v, d) := (\delta_{\mathcal{B}\multimap\mathcal{A}}(q, \mathtt{a}, f, v, d)\,,\, \delta_{\tau}(q_{\tau}, \mathtt{a}, f(v), v))$, and with $\Omega_{\mathcal{C}} := (Q_{\mathcal{B}\multimap\mathcal{A}} \times Q_{\tau}).\mathbb{t}^{\omega} + \pi^{-1}(\Omega_{\mathcal{B}\multimap\mathcal{A}})$ (where $\pi : Q_{\mathcal{B}\multimap\mathcal{A}} \times Q_{\tau} \to Q_{\mathcal{B}\multimap\mathcal{A}}$ is a projection). Note that $\mathcal{L}(\mathcal{B} \multimap \mathcal{A}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$. If $\mathcal{A}$ and $\mathcal{B}$ are parity automata both with colorings of range $\{0, \dots, n\}$ for some even n, then (as in [36, §2.2.2]) since τ is winning, $\Omega_{\mathcal{C}}$ can be described with a coloring c of range $\{0, \dots, n\}$ (with $c(\mathbb{t}) = n$) and such that in each strongly connected component of $\mathcal{C}$ (for $q \to q'$ iff $q' = \delta_{\mathcal{C}}(q, \mathtt{a}, f, v, d)$ for some $\mathtt{a}, f, v, d$), c has range either $\{1, \dots, n\}$ or $\{0, \dots, n-1\}$.

3.4 Alternating v.s. Non-Deterministic Tree Automata

We have seen in §3.2 that similarly to usual alternating automata, our automata have linear complements. Moreover, we have seen in Prop. 2.15 that the projection operation is correct on non-deterministic automata. However, complementation is not linear on non-deterministic automata and projection is not correct in general on alternating automata.

Actually, regular alternating and non-deterministic automata are equivalent in expressive power:

Theorem 3.8 (Simulation [30])**.** *Given a regular automaton $\mathcal{A}$, we can effectively build a non-deterministic automaton $!\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(!\mathcal{A})$.*

The automaton $!\mathcal{A}$ is in general exponentially larger than $\mathcal{A}$. This can be seen as a reason for the non-elementary complexity of MSO: each alternation of quantifiers costs an exponential.

We will see in §7 that in our context, the non-determinization operation $!(-)$ satisfies the *deduction* rules of the usual exponential modality ! of intuitionistic linear logic (see e.g. [27]). It follows that the exponential ! allows to define, using Girard's decomposition, an intuitionistic implication $\Rightarrow$ (*i.e.* satisfying the deduction rules of intuitionistic logic) as $\mathcal{A} \Rightarrow \mathcal{B} := !\mathcal{A} \multimap \mathcal{B}$.

Example 3.9.
(i) There is a winning P-strategy on $((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \Longrightarrow ?\mathcal{A}$, where $?\mathcal{A} := (!(\mathcal{A}^{\perp}))^{\perp}$.
(ii) Extending Ex. 3.4, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, then there is a winning P-strategy on $!\mathcal{A} \multimap ?\mathcal{B}$.

4. Simple Zig-Zag Games

In this section, we decompose the synchronicity constraints imposed on the linear synchronous arrow games of [35] (see §2.2 and Fig. 2) using a monad of monoid indexing and comonads of comonoid indexing in a category $\mathbf{DZ}$ of *simple zig-zag games*. This leads to a fibration DialZ, with existential and universal quantifications and with fibrewise symmetric monoidal closed structure. It is the base of the fibrations of automata and acceptance games presented in §5.

4.1 Simple Zig-Zag Games

The synchronicity constraints of [35] presented in §2.2 impose the P-strategies on games of the form $\mathcal{A}(M) \multimap \mathcal{B}(N)$ depicted in Fig. 2 to be *zig-zag* strategies, in the sense that for all (even-length) play s of such a strategy, the projections of s to components $\mathcal{A}(M)$ and $\mathcal{B}(N)$ must have the same length.

Definition 4.1. *The category* $\mathbf{DZ}$ *of* simple zig-zag games *has pairs of non-empty sets $A = (U, X)$, $B = (V, Y)$, etc as objects. Morphisms from A to B are zig-zag strategies $\sigma : A \multimap B$ where A and B are seen as simple full positive games[4], see Fig. 4 (left), where $u \in U$, $v \in V$, $y \in Y$ and $x \in X$.*

[4] See App. A.

$\mathbf{DZ}$	A	$\overset{\sigma}{\multimap}$	B
O	u		
P			v
O			y
P	x		

$\mathbf{DZ}_D$	A	$\overset{\sigma}{\multimap}_{\mathbf{DZ}_D}$	B
O	u		
P			v
O			(y, d)
P	x		

Figure 4. Zig-Zag Strategies

The category $\mathbf{DZ}$ is equipped with a very simple synchronous monoidal closed structure which is different from the asynchronous usual ones in game semantics.

Proposition 4.2. *The category $\mathbf{DZ}$ is symmetric monoidal. The unit is $\mathbf{I} = (\mathbf{1}, \mathbf{1})$, and given $A = (U, X)$ and $B = (V, Y)$ we let*

$$A \otimes B \quad := \quad (U \times V \ , \ X \times Y)$$

4.2 Monoidal Closed Structure

The monoidal closed structure follows the pattern of the Gödel's Dialectica interpretation: A full positive game $A = (U, X)$ is seen as a succession of $\vee\wedge$-forms, where P plays from the $\vee$'s by choosing the $u \in U$ and O plays from the $\wedge$'s by choosing the $x \in X$. Given another game $B = (V, Y)$, a (total) zig-zag strategy $\sigma : A \multimap B$ as in Fig. 4 (left) can be seen as providing a succession of maps $f : U \to V$ (corresponding to the P-move v in component B following the O-move u in A) and $F : U \times Y \to X$ (corresp. to the P-move x in A following the O-moves u and y).

Proposition 4.3. *The category $\mathbf{DZ}$ is symmetric monoidal closed. The linear exponent of $A = (U, X)$ and $B = (V, Y)$ is*

$$A \multimap B \quad := \quad (V^U \times X^{U \times Y} \ , \ U \times Y)$$

The monoidal closed structure of $\mathbf{DZ}$ thus departs from traditional game semantics since the natural isomorphism $A \otimes B \multimap C \simeq B \multimap (A \multimap C)$ relates only strategies, but not *plays*.

4.3 D-Synchronicity

We express the D-synchronicity constraint of [35] using a monad of monoid indexing.

Monoid Indexing [16, 17]. Let $(\mathbb{C}, \otimes, \mathbf{I})$ be a symmetric monoidal category and let $\mathbf{Mon}(\mathbb{C})$ be its category of commutative monoids. Its objects are objects M of $\mathbb{C}$ equipped with structure maps

$$\mathbf{I} \quad \xrightarrow{u} \quad M \quad \xleftarrow{m} \quad M \otimes M$$

subject to some coherence conditions (see e.g. [27]). A morphism from (M, u, m) to (N, u', m') is a $\mathbb{C}$-morphism $M \to N$ which commutes with the structure maps.

Given a commutative monoid (M, u, m) in $\mathbb{C}$, define the monad $M = (M, \eta, \mu)$ as follows. The functor M acts on objects by tensoring with M on the right and on morphisms by

$$(f : A \to B) \quad \longmapsto \quad (f \otimes \mathrm{id}_M : A \otimes M \to B \otimes M)$$

The natural maps η and μ are

$$\begin{array}{lllrcl} \eta_A & := & (\mathrm{id}_A \otimes u) \circ \rho^{-1} & : & A & \longrightarrow & A \otimes M \\ \mu_A & := & (\mathrm{id}_A \otimes m) \circ \alpha & : & (A \otimes M) \otimes M & \longrightarrow & A \otimes M \end{array}$$

(where $\rho : A \otimes \mathbf{I} \to A$ and $\alpha : (A \otimes M) \otimes M \to A \otimes (M \otimes M)$ are structural isos of $(\mathbb{C}, \otimes, \mathbf{I})$). It is easy to see that M is a (lax) monoidal monad (see e.g. [27] for definitions).

We now turn to the monoid of D-synchronicity in $\mathbf{DZ}$.

Proposition 4.4. *In $\mathbf{DZ}$, the object $D := (\mathbf{1}, D)$ is a commutative monoid with structure:*

	$\mathbf{I}$	$\overset{u}{\multimap}$	D	
O	$\bullet$			
			$\bullet$	P
			d	O
P	$\bullet$			

	$D \otimes D$	$\overset{m}{\multimap}$	D	
O	$(\bullet, \bullet)$			
			$\bullet$	P
			d	O
P	(d, d)			

We let $\mathbf{DZ}_D$ be the Kleisli category of the monad D. A $\mathbf{DZ}_D$-map from A to B is a $\mathbf{DZ}$-strategy $\sigma : A \multimap B \otimes D$ (see Fig. 4 right).

The symmetric monoidal closed structure of $\mathbf{DZ}$ lifts to $\mathbf{DZ}_D$. For the symmetric monoidal structure this follows from the fact that the monad D is (lax) monoidal. For monoidal closure, since

$$\begin{array}{rcl} \mathbf{DZ}_D[A \otimes B, C] & = & \mathbf{DZ}[A \otimes B, C \otimes D] \\ & \simeq & \mathbf{DZ}[A, B \multimap_{\mathbf{DZ}} C \otimes D] \end{array}$$

we should have $(A \multimap_{\mathbf{DZ}_D} B) \otimes D \simeq (A \multimap_{\mathbf{DZ}} B \otimes D)$. This leads to $((U, X) \multimap_{\mathbf{DZ}_D} (V, Y)) = (W, Z)$ with

$$(W, Z \times D) \quad \simeq \quad (V^U \times X^{U \times Y \times D}, U \times Y \times D)$$

We therefore let

$$(U, X) \multimap_{\mathbf{DZ}_D} (V, Y) \quad := \quad (V^U \times X^{U \times Y \times D}, U \times Y)$$

Proposition 4.5. *$\mathbf{DZ}_D$ is symmetric monoidal closed.*

4.4 Fibred Structure in $\mathbf{DZ}_D$

We now turn to the synchronicity constraint of [35] imposing P to play the same input character a as chosen by O (see Fig. 2). We express this constraint in $\mathbf{DZ}_D$ by a comonad of comonoid indexing (dual to monoid indexing), which leads to a split fibred structure similar to the usual *simple fibrations* (see e.g. [18]).

4.4.1 The Fibred Structure of Comonoid Indexing

The fibred structure of $\mathbf{DZ}_D$ is added along a pattern similar to the *simple fibration* $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ over a Cartesian base category $\mathbb{B}$ (see e.g. [13, 15]). Recall (from e.g. [18]) that $\mathsf{s}(\mathbb{B})$ has pairs (I, X) of $\mathbb{B}$-objects as objects, with maps $(I, X) \to (J, Y)$ given by a pairs of $\mathbb{B}$-maps $f_0 : I \to J$ and $f : I \times X \to Y$. The functor $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ is the first projection. We would like to use as base the category $\mathbf{DZ}_D$ and its monoidal product, which is not Cartesian. However, it is well-known (1) that the fibre category of $\mathsf{s}(\mathbb{B})$ over say I, is the co-Kleiseli category of a co-monad whose functor is $I \times (-)$ (see e.g. [18, Ex. 1.3.4]), and (2) that commutative comonoids form a Cartesian category. *Comonoid indexing* [16, 17], allows to get a fibration whose fibre over K, for K a commutative comonoid, is the co-Kleiseli category for $K \otimes (-)$.

Comonoid Indexing [16, 17]. This is dual to monoid indexing used above. Let $\mathbf{Comon}(\mathbb{C})$ be the category of commutative comonoids on a symmetric monoidal category $\mathbb{C}$. Its objects are objects K of $\mathbb{C}$ equipped with structure maps

$$\mathbf{I} \quad \xleftarrow{e} \quad K \quad \xrightarrow{m} \quad K \otimes K$$

subject to some condition dual to those of $\mathbf{Mon}(\mathbb{C})$. A morphism from (K, e, m) to (L, e', m') is also a $\mathbb{C}$-morphism $K \to L$ compatible with the structure maps. It is well-known (see e.g. [27, Cor. 18, §6.5]) that $\mathbf{Comon}(\mathbb{C})$ is Cartesian.

Given a commutative comonoid (K, e, m) in $\mathbb{C}$, define a comonad $K = (K, \epsilon, \delta)$ whose functor K acts on objects by tensoring on the left and on morphisms by

$$(f : A \to B) \quad \longmapsto \quad (\mathrm{id}_K \otimes f : K \otimes A \to K \otimes B)$$

The natural maps ϵ and δ are given by (modulo structural isos)

$$\begin{array}{lllrcl} \epsilon_A & \simeq & e \otimes \mathrm{id}_A & : & K \otimes A & \longrightarrow & A \\ \delta_A & \simeq & m \otimes \mathrm{id}_A & : & K \otimes A & \longrightarrow & K \otimes K \otimes A \end{array}$$

DialZ_Σ	A	$\xrightarrow{\sigma}\!\!\circ$	B
O	(a, u)		
P			v
O			(y, d)
P	x		

$\mathbf{T}$	Σ	$\xrightarrow{M}$	Γ
O	a		
P			b
O			d
P	$\bullet$		

Figure 5. The Fibred Structure of DialZ

Grothendieck Completion. A comonoid morphism $u : K \to L$ induces a functor $u^* : \mathbf{Kl}(L) \to \mathbf{Kl}(K)$ acting as the identity on objects and taking $f : L \otimes A \to B$ to $f \circ (u \otimes \mathrm{id}_A) : K \otimes A \to B$. It readily follows that $\mathrm{id}_K^* = \mathrm{id}_{\mathbf{Kl}(K)}$ and that $(u \circ v)^* = v^* \circ u^*$. In other words, we have a functor $\mathsf{Cl}(\mathbb{C}) : \mathbf{Comon}(\mathbb{C})^{\mathrm{op}} \to \mathbf{Cat}$. Its Grothendieck completion $\int \mathsf{Cl}(\mathbb{C})$ (see e.g. [18]) is the category whose objects are pairs (K, A) of an object K of $\mathbf{Comon}(\mathbb{C})$ and an object A of $\mathbb{C}$, and whose morphisms from (K, A) to (L, B) are pairs (u, f) where $u : K \to L$ is a comonoid morphism and $f : K \otimes A \to B$. The category $\int \mathsf{Cl}(\mathbb{C})$ is fibred over $\mathbf{Comon}(\mathbb{C})$ *via* the first projection, that we denote

$$\mathsf{s}_{\mathsf{Cl}}(\mathbb{C}) \quad : \quad \int \mathsf{Cl}(\mathbb{C}) \quad \longrightarrow \quad \mathbf{Comon}(\mathbb{C})$$

Its is a *split* fibration since $\mathsf{Cl}(\mathbb{C})$ is strict, and its fibre over K is the category $\mathbf{Kl}(K)$.

4.4.2 Comonoid Indexing in $\mathbf{DZ}_D$

In $\mathbf{DZ}_D$, objects of the form $\Sigma = (\Sigma, \mathbf{1})$ can be equipped with a commutative comonoid structure.

Proposition 4.6. *In $\mathbf{DZ}_D$, each object $\Sigma = (\Sigma, \mathbf{1})$ is a commutative monoid with structure:*

	Σ	$\xrightarrow{e_\Sigma}\!\!\circ$	$\mathbf{I}$	
O	a			
			$\bullet$	P
			d	O
P	$\bullet$			

	Σ	$\xrightarrow{d_\Sigma}\!\!\circ$	$\Sigma \otimes \Sigma$	
O	a			
			(a, a)	P
			d	O
P	$\bullet$			

Hence for each $\Sigma = (\Sigma, \mathbf{1})$ there is a comonad Σ_D in $\mathbf{DZ}_D$. We denote by DialZ_Σ its Kleisli category. A typical play of a strategy $\sigma \in \mathsf{DialZ}_\Sigma[A, B]$ is depicted on Fig. 5 (left).

The symmetric monoidal structure of $\mathbf{DZ}_D$ lifts to DialZ_Σ since (dually to D), the comonad Σ_D is oplax symmetric monoidal. The closed structure of DialZ is the same as that of $\mathbf{DZ}_D$ since

$$\begin{array}{rcl} \mathsf{DialZ}_\Sigma[A \otimes B, C] & = & \mathbf{DZ}_D[\Sigma \otimes A \otimes B, C] \\ & \simeq & \mathbf{DZ}_D[\Sigma \otimes A, B \multimap_{\mathbf{DZ}_D} C] \end{array}$$

Proposition 4.7. DialZ_Σ *is symmetric monoidal closed.*

4.4.3 The Fibred Category DialZ

We now turn to DialZ. It is fibred over a base category $\mathbf{T}$ whose objects are alphabets and which embeds in $\mathbf{Comon}(\mathbf{DZ}_D)$. DialZ is obtained by change-of-base of fibrations of $\mathsf{s}_{\mathsf{Cl}}(\mathbf{DZ}_D)$ along this embedding. The fibre of DialZ over Σ will thus be DialZ_Σ.

Definition 4.8 (The Base Category $\mathbf{T}$)**.** *The objects of $\mathbf{T}$ are alphabets (Σ, Γ, etc) and the morphisms $M \in \mathbf{T}[\Sigma, \Gamma]$ are the strategies $M \in \mathbf{DZ}_D[\Sigma, \Gamma]$ (see Fig. 5, right).*

Remark 4.9. Note that $\mathbf{T}$-maps $\Sigma \to \Gamma$ are determined by functions of the form $\left(\bigcup_{n \in N} D^n \times \Sigma^{n+1}\right) \to \Gamma$. It follows that, as required in §2.2, each function $D^* \to (\Sigma \to \Gamma)$ induces a map in $\mathbf{T}[\Sigma, \Gamma]$ and that $D^* \to \Sigma \simeq \mathbf{T}[\mathbf{1}, \Sigma]$.

It is easy to see that $\mathbf{T}$-maps induce commutative comonoid morphisms in $\mathbf{DZ}_D$.

Proposition 4.10. *The category $\mathbf{T}$ embeds to $\mathbf{Comon}(\mathbf{DZ}_D)$ via the functor $\mathsf{E}_{\mathbf{T}}$ mapping an object Σ of $\mathbf{T}$ to the comonoid $(\Sigma, e_\Sigma, d_\Sigma)$ and a morphism $M : \mathbf{T}[\Gamma, \Sigma]$ to itself.*

We now define the fibred category DialZ by change-of-base of fibrations of $\mathsf{s}_{\mathsf{Cl}(\mathbf{DZ}_D)}$ along $\mathsf{E}_{\mathbf{T}}$:

$$\begin{tikzcd} \mathsf{DialZ} \arrow[r] \arrow[d, "\mathsf{dz}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathsf{Cl}(\mathbf{DZ}_D) \arrow[d, "\mathsf{s}_{\mathsf{Cl}}(\mathbf{DZ}_D)"] \\ \mathbf{T} \arrow[r, "\mathsf{E}_{\mathbf{T}}"] & \mathbf{Comon}(\mathbf{DZ}_D) \end{tikzcd}$$

Explicitly, the objects of DialZ are pairs (Σ, A) of an alphabet Σ and a $\mathbf{DZ}$-object A, and a morphism from (Σ, A) to (Γ, B) is given by a pair (L, σ) where $L \in \mathbf{T}[\Sigma, \Gamma]$ and σ is a DialZ_Σ-map from A to B (recall from §4.4.1 that $L^* : \mathsf{DialZ}_\Gamma \to \mathsf{DialZ}_\Sigma$ is the identity on objects). Composition in DialZ is induced by composition in $\int \mathsf{Cl}(\mathbf{DZ}_D)$ (see e.g. [18]).

Proposition 4.11. $\mathsf{dz} : \mathsf{DialZ} \to \mathbf{T}$ *is symmetric monoidal closed.*

4.5 Quantification in DialZ

We now sketch existential and universal quantifications in DialZ.

Existential Quantification. A fibration $\mathsf{p} : \mathbb{E} \to \mathbb{B}$ has existential quantifications (also called simple coproducts [18]) when the weakening functors $\pi^* : \mathbb{E}_J \to \mathbb{E}_{I \times J}$ (induced by the $\mathbb{B}$-projections $\pi : I \times J \to I$) have left adjoints $\coprod_{I,J} : \mathbb{E}_{I \times J} \to \mathbb{E}_I$ satisfying some coherence conditions, called the *Beck-Chevalley* conditions, insuring that the adjunction $\coprod_{I,J} \dashv \pi^*$ is preserved by substitution (see e.g. [18]).

The simple fibration $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ has simple coproducts (see e.g. [18, Prop. 1.9.3]). They are induced by

$$\coprod_{I,J} (I \times J, X) \quad := \quad (I, J \times X)$$

So, for $\mathbb{C}$ symmetric monoidal, $\mathsf{s}_{\mathsf{Cl}} : \int \mathsf{Cl}(\mathbb{C}) \to \mathbf{Comon}(\mathbb{C})$ has coproducts induced, recalling that $\mathbf{Comon}(\mathbb{C})$ is Cartesian, by

$$\coprod_{I,J} (I \times J, X) \quad := \quad (I, J \otimes X)$$

This leads in DialZ to $\coprod_{\Sigma,\Gamma}(\Sigma \times \Gamma, (U, X)) := (\Sigma, (\Gamma \times U, X))$. The Beck-Chevalley condition amounts to (for $L \in \mathbf{T}[\Delta, \Sigma]$):

$$\coprod_{\Delta,\Gamma} (L \times \mathrm{id}_\Gamma)^*(\Sigma \times \Gamma, (U, X)) \quad = \quad L^*(\coprod_{\Sigma,\Gamma} (\Sigma \times \Gamma, (U, X)))$$

Universal Quantification. Universal quantifications (simple products [18]) are given by a right adjoint $\prod_{I,J}$ to π^* (also with a Beck-Chevalley condition). It is also well-known (see e.g. [18, Prop. 1.9.3.(ii)]) that the simple fibration $\mathsf{s}(\mathbb{B})$ has products $\prod_{I,J} : \mathsf{s}(\mathbb{B})_{/\!\!/ I \times J} \to \mathsf{s}(\mathbb{B})_{/\!\!/ I}$ iff $\mathbb{B}$ is Cartesian closed. They are given by

$$\prod_{I,J} (I \times J, X) \quad := \quad (I, X^J)$$

So, for $\mathbb{C}$ symmetric monoidal closed, recalling that $\mathbf{Comon}(\mathbb{C})$ is Cartesian, the fibration $\mathsf{s}_{\mathsf{Cl}} : \int \mathsf{Cl}(\mathbb{C}) \to \mathbf{Comon}(\mathbb{C})$ has simple products induced by

$$\prod_{I,J} (I \times J, X) \quad := \quad (I, J \multimap X)$$

In the case of DialZ, for $\Gamma = (\Gamma, \mathbf{1})$ this gives

$$\begin{array}{rcl} \prod_{\Sigma,\Gamma}(\Sigma \times \Gamma, (U, X)) & := & (\Sigma,\ \Gamma \multimap_{\mathbf{DZ}_D} (U, X)) \\ & \simeq & (\Sigma, (U^\Gamma, \Gamma \times X)) \end{array}$$

The Beck-Chevalley condition amounts to (for $L \in \mathbf{T}[\Delta, \Sigma]$):

$$L^*(\prod_{\Sigma,\Gamma}(\Sigma \times \Gamma, (U, X))) \quad = \quad \prod_{\Delta,\Gamma}(L \times \mathrm{id}_\Gamma)^*(\Sigma \times \Gamma, (U, X))$$

4.6 The Distributive Law of Comonoid over Monoid Indexing

We note here that the comonoids Σ_D on $\mathbf{DZ}_D$ are actually generated (by the free functor F_D of D) from comonoids Σ on $\mathbf{DZ}$. Moreover, on $\mathbf{DZ}$, the comonad Σ is related to the monad D by a distributive law.

The comonoid $\Sigma = (\Sigma, \mathbf{1})$ on $\mathbf{DZ}$ is dual to the monoid D.

Proposition 4.12. *In $\mathbf{DZ}$, the objects $\Sigma = (\Sigma, \mathbf{1})$ can be equipped with a commutative comonoid structure $(\tilde{e}_\Sigma, \tilde{d}_\Sigma)$ such that $e_\Sigma = \mathsf{F}_D(\tilde{e}_\Sigma)$ and $d_\Sigma = \mathsf{F}_D(\tilde{e}_\Sigma)$.*

Recall from e.g. [27] that F_D is the identity on objects and takes $\sigma \in \mathbf{DZ}[A, B]$ to $\eta_B \circ \sigma \in \mathbf{DZ}_D[A, B]$.

A distributive law Λ of a comonad G over a monad T on a category $\mathbb{C}$ is given by a natural transformation

$$\Lambda \quad : \quad G \circ T \quad \Longrightarrow \quad T \circ G$$

subject to some coherence conditions, which can be found e.g. in [11]. These coherence conditions ensure that we can define a category $\mathbf{Kl}(\Lambda)$, whose objects are the objects of $\mathbb{C}$, and whose morphisms are given by $\mathbf{Kl}(\Lambda)[A, B] := \mathbb{C}[GA, TB]$.

We therefore take for G the comonad of indexing by the comonoid $\Sigma = (\Sigma, \mathbf{1})$, and for T the monad of indexing by the monoid $D = (\mathbf{1}, D)$. For the natural transformation, we take the natural associativity map of the monoidal structure of $\mathbf{DZ}$:

$$\Phi^\Sigma_A \quad := \quad \alpha^{-1}_{\Sigma,A,D} \quad : \quad \Sigma \otimes (A \otimes D) \quad \multimap \quad (\Sigma \otimes A) \otimes D$$

Proposition 4.13. *The family of maps $\Phi^\Sigma_A : \Sigma \otimes (A \otimes D) \multimap (\Sigma \otimes A) \otimes D$ forms a distributive law.*

Remark 4.14. The Kleisli category $\mathbf{Kl}(\Phi^\Sigma)$ of Φ^Σ is equivalent to the category DialZ_Σ. It allows to describe the homset $\mathsf{DialZ}_\Sigma[A, B]$ in the homset $\mathbf{DZ}[\Sigma \otimes (A \otimes D)\ ,\ \Sigma \otimes (B \otimes D)]$: A strategy $\sigma \in \mathsf{DialZ}_\Sigma[A, B]$ can be lifted to the strategy $\sigma^\uparrow \in \mathbf{DZ}[\Sigma \otimes (A \otimes D), \Sigma \otimes (B \otimes D)]$ defined as

$$\sigma^\uparrow \quad := \quad (\mathrm{id}_\Sigma \otimes (\mu_B \circ (\sigma \otimes \mathrm{id}_D) \circ \Phi^\Sigma_A)) \circ \delta_A$$

It moreover follows from the coherence laws of Φ^Σ and of D and Σ that $\sigma^\uparrow \circ \tau^\uparrow = (\sigma \circ \tau)^\uparrow$ and that $\mathrm{id}_A{}^\uparrow = \mathrm{id}_{\Sigma \otimes (A \otimes D)}$.

5. Fibrations of Tree Automata

We present here an adaptation of the fibrations $\mathsf{SAG}^{(\mathrm{W})}$ and $\mathsf{Aut}^{(\mathrm{W})}$ of [35]. They are based on the fibration DialZ of §4.

The fibrations $\mathsf{Aut}^{(\mathrm{W})}$ are made of finite objects only (but their morphisms can be arbitrary strategies), and can be seen as fibrations of tree automata. The fibrations $\mathsf{SAG}^{(\mathrm{W})}$ can be seen as the result of saturating $\mathsf{Aut}^{(\mathrm{W})}$ by precomposition of games with $\mathbf{T}$-maps, so that to incorporate infinite objects such as $\mathcal{A}(T)$ for an arbitrary labelled tree $T : D^* \to \Sigma$.

We begin with fibrations $\mathsf{DialAut}^{(\mathrm{W})}$ which generalize $\mathsf{SAG}^{(\mathrm{W})}$ so that to have existential and universal quantifications (in [35], existential quantification is restricted to $\mathsf{Aut}^{(\mathrm{W})}$). We then present $\mathsf{SAG}^{(\mathrm{W})}$ and $\mathsf{Aut}^{(\mathrm{W})}$ and finally discuss quantifiers.

5.1 The Fibred Categories $\mathsf{DialAut}$ and $\mathsf{DialAut}^{\mathrm{W}}$

Given an alphabet Σ the (fibre) category $\mathsf{DialAut}_\Sigma$ is defined as follows. Its objects are tuples $A = (Q_A, U, X, \alpha)$ where

$$\alpha \quad : \quad \bigcup_{n \in N}(\Sigma^n \times U^n \times X^n \times D^n) \quad \longrightarrow \quad Q_A$$

A morphism from $A = (Q_A, U, X, \alpha)$ to $B = (Q_B, V, Y, \beta)$ is a DialZ_Σ-strategy $\sigma : (U, X) \multimap (V, Y)$.

The categories $\mathsf{DialAut}_\Sigma$ are indexed over $\mathbf{T}$: It follows from Rem. 4.9 that a strategy $L \in \mathbf{T}[\Sigma, \Gamma]$ can be seen as a function

$$\overline{L} \quad : \quad \prod_{n \in \mathbb{N}}((D^n \times \Sigma^{n+1}) \to \Gamma^{n+1})$$

The action of L^* on $A = (Q_A, U, X, \alpha)$ is $(Q_A, U, X, L^*(\alpha))$ where $L^*(\alpha)(\varepsilon, \varepsilon, \varepsilon, \varepsilon) := \alpha(\varepsilon, \varepsilon, \varepsilon, \varepsilon)$ and

$$L^*(\alpha)(\overline{\mathsf{a}}, \overline{u}, \overline{x}, p.d) \quad := \quad \alpha(\overline{L}(p, \overline{\mathsf{a}}), \overline{u}, \overline{x}, p.d)$$

On maps, L^* takes $\sigma : A \to B$ to the action of L on σ in DialZ.

We get a strict indexed category, of which $\mathsf{da} : \mathsf{DialAut} \to \mathbf{T}$ is the Grothendieck completion. Explicitly, the objects of $\mathsf{DialAut}$ are pairs (Σ, A), where A is an object of $\mathsf{DialAut}_\Sigma$. Morphisms from (Σ, A) to (Γ, B) are pairs (L, σ) where $L \in \mathbf{T}[\Sigma, \Gamma]$ and σ is a $\mathsf{DialAut}_\Sigma$-map from A to $L^*(B)$. The functor $\mathsf{da} : \mathsf{DialAut} \to \mathbf{T}$ is the first projection.

Winning. We equip objects of $\mathsf{DialAut}_\Sigma$ with acceptance conditions, leading to winning conditions on games[5], and to the fibration $\mathsf{da}^{\mathrm{W}} : \mathsf{DialAut}^{\mathrm{W}} \to \mathbf{T}$. The objects of the fibre category $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$ are pairs (A, Ω_A) where A is an object of $\mathsf{DialAut}_\Sigma$ and $\Omega_A \subseteq Q^\omega_A$. The morphisms are defined as follows.

Given a $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$-object $A = ((Q_A, U, X, \alpha), \Omega_A)$, define the $\mathbf{DZ}$-object $\eth(A) := (\Sigma \times U, X \times D)$ (see Rem. 4.14). We equip $\eth(A)$ with the winning condition $\mathcal{W}_A$ consisting of the infinite plays ϖ in $\eth(A)$ such that $(\alpha(\partial(\varpi(0). \cdots .\varpi(2n))))_n \in \Omega_A$.[6]

Given now another $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$-object $B = ((Q_B, V, Y, \beta), \Omega_B)$, a $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$-map from A to B is a DialZ_Σ-strategy $\sigma : (U, X) \multimap (V, Y)$ whose lift $\sigma^\uparrow$ (see Rem. 4.14) is (total) winning on $\eth(A) \multimap \eth(B)$. Recall from Rem. 4.14 that $(-)^\uparrow$ preserves composition. It follows that $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$ is a category since the identity is (total) winning and since (total) winning strategies compose. Moreover,

Proposition 5.1. *Given $L \in \mathbf{T}[\Sigma, \Gamma]$, L^* restricts to a functor $\mathsf{DialAut}^{\mathrm{W}}_\Gamma \to \mathsf{DialAut}^{\mathrm{W}}_\Sigma$.*

Then define $\mathsf{da}^{\mathrm{W}} : \mathsf{DialAut}^{\mathrm{W}} \to \mathbf{T}$ by Grothendieck completion.

Symmetric Monoidal Closed Structure. The fibred symmetric monoidal closed structure of DialZ extends to $\mathsf{DialAut}^{(\mathrm{W})}$.

The unit is $\mathbf{I} := (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. Consider $\mathsf{DialAut}_\Sigma$ objects $A = (Q_A, U, X, \alpha)$ and $B = (Q_B, V, Y, \beta)$. Following the structure in DialZ, let

$$\begin{array}{lcl} A \otimes B & := & (\Sigma, Q, U \times V, X \times Y, \alpha \sqcap \beta) \\ A \multimap B & := & (\Sigma, Q, V^U \times X^{U \times Y \times D}, U \times Y, \alpha \sqsupset \beta) \end{array}$$

where $Q := Q_A \times Q_B$ and

$$\begin{array}{rcl} (\alpha \sqcap \beta)(\overline{\mathsf{a}}, \overline{(u, v)}, \overline{(x, y)}, p) & := & (\alpha(\overline{\mathsf{a}}, \overline{u}, \overline{x}, p)\ ,\ \beta(\overline{\mathsf{a}}, \overline{v}, \overline{y}, p)) \\ (\alpha \sqsupset \beta)(\overline{\mathsf{a}}, \overline{(f, F)}, \overline{(u, y)}, \overline{d}) & := & (q_A, q_B) \end{array}$$

where $q_A := \alpha(\overline{\mathsf{a}}, \overline{u}, \overline{F(u, d)}, \overline{d})$ and $q_B := \beta(\overline{\mathsf{a}}, \overline{f(v)}, \overline{y}, \overline{d})$.

On $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$-objects, define $\Omega_{A \otimes B}$ as in Def. 2.18 and $\Omega_{A \multimap B}$ as in Def. 3.1.

Proposition 5.2. *The fibrations $\mathsf{DialAut}^{(\mathrm{W})}$ are fibrewise monoidal closed.*

5.2 Automata and Substituted Acceptance Games

We now discuss how the fibrations $\mathsf{SAG}^{(\mathrm{W})}$ and $\mathsf{Aut}^{(\mathrm{W})}$ of [35] (adapted to the base category $\mathbf{T}$) embed into $\mathsf{DialAut}^{(\mathrm{W})}$.

[5] See App. A

[6] ∂ is defined in App. A.

The Fibred Categories $\mathsf{SAG}^{(\mathrm{W})}$**.** An object of the fibres $\mathsf{SAG}^{(\mathrm{W})}_{\Sigma}$ is a pair $(\mathcal{A}, M)$ of an automaton $\mathcal{A} : \Gamma$ and an $M \in \mathbf{T}[\Sigma, \Gamma]$. It leads to the $\mathsf{DialAut}_{\Sigma}$-object $\mathcal{A}(M) := (Q_{\mathcal{A}}, U, X, \alpha)$ where α is defined by induction as $\alpha(\varepsilon, \varepsilon, \varepsilon, \varepsilon) := q^{\imath}_{\mathcal{A}}$, and

$$\alpha(\overline{\mathsf{a}}.\mathsf{a}, \overline{u}.u, \overline{x}.x, p.d) \;:=\; \delta_{\mathcal{A}}(\alpha(\overline{\mathsf{a}}, \overline{u}, \overline{x}, p),\; M(p, \overline{\mathsf{a}}.\mathsf{a}),\; u,\; x,\; d)$$

where, by Rem. 4.9, M is seen as a map $\cup_{n\in\mathbb{N}}(D^n \times \Sigma^{n+1}) \to \Gamma$. The corresponding $\mathsf{DialAut}^{\mathrm{W}}_{\Sigma}$-object is $\mathcal{A}(M) := (\mathcal{A}(M), \Omega_{\mathcal{A}})$.

A fibre $\mathsf{SAG}^{(\mathrm{W})}_{\Sigma}$-morphism from $(\mathcal{A}, M)$ to $(\mathcal{B}, N)$ is then just a $\mathsf{DialAut}^{(\mathrm{W})}_{\Sigma}$-strategy

$$\sigma \quad : \quad \mathcal{A}(M) \quad \multimap \quad \mathcal{B}(N)$$

Given $L \in \mathbf{T}[\Gamma, \Sigma]$, the functor $L^* : \mathsf{SAG}^{(\mathrm{W})}_{\Sigma} \to \mathsf{SAG}^{(\mathrm{W})}_{\Gamma}$ maps $(\mathcal{A}, M)$ to $(\mathcal{A}, M \circ L)$, and acts on strategies as in $\mathsf{DialAut}^{(\mathrm{W})}$. This gives fibrations $\mathsf{sag}^{(\mathrm{W})} : \mathsf{SAG}^{(\mathrm{W})} \to \mathbf{T}$.

The Fibred Categories $\mathsf{Aut}^{(\mathrm{W})}$**.** The fibred categories $\mathsf{Aut}^{(\mathrm{W})}$ can be seen as the restrictions of $\mathsf{sag}^{(\mathrm{W})} : \mathsf{SAG}^{(\mathrm{W})} \to \mathbf{T}$ to automata and to indexing by *finite functions* between alphabets.

Write $\mathbf{A}$ for the category of non-empty finite sets: its objects are alphabets (Σ, Γ, etc) and its morphisms are functions $\mathtt{f} : \Sigma \to \Gamma$.

Note that $\mathtt{f} : \Sigma \to \Gamma$ induces (*via* Rem. 4.9) the $\mathbf{T}$-map $M_{\mathtt{f}} := \boldsymbol{\lambda}\_.\boldsymbol{\lambda}\overline{\mathsf{a}}.\boldsymbol{\lambda}\mathsf{a}.\mathtt{f}(\mathsf{a})$. The action of $M^*_{\mathtt{f}}$ on a SAG_{Γ}-object $\mathcal{A} = (\mathcal{A}, \mathrm{Id}_{\Gamma})$ gives the object $(\mathcal{A}, M_{\mathtt{f}})$, and thus the $\mathsf{DialAut}_{\Sigma}$-object $\mathcal{A}(M_{\mathtt{f}})$. The action of α can be internalized in $\mathcal{A}$. Since

$$\alpha(\overline{\mathsf{a}}.\mathsf{a}, \overline{u}.u, \overline{x}.x, p.d) \quad = \quad \delta_{\mathcal{A}}(\alpha(\overline{\mathsf{a}}, \overline{u}, \overline{x}, p)\,,\, \mathtt{f}(\mathsf{a})\,,\, u\,,\, x\,,\, d)$$

we can define an automaton $\mathcal{A}[\mathtt{f}]$ such that $\mathcal{A}[\mathtt{f}](\mathrm{Id}_{\Sigma}) = \mathcal{A}(M_{\mathtt{f}})$.

Definition 5.3. *Given $\mathcal{A} : \Gamma$ as in (1) and a function $\mathtt{f} : \Sigma \to \Gamma$, define the automaton $\mathcal{A}[\mathtt{f}] : \Sigma$ as*

$$\mathcal{A}[\mathtt{f}] \quad = \quad (Q_{\mathcal{A}}\,,\, q^{\imath}_{\mathcal{A}}\,,\, U\,,\, X\,,\, \delta_{\mathcal{A}[\mathtt{f}]}\,,\, \Omega_{\mathcal{A}})$$

where $\delta_{\mathcal{A}[\mathtt{f}]}(q, \mathsf{a}, u, x, d) := \delta_{\mathcal{A}}(q, \mathtt{f}(\mathsf{a}), u, x, d)$.

The categories $\mathsf{Aut}^{(\mathrm{W})}_{\Sigma}$ are then the full subcategories of $\mathsf{SAG}^{(\mathrm{W})}_{\Sigma}$ with automata $\mathcal{A} : \Sigma$ (seen as $(\mathcal{A}, \mathrm{Id}_{\Sigma})$) as objects. We thus get fibrations $\mathsf{aut}^{(\mathrm{W})} : \mathsf{Aut}^{(\mathrm{W})} \to \mathbf{A}$.

Symmetric Monoidal Closed Structure. The fibrewise symmetric monoidal closed structure of $\mathsf{Aut}^{(\mathrm{W})}$ is given by Def. 2.18 for the monoidal product (with unit $\mathbf{I}_{\Sigma}$ as in Ex. 2.4.(i)) and by Def. 3.1 for the closed structure.

This structure lifts to $\mathsf{SAG}^{(\mathrm{W})}$ as follows. First, given $\mathcal{A} : \Sigma$ and $\mathcal{B} : \Gamma$, define the automaton $\mathcal{A} \otimes \mathcal{B}$ over $\Sigma \times \Gamma$ as $\mathcal{A}[\mathsf{p}_{\Sigma}] \otimes \mathcal{B}[\mathsf{p}_{\Gamma}]$ (and similarly for $\mathcal{A} \multimap \mathcal{B}$), where $\mathsf{p}_{\Sigma} : \Sigma \times \Gamma \to \Sigma$ and p_{Γ} are projections (as in Ex. 2.11). The fibrewise symmetric monoidal closed structure of $\mathsf{SAG}^{(\mathrm{W})}$ is then issued from

$$(\mathcal{A}, M) \,\square\, (\mathcal{B}, N) \;:=\; (\mathcal{A} \,\square\, \mathcal{B})\langle M, N\rangle \qquad (\text{for } \square \in \{\otimes, \multimap\})$$

Note that $(\mathcal{A}, M) \,\square\, (\mathcal{B}, N) = \mathcal{A}(M) \,\square\, \mathcal{B}(N)$.

Proposition 5.4. *The fibrations $\mathsf{Aut}^{(\mathrm{W})}$ and $\mathsf{SAG}^{(\mathrm{W})}$ are fibrewise symmetric monoidal closed.*

5.3 Quantification

$\mathsf{DialAut}^{(\mathrm{W})}$ and $\mathsf{Aut}^{(\mathrm{W})}$ (but not $\mathsf{SAG}^{(\mathrm{W})}$) have existential and universal quantifications, based on that of DialZ (see §4.5).

In $\mathsf{DialAut}^{(\mathrm{W})}$, existential and universal quantifications are given by

$$\coprod_{\Sigma,\Gamma}(\Sigma \times \Gamma, (Q_{\mathcal{A}}, U, X, \alpha)) \;\;:=\;\; (\Sigma, (Q_{\mathcal{A}}, \Gamma \times U, X, \coprod_{\Sigma,\Gamma} \alpha))$$

$$\prod_{\Sigma,\Gamma}(\Sigma \times \Gamma, (Q_{\mathcal{A}}, U, X, \alpha)) \;\;:=\;\; (\Sigma, (Q_{\mathcal{A}}, U^{\Gamma}, \Gamma \times X, \prod_{\Sigma,\Gamma} \alpha))$$

where

$$\begin{array}{rcl} \coprod_{\Sigma,\Gamma}(\alpha)(\overline{\mathsf{a}}, \overline{(\mathsf{b}, u)}, \overline{x}, p) & := & \alpha(\overline{(\mathsf{a}, \mathsf{b})}, \overline{u}, \overline{x}, p) \\ \prod_{\Sigma,\Gamma}(\alpha)(\overline{\mathsf{a}}, \overline{f}, \overline{(\mathsf{b}, x)}, p) & := & \alpha(\overline{(\mathsf{a}, \mathsf{b})}, \overline{f(\mathsf{b})}, \overline{x}, p) \end{array}$$

The action on maps is inherited from DialZ. The Beck-Chevalley condition amounts, for $L \in \mathbf{T}[\Delta, \Sigma]$, to

$$\coprod_{\Delta,\Gamma}(L \times \mathrm{id})^*(\alpha) = L^*(\coprod_{\Sigma,\Gamma} \alpha) \qquad L^*(\prod_{\Sigma,\Gamma} \alpha) = \prod_{\Delta,\Gamma}(L \times \mathrm{id})^*(\alpha)$$

Quantifications in $\mathsf{Aut}^{(\mathrm{W})}$ are given by Def. 2.14 and Def. 3.5. Note that as $\mathsf{DialAut}$-objects, $\coprod_{\Sigma,\Gamma} \mathcal{A} = \exists_{\Gamma}\mathcal{A}$ and $\prod_{\Sigma,\Gamma} \mathcal{A} = \forall_{\Gamma}\mathcal{A}$.

Proposition 5.5. *The fibrations* $\mathsf{DialAut}^{(\mathrm{W})}$ *and* $\mathsf{Aut}^{(\mathrm{W})}$ *have existential and universal quantifications.*

Remark 5.6. The isomorphisms (4) and (7) of Rem. 2.17 and Rem. 3.6:

$$\begin{array}{rcl} (\exists_{\Gamma}\mathcal{A})(M) \;\multimap\; \mathcal{B}(N) & \simeq & \mathcal{A}(M \times \mathrm{Id}_{\Gamma}) \;\multimap\; \mathcal{B}(N \circ \mathsf{p}_{\Sigma}) \\ \mathcal{B}(N) \;\multimap\; (\forall_{\Gamma}\mathcal{A})(M) & \simeq & \mathcal{B}(N \circ \mathsf{p}_{\Sigma}) \;\multimap\; \mathcal{A}(M \times \mathrm{Id}_{\Gamma}) \end{array}$$

follow from the Beck-Chevalley conditions in $\mathsf{DialAut}^{\mathrm{W}}$ since

$$\begin{array}{rcccl} \coprod_{\Sigma,\Gamma} \mathcal{A}(M \times \mathrm{Id}_{\Gamma}) & = & M^*(\coprod_{\Delta,\Gamma} \mathcal{A}) & = & (\exists_{\Gamma}\mathcal{A})(M) \\ \prod_{\Sigma,\Gamma} \mathcal{A}(M \times \mathrm{Id}_{\Gamma}) & = & M^*(\prod_{\Delta,\Gamma} \mathcal{A}) & = & (\forall_{\Gamma}\mathcal{A})(M) \end{array}$$

6. A Deduction System for Automata

We now present a deduction system for tree automata. It allows to derive judgments of the form

$$M\,;\,\overline{\mathcal{A}} \vdash \mathcal{A}$$

where $\overline{\mathcal{A}} = \mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{A}$ have the same input alphabet Γ (say), and $M \in \mathbf{T}[\Sigma, \Gamma]$. The deduction system is interpreted in the fibration $\mathsf{SAG}^{\mathrm{W}}$ as follows: If $M\,;\,\overline{\mathcal{A}} \vdash \mathcal{A}$ is derivable, then there is a winning P-strategy in $\mathcal{A}_1(M) \otimes \ldots \otimes \mathcal{A}_n(M) \multimap \mathcal{A}(M)$.

Note that a game of the form $\mathcal{A}(M) \multimap \mathcal{B}(N)$ is represented as the judgment $\langle M, N\rangle\,;\,\mathcal{A}[\mathsf{p}] \vdash \mathcal{B}[\mathsf{q}]$ where p and q are suitable projections.

The rules are given in Figs. 6, 7, 8 and 9 (where $\mathcal{N}$ is non-deterministic). The properties of $\mathsf{SAG}^{\mathrm{W}}$ seen up to now allows to show the adequacy (existence of a winning strategy for derivable judgments) of the system made of the rules of Figs. 6, 7 and 8. The interpretation of the exponential rules of Fig. 9 is presented in §7.

Example 6.1.
(i) If $\mathcal{A}$ is non-deterministic, then using rules of Fig. 6 one can derive $\mathcal{A} \vdash \mathcal{A} \otimes \mathcal{A}$.
(ii) Continuing (i), if $\mathcal{B}$ is non-deterministic, then one can derive $\mathcal{B} \otimes (\mathcal{B} \multimap \mathcal{A}) \vdash \mathcal{A} \otimes \mathcal{B}$, so that by adequacy, P has a winning strategy on $\mathcal{B} \otimes (\mathcal{B} \multimap \mathcal{A}) \multimap \mathcal{A} \otimes \mathcal{B}$.
It then follows from Ex. 3.4 that if $\mathcal{A}$ and $\mathcal{B}$ are both non-deterministic and Borel with $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, then P has winning strategy on $(\mathcal{B} \multimap \mathcal{A}) \multimap \mathcal{B}^{\perp}$.

Remark 6.2 (Cut-Elmination). The rules of Fig. 6, Fig. 7 and Fig. 8 are compatible with cut-elimination (see e.g. [27]). For instance, the following two derivations are interpreted by the same strategy

$$\dfrac{\dfrac{\dfrac{\Delta_1}{\mathcal{A} \vdash \mathcal{B}}}{\mathbf{I} \vdash \mathcal{A} \multimap \mathcal{B}} \qquad \dfrac{\dfrac{\Delta_2}{\mathbf{I} \vdash \mathcal{A}} \qquad \overline{\mathcal{B} \vdash \mathcal{B}}}{\mathcal{A} \multimap \mathcal{B} \vdash \mathcal{B}}}{\mathbf{I} \vdash \mathcal{B}} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \Delta_1[\Delta_2/\mathcal{A}]\end{array}}{\mathbf{I} \vdash \mathcal{B}}$$

7. The Exponential Modality

We now discuss the interpretation of the exponential rules of Fig. 9, where $!(-)$ is given by an adaptation of Thm. 3.8 to our context.

Note that these rules are restricted to *regular* automata and that the usual weakening rule is part of the basic system (Fig. 6),

$$\frac{}{M \,;\, \overline{\mathcal{A}},\, \mathcal{A} \vdash \mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}} \vdash \mathcal{B} \qquad M \,;\, \overline{\mathcal{C}},\, \mathcal{B} \vdash \mathcal{C}}{M \,;\, \overline{\mathcal{A}},\, \overline{\mathcal{C}} \vdash \mathcal{C}}$$

$$\frac{M \,;\, \overline{\mathcal{A}},\, \mathcal{B},\, \mathcal{C} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}},\, \mathcal{B} \otimes \mathcal{C} \vdash \mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}} \vdash \mathcal{A} \qquad M \,;\, \overline{\mathcal{B}} \vdash \mathcal{B}}{M \,;\, \overline{\mathcal{A}},\, \overline{\mathcal{B}} \vdash \mathcal{A} \otimes \mathcal{B}}$$

$$\frac{M \,;\, \overline{\mathcal{B}} \vdash \mathcal{B} \qquad M \,;\, \overline{\mathcal{C}},\, \mathcal{C} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{B}}, \overline{\mathcal{C}}, \mathcal{B} \multimap \mathcal{C} \vdash \mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{C}}{M \,;\, \overline{\mathcal{A}} \vdash \mathcal{B} \multimap \mathcal{C}}$$

$$\frac{}{M \,;\, \overline{\mathcal{A}} \vdash \mathbf{I}} \qquad \frac{M \,;\, \overline{\mathcal{A}} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}},\, \mathcal{N},\, \mathcal{N} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}},\, \mathcal{N} \vdash \mathcal{A}}$$

Figure 6. Propositional rules (for $\mathcal{N}$ non-deterministic)

$$\frac{M \,;\, \overline{\mathcal{A}} \vdash \mathcal{A}}{M \circ N \,;\, \overline{\mathcal{A}} \vdash \mathcal{A}} \qquad \frac{M \times \mathrm{Id}_\Gamma \,;\, \overline{\mathcal{A}} \vdash \mathcal{A} \qquad \mathsf{a} \in \mathbf{A}[\Sigma, \Gamma]}{M \times \mathrm{Id}_\Sigma \,;\, \overline{\mathcal{A}[\mathsf{a}]} \vdash \mathcal{A}[\mathsf{a}]}$$

Figure 7. Substitution rules (where M and N are composable)

$$\frac{M \times \mathrm{Id}_\Gamma \,;\, \overline{\mathcal{A}[\mathsf{p}]},\, \mathcal{B} \vdash \mathcal{A}[\mathsf{p}]}{M \,;\, \overline{\mathcal{A}},\, \exists_\Gamma \mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \times N \,;\, \overline{\mathcal{A}} \vdash \mathcal{A}}{M \times N \,;\, \overline{\mathcal{A}} \vdash (\exists_\Gamma \mathcal{A})[\mathsf{p}]}$$

$$\frac{M \times N \,;\, \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{A}}{M \times N \,;\, \overline{\mathcal{A}},\, (\forall_\Gamma \mathcal{B})[\mathsf{p}] \vdash \mathcal{A}} \qquad \frac{M \times \mathrm{Id}_\Gamma \,;\, \overline{\mathcal{A}[\mathsf{p}]} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}} \vdash \forall_\Gamma A}$$

Figure 8. Quantification rules (where p is a suitable projection)

$$\frac{M \,;\, !\overline{\mathcal{A}} \vdash \mathcal{A}}{M \,;\, !\overline{\mathcal{A}} \vdash !\mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}},\, !\mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \,;\, \overline{\mathcal{A}},\, !\mathcal{B},\, !\mathcal{B} \vdash \mathcal{A}}{M \,;\, \overline{\mathcal{A}},\, !\mathcal{B} \vdash \mathcal{A}}$$

Figure 9. Exponentials (for regular automata)

thanks to Ex. 2.19.(i). The exponential automaton $!\mathcal{A}$ is a non-deterministic automaton obtained by an adaptation of a known construction [42].

The most difficult rule is the first one (*Promotion*). It relies on the existence of positional strategies in suitable games. Unfortunately, these strategies do not compose, so that the rules of Fig. 9 are not compatible with cut-elimination (in the sense of Rem. 6.2).

Example 7.1. The law of Peirce $!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \vdash ?\mathcal{A}$, (where $?\mathcal{A} = (!(\mathcal{A})^{\perp})^{\perp}$, see Ex 3.9. (i)) can be derived using the exponential rules.

Game Graphs and Positionality. Fix $\mathcal{A}(M) : \Sigma$ and $\mathcal{B}(N) : \Sigma$. The *game graph* of $\mathcal{A}(M) \multimap \mathcal{B}(N)$ is the graph G with vertices:

$$(A_{\mathsf{P}} \times B_{\mathsf{P}}) + (A_{\mathsf{O}} \times B_{\mathsf{P}}) + (A_{\mathsf{O}} \times B_{\mathsf{O}})$$

where

$$\begin{array}{lll@{\qquad}lll} A_{\mathsf{P}} & := & D^* \times \Sigma^* \times Q_{\mathcal{A}} & A_{\mathsf{O}} & := & D^* \times \Sigma^* \times Q_{\mathcal{A}} \times U \\ B_{\mathsf{P}} & := & D^* \times \Sigma^* \times Q_{\mathcal{B}} & B_{\mathsf{O}} & := & D^* \times \Sigma^* \times Q_{\mathcal{B}} \times V \end{array}$$

and edges as in Fig. 10, with $q'_{\mathcal{A}} := \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(\bar{\mathsf{a}}.\mathsf{a}, p), u, x, d)$ (for some $x \in X$) and $q'_{\mathcal{B}} := \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(\bar{\mathsf{a}}.\mathsf{a}, p), v, y, d)$ (for some $y \in Y$). Write pos for the graph morphism from the set of plays of the game $\mathcal{A}(M) \multimap \mathcal{B}(N)$ (seen as a tree) to G. We say that a

$$\begin{array}{c|ccc|c} & ((p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{A}}) & , & (p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{B}})) & A_{\mathsf{P}} \times B_{\mathsf{P}} \\ \mathsf{O} & \downarrow & & & \\ & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{B}})) & A_{\mathsf{O}} \times B_{\mathsf{P}} \\ \mathsf{P} & & & \downarrow & \\ & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{B}}\,,\,v)) & A_{\mathsf{O}} \times B_{\mathsf{O}} \\ \mathsf{O} & & & \downarrow & \\ & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{B}})) & A_{\mathsf{O}} \times B_{\mathsf{P}} \\ \mathsf{P} & \downarrow & & & \\ & ((p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{A}}) & , & (p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{B}})) & A_{\mathsf{P}} \times B_{\mathsf{P}} \end{array}$$

Figure 10. Edges of the graph G

strategy σ is *positional* if it agrees on plays with the same position, *i.e.* if $s.m \in \sigma$, $t.m' \in \sigma$ with $\mathsf{pos}(s) = \mathsf{pos}(t)$ implies $m = m'$.

Consider now parity automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{B}$. Then the winning condition of a game of the form $\mathcal{A}_1(M_1) \otimes \ldots \otimes \mathcal{A}_n(M_n) \multimap \mathcal{B}(N)$ is a disjunction of parity conditions, also called a *Rabin* condition, which is induced by colorings depending only on the vertices of its game graph G. It has been shown in [20–22, 43] that if P has a winning strategy σ in such a game, then he has a winning *positional* strategy (w.r.t. G), which according to [43] is recursive in σ.

Non-Determinization (or Simulation [30]). Our exponential construction $!(-)$ is an adapatation of the one used in the proof of Thm. 3.8 by [42]. Given a parity automaton $\mathcal{A} : \Sigma$, we let

$$!\mathcal{A} \quad := \quad (Q_{!\mathcal{A}}\,,\, q^{\imath}_{!\mathcal{A}}\,,\, U^{Q_{\mathcal{A}}}\,,\, \mathbf{1}\,,\, \delta_{!\mathcal{A}}\,,\, \Omega_{!\mathcal{A}})$$

where $Q_{!\mathcal{A}} := \mathcal{P}(Q_{\mathcal{A}} \times Q_{\mathcal{A}})$, $q^{\imath}_{!\mathcal{A}} := \{(q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{A}})\}$ and the transition function $\delta_{!\mathcal{A}}$ is defined as follows: Given $\mathsf{a} \in \Sigma$, $f \in U^{Q_{\mathcal{A}}}$, $d \in D$ and $S = \{(\_, q_1), \ldots, (\_, q_n)\} \in Q_{!\mathcal{A}}$, let

$$\delta_{!\mathcal{A}}(S, \mathsf{a}, f, \bullet, d) \quad := \quad T_1 \cup \cdots \cup T_n$$

where, for each $k \in \{1, \ldots, n\}$,

$$T_k \quad := \quad \{(q_k, q) \mid \exists x \in X.\ q = \delta_{\mathcal{A}}(q_k, \mathsf{a}, f(q_k), x, d)\}$$

Let a *trace* in an infinite sequence $(S_n)_n \in Q^{\omega}_{!\mathcal{A}}$ be a sequence $(q_n)_n$ such that for all n, $(q_n, q_{n+1}) \in S_{n+1}$. We let $\Omega_{!\mathcal{A}}$ be the set of sequences $(S_n)_n$ whose traces all belong to $\Omega_{\mathcal{A}}$. Note that $\Omega_{!\mathcal{A}}$ is ω-regular since $\Omega_{\mathcal{A}}$ is ω-regular.

If $\mathcal{A}$ is a regular automaton, let $!\mathcal{A} := !(\mathcal{A}^{\dagger})$ (see Ex. 2.10.(ii)).

Interpretation of the* '!' *Rules. We now discuss the interpretation of the rules of Fig. 9. The first rule (called *Promotion*) follows from:

Proposition 7.2. *If $\mathcal{N}$ is regular non-deterministic and $\mathcal{A}$ is regular, and if there is a winning P-strategy on $\mathcal{N}(L) \multimap \mathcal{A}(M)$ then there is a winning P-strategy on $\mathcal{N}(L) \multimap !\mathcal{A}(M)$.*

Prop. 7.2 relies on the existence of positional winning strategies in Rabin games. The second rule (called *Dereliction*) is given by:

Proposition 7.3. *If $\mathcal{A}$ is regular, there is a winning P-strategy $\epsilon_{\mathcal{A}(M)}$ on $!\mathcal{A}(M) \multimap \mathcal{A}(M)$.*

Corollary 7.4 (Thm. 3.8)**.** *$\mathcal{L}(\mathcal{A}) = \mathcal{L}(!\mathcal{A})$ for a regular $\mathcal{A}$.*

The last rule (*Contraction*) follows from Ex. 2.19.(ii) and the fact that $!\mathcal{A}$ is non-deterministic.

Weak Completeness. The exponentials give the following *weak completeness* property (see Ex. 3.9.(ii)).

Proposition 7.5 (Weak Completeness)**.** *Given regular automata $\mathcal{A}$ and $\mathcal{B}$ on Σ, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then there is an effective winning P-strategy on $!\mathcal{A} \multimap (!(\mathcal{B}^{\perp}))^{\perp}$.*

8. Conclusion

We proposed fibred monoidal closed categories of tree automata. They handle their basic constructs (closure under Boolean operations and equivalence with non-deterministic automata). Our model is based on games, which provide a realizability semantics for tree automata. Further work will include the interpretation of deduction systems for MSO.

A. Simple Games

Simple Games. Simple games are two-player games where the *Proponent* P ($\exists$loïse) and the *Opponent* O ($\forall$belard) play in turn moves from a specific set, producing sequences of moves which may be subject to specified rules. Formally, a simple game A has the form

$$A \quad = \quad (A^+, A^-, \xi_A, L_A)$$

where A^+ ($=: A_\mathsf{P}$) and A^- ($=: A_\mathsf{O}$) are resp. the sets of P-moves and O-moves, $\xi_A \in \{+,-\}$ is the *polarity* of A, and $L_A \subseteq \wp_A^{\xi_A}$ is a non-empty prefix-closed set of *legal plays*, where the sets $\wp_A^+$ and $\wp_A^-$ of positive and negative plays are given by

$$\wp_A^\xi \quad := \quad (A^\xi \cdot A^{-\xi})^* + (A^\xi \cdot A^{-\xi})^* \cdot A^\xi \qquad \text{for } \xi \in \{+,-\}$$

So P starts in a positive game and O starts in a negative one. We let $s, t, \dots$ range of over plays and $m, n, \dots$ range over moves. We write $\wp_A^{\mathsf{even}}$ for the set of even length plays in $\wp_A^{\xi_A}$. The *dual* of A is the game $\overline{A} := (A_\mathsf{O}, A_\mathsf{P}, -\xi_A, L_A)$.

A game A is *full* if $L_A = \wp_A^{\xi_A}$. We write

$$A \quad = \quad (U, X)$$

to denote a full positive game with $A_\mathsf{P} := U$ and $A_\mathsf{O} := X$. Note that there is a bijection

$$\partial \quad = \quad \langle \partial_U, \partial_X \rangle \quad : \quad \wp_A^{\mathsf{even}} \quad \longrightarrow \quad \cup_{n \in \mathbb{N}} (U^n \times X^n)$$

with $\partial(\varepsilon) = (\bullet, \bullet)$ and $\partial(s.u.x) = (\partial_U(s).u, \partial_X(s).x)$.

A play is a P-play (resp. an O-play) if it is either empty or ends with a P-move (resp. an O-move). A P-*strategy* σ is a non-empty set of legal P-plays which is

P-prefix-closed: if $s.t \in \sigma$ and s is a P-play then $s \in \sigma$, and

P-deterministic: if $s.n \in \sigma$ and $s.m \in \sigma$ then $n = m$.

Linear Arrow Games. Simple games form a category $\mathbf{SG}$, in which, given games A and B of the same polarity, the morphisms from A to B are P-strategies in the negative *linear arrow game*

$$A \multimap B \quad = \quad (B_\mathsf{P} + A_\mathsf{O}, B_\mathsf{O} + A_\mathsf{P}, -, L_{A \multimap B})$$

where $L_{A \multimap B} \subseteq \wp_{A \multimap B}^-$ consists of those negative plays s such that $s \restriction A \in L_A$ and $s \restriction B \in L_B$, where $s \restriction A$ is the restriction of s to $A_\mathsf{P} + A_\mathsf{O}$, and similarly for $s \restriction B$.

Note that the polarity of moves in component B is preserved while the polarity of moves in A is reversed. The plays of $A \multimap B$ start in component A iff A and B are both positive. Moreover, plays satisfy the *switching condition*: given $s.m.n \in L_{A \multimap B}$, with $n \in (A \multimap B)_\mathsf{O}$, then m and n are in the same component (*i.e.* only P is allowed to switch between A and B).

The Hyland-Schalk Functor [16]. There is a faithful functor $\mathrm{HS} : \mathbf{SG} \longrightarrow \mathbf{Rel}$. mapping a simple game to its set of legal plays, and a strategy $\sigma : A \multimap B$ to

$$\mathrm{HS}(\sigma) \quad := \quad \{(s \restriction A, s \restriction B) \mid s \in \sigma\} \quad \subseteq \quad L_A \times L_B$$

Hence strategies $\sigma : A \multimap B$ can be represented as spans

$$L_A \quad \longleftarrow \quad \mathrm{HS}(\sigma) \quad \longrightarrow \quad L_B$$

In particular, the identity strategy id_A is the unique strategy such that $\mathrm{HS}(\mathrm{id}_A) = L_A \times_{L_A} L_A$, where $L_A \times_{L_A} L_A$ is the pullback of the identity $L_A \to L_A$ with itself in $\mathbf{Set}$.

Zig-Zag Strategy Given simple games A and B, a strategy $\sigma : A \multimap B$ is a *zig-zag strategy* if for all play $s \in \sigma$, the restrictions $s \restriction A$ and $s \restriction B$ have the same length. It is easy to check that zig-zag strategies form a categories.

Note that synchronous strategies are zig-zag strategies. A zig-zag strategy $\sigma : A \multimap B$ always has to switch component (recall that O can never switch). In particular, if A and B are positive, then the plays of σ have the same shape as those of Fig. 4 (left).

Totality. Given games A and B, a strategy $s : A \multimap B$ is *total* if given $s \in \sigma$, if $s.n$ is legal then $s.n.m \in \sigma$ for some move m.

It is easy to see that if $\sigma : A \multimap B$ and $\tau : B \multimap C$ are both zig-zag and total, then $\tau \circ \sigma$ is zig-zag and total.

Indeed, consider $(s, t) \in \mathrm{HS}(\tau \circ \sigma) = \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$, and u such that $(s, u) \in \mathrm{HS}(\sigma)$ and $(u, t) \in \mathrm{HS}(\tau)$. Given a legal $(A \multimap C)_\mathsf{O}$-move m in (say) component A, since σ is zig-zag and total, there is some n such that $(s.m, u.n) \in \mathrm{HS}(\sigma)$. Since $n \in B_\mathsf{P} \subseteq (B \multimap C)_\mathsf{O}$, and since τ is zig-zag and total, there is some $r \in C_\mathsf{P}$ such that $(u.n, t.r) \in \mathrm{HS}(\tau)$, from which it follows that $(s.m, t.r) \in \mathrm{HS}(\tau \circ \sigma)$. The case of $m \in C_\mathsf{O}$ is similar.

Since identity strategies are total, it follows that simple games and total zig-zag strategies form a category.

Winning. Simple games can be equipped with *winning conditions*, which are infinite sequences of moves.

It is well-known (see e.g. [1, 14]), that total and winning strategies compose and form a category. The case of zig-zag strategies is particularly simple. Given $(A, \mathcal{W}_A)$ and $(B, \mathcal{W}_B)$, a total zig-zag strategy $\sigma : A \multimap B$ is *winning* if for all infinite sequences of moves ϖ such that $\varpi(0). \cdots .\varpi(n) \in \sigma$ for infinitely many n, $\varpi \restriction A \in \mathcal{W}_A$ implies $\varpi \restriction B \in \mathcal{W}_B$.

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B. Proofs of §4 (Simple Zig-Zag Games)

In this appendix we give the proofs of §4. The monoidal *closed* structure is proven directly here. The other parts (concerning the symmetric monoidal structure, as well as §4.3, §4.4 and §4.6) are based on an interpretation of $\mathbf{DZ}$ as a subcategory of the simple self-dualization $\mathbf{G}(\mathscr{S})$ of the topos of trees $\mathscr{S}$. The construction of simple self dualization is presented in §F. The representation of zig-zag strategies in $\mathbf{G}(\mathscr{S})$ is presented in §G. Some material on monoidal categories is recalled in §J.

B.1 The Monoidal Structure of DZ

The monoidal structure of $\mathbf{DZ}$ is given by the following data, (where $A = (U, X)$, $B = (V, Y)$ and $C = (W, Z)$):

$$A \otimes B \quad := \quad (U \times V, X \times Y) \qquad \text{with unit} \quad \mathbf{I} := (\mathbf{1}, \mathbf{1})$$

and the natural structure maps:

$$\begin{array}{c|ccc|c}
 & (A \otimes B) \otimes C & \overset{\alpha_{A,B,C}}{\multimap} & A \otimes (B \otimes C) & \\
\hline
\mathsf{O} & ((u, v), w) & & & \\
 & & & (u, (v, w)) & \mathsf{P} \\
 & & & (x, (y, z)) & \mathsf{O} \\
\mathsf{P} & ((x, y), z) & & &
\end{array}$$

$$\begin{array}{c|ccc|c}
 & \mathbf{I} \otimes A & \overset{\lambda_A}{\multimap} & A & \\
\hline
\mathsf{O} & (\bullet, u) & & & \\
 & & & u & \mathsf{P} \\
 & & & x & \mathsf{O} \\
\mathsf{P} & (\bullet, x) & & &
\end{array}
\qquad
\begin{array}{c|ccc|c}
 & A \otimes \mathbf{I} & \overset{\rho_A}{\multimap} & A & \\
\hline
\mathsf{O} & (u, \bullet) & & & \\
 & & & u & \mathsf{P} \\
 & & & x & \mathsf{O} \\
\mathsf{P} & (x, \bullet) & & &
\end{array}
\qquad
\begin{array}{c|ccc|c}
 & A \otimes B & \overset{\gamma_{A,B}}{\multimap} & B \otimes A & \\
\hline
\mathsf{O} & (u, v) & & & \\
 & & & (v, u) & \mathsf{P} \\
 & & & (y, x) & \mathsf{O} \\
\mathsf{P} & (x, y) & & &
\end{array}$$

The following is shown in Prop. G.5 using a represention of total zig-zag strategies in the topos of trees.

Proposition B.1 (Prop. 4.2)**.** *The category $\mathbf{DZ}$ equipped with the above data is symmetric monoidal.*

B.2 The Monoidal Closed Structure of DZ

Proposition B.2 (Prop. 4.3)**.** *The category $\mathbf{DZ}$ is symmetric monoidal closed.*

We rely on the faithfulness of HS : $\mathbf{SG} \to \mathbf{Rel}$ (see [16], but also Lemma 4.6 in the Appendix of the long version of [35].[7]

Recall from e.g. [27] that a symmetric monoidal category $\mathbb{C}$ is *closed* if for every object A, the functor $A \otimes (-)$ has a right adjoint $(-)^A$. Since $A \otimes (-)$ is already a functor, according to [24, Thm. IV.1.2] it is sufficient to show that for every object C there is an object C^A and map

$$\mathsf{eval}_C \quad : \quad A \otimes C^A \quad \longrightarrow \quad C$$

such that for every $f : A \otimes B \to C$ there is a unique $\mathbf{\Lambda}(f) : B \to C^A$ such that

$$\begin{tikzcd}
A \otimes C^A \arrow[r, "\mathsf{eval}_C"] & C \\
A \otimes B \arrow[u, dotted, "\mathrm{id}_A \otimes \mathbf{\Lambda}(f)"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof of Prop. B.2. Let $A = (U, X)$, $B = (V, Y)$ and $C = (W, Z)$. Recall that $A \multimap_{\mathbf{DZ}} C = (W^U \times X^{U \times Z}, U \times Z)$. We define the total zig-zag strategy $\mathsf{eval}_C : A \otimes (A \multimap_{\mathbf{DZ}} C) \multimap C$ as follows:

$$\begin{array}{c|ccc|c}
 & A \otimes (A \multimap_{\mathbf{DZ}} C) & \overset{\mathsf{eval}_C}{\multimap} & C & \\
\hline
\mathsf{O} & (u, (f, F)) & & & \\
 & & & f(u) & \mathsf{P} \\
 & & & z & \mathsf{O} \\
\mathsf{P} & (F(u, z), (u, z)) & & &
\end{array}$$

Given any $\tau' : B \multimap (A \multimap_{\mathbf{DZ}} C)$, the composition $\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau')$ is given by:

$$\begin{array}{c|ccccc|c}
 & A \otimes B & \overset{\mathrm{id}_A \otimes \tau'}{\multimap} & A \otimes (A \multimap_{\mathbf{DZ}} C) & \overset{\mathsf{eval}_C}{\multimap} & C & \\
\hline
\mathsf{O} & (u, v) & & & & & \\
 & & & (u, (f', F')) & & & \\
 & & & & & f'(u) & \mathsf{P} \\
 & & & & & z & \mathsf{O} \\
 & & & (F'(u, z), (u, z)) & & & \\
\mathsf{P} & (F'(u, z), y') & & & & &
\end{array}$$

It follows that $\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau') = \mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau'')$ implies $\tau' = \tau''$.

[7] Available at `https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf`.

We show this by induction on pairs of even-length plays $(s,t) \in \wp_A^{\mathsf{even}} \times \wp_{A \multimap_{\mathbf{DZ}} C}^{\mathsf{even}}$. Assume toward a contradiction that for some such $(s,t) \in \mathrm{HS}(\tau') \cap \mathrm{HS}(\tau'')$, for some $v \in V$ we have $(s.v, t.(f', F')) \in \mathrm{HS}(\tau')$ and $(s.v, t.(f'', F'')) \in \mathrm{HS}(\tau'')$ with $f' \neq f''$. Then for some $u \in U$, we have say $f'(u) \neq f''(u)$. Then, for some r we have

$$\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau') \quad \ni \quad r.(u,v).f'(u) \quad \neq \quad r.(u,v).f''(u) \quad \in \quad \mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau'')$$

Hence a contradiction. The case of $F' \neq F''$ is dealt-with similarly.

Fix now some total zig-zag $\sigma : A \otimes B \multimap C$.

We define $\tau = \mathbf{\Lambda}(\sigma) : B \multimap (A \multimap_{\mathbf{DZ}} C)$ by induction on plays. To each $(s,t) \in \mathrm{HS}(\tau)$, with s and t even-length, we associate $(s', t') \in \mathrm{HS}(\sigma)$, with s' and t' of the same length, and such that, for $(\overline{v}, \overline{y}) = \partial(s)$ and $((\overline{f}, \overline{F}), (\overline{u}, \overline{z})) = \partial(t)$, we have $\partial(s') = ((\overline{u}, \overline{v}), (\overline{F}(\overline{u}, \overline{z}), \overline{y}))$ and $\partial(t') = (\overline{f}(\overline{u}), \overline{z})$, where we take the pointwise application of sequences of functions and the map ∂ is defined in App. A.

For the base case, we put $(\varepsilon, \varepsilon) \in \mathrm{HS}(\tau)$, and associate it to $(\varepsilon, \varepsilon) \in \mathrm{HS}(\sigma)$.

Assume now $(s,t) \in \mathrm{HS}(\tau)$, associated to $(s', t') \in \mathrm{HS}(\sigma)$. For each $v \in V$, we define the functions $f_v : U \to W$ and $F_v : U \times Z \to X$ as follows: given $u \in U$, let w such that $(s'.(u,v), t'.w) \in \mathrm{HS}(\sigma)$, and for each $z \in Z$, let x and $y_{u,z}$ such that $(s'.(u,v).(x, y_{u,z}), t'.w.z) \in \mathrm{HS}(\sigma)$. We then let $f_v(u) := w$ and $F_v(u,z) := x$. We now let $(s.v.y_{u,z}, t.(f_v, F_v).(u,z)) \in \mathrm{HS}(\tau)$, and associate it to $(s'.(u,v).(x, y_{u,z}), t'.w.z) = (s'.(u,v).(F_v(u,z), y_{u,z}), t'.f_v(u).z)$ so that the invariant is satisfied.

This conclude the definition of τ.

It then follows from the invariant that we indeed have $\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau = \sigma$.

First note that the map $(s,t) \in \mathrm{HS}(\tau) \mapsto (s', t') \in \mathrm{HS}(\sigma)$ is surjective. The property then follows from the fact that $(s,t) \in \mathrm{HS}(\tau)$ iff $(s', t') \in \mathrm{HS}(\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau)$. This is shown by induction on pairs of plays $(s,t) \in \wp_B^{\mathsf{even}} \times \wp_{A \multimap_{\mathbf{DZ}} C}^{\mathsf{even}}$. The base case is trivial. For the induction step, given such $(s.v.y_{u,z}, t.(f_v, F_v).(u,z))$, we have $(s.v.y_{u,z}, t.(f_v, F_v).(u,z)) \in \mathrm{HS}(\tau)$ if and only if $(s'.(u,v).(F_v(u,z), y_{u,z}), t'.f)v(u).z) \in \mathrm{HS}(\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau)$.

This concludes the proof of Prop. B.2. □

B.3 D-Synchronicity (§4.3)

The following is shown in Prop. G.6.(i) using a represention of total zig-zag strategies in the topos of trees.

Proposition B.3 (Prop. 4.4). *In $\mathbf{DZ}$, the object $D := (\mathbf{1}, D)$ is a commutative monoid with structure:*

	$\mathbf{I}$	$\stackrel{u}{\multimap}$	D	
O	$\bullet$			
			$\bullet$	P
			d	O
P	$\bullet$			

	$D \otimes D$	$\stackrel{m}{\multimap}$	D	
O	$(\bullet, \bullet)$			
			$\bullet$	P
			d	O
P	(d, d)			

Proposition B.4 (Prop. 4.5). *$\mathbf{DZ}_D$ is symmetric monoidal closed.*

Proof. The *closed* structure is presented in §4.3. The symmetric monoidal structure, which follows from the fact that the monad D is (lax) symmetric monoidal, is given by Cor. J.11 applied to Prop. B.3. □

B.4 Comonoid Indexing in $\mathbf{DZ}_D$ (§4.4.2)

The following is shown in Prop. G.7 using a represention of total zig-zag strategies in the topos of trees.

Proposition B.5 (Prop. 4.6). *In $\mathbf{DZ}_D$, each object $\Sigma = (\Sigma, \mathbf{1})$ is a commutative monoid with structure:*

	Σ	$\stackrel{e_\Sigma}{\multimap}$	$\mathbf{I}$	
O	a			
			$\bullet$	P
			d	O
P	$\bullet$			

	Σ	$\stackrel{d_\Sigma}{\multimap}$	$\Sigma \otimes \Sigma$	
O	a			
			(a, a)	P
			d	O
P	$\bullet$			

The proof of the following is dual to that of Prop. B.4. We apply Cor. J.13 instead of Cor. J.11 and Prop. B.5 instead of Prop. B.3.

Proposition B.6 (Prop. 4.7). *DialZ_Σ is symmetric monoidal closed.*

B.5 The Fibred Category DialZ (§4.4.3)

The following is shown in Prop. G.8.

Proposition B.7 (Prop. 4.10). *The category $\mathbf{T}$ embeds to $\mathbf{Comon}(\mathbf{DZ}_D)$ via the functor $\mathsf{E}_{\mathbf{T}}$ mapping an object Σ of $\mathbf{T}$ to the comonoid $(\Sigma, e_\Sigma, d_\Sigma)$ and a morphism $M : \mathbf{T}[\Gamma, \Sigma]$ to itself.*

Proposition B.8 (Prop. 4.11). *$\mathsf{dz} : \mathsf{DialZ} \to \mathbf{T}$ is symmetric monoidal closed.*

Proof. For the symmetric monoidal structure, we use the known fact that change-of-base of fibrations preserves fibrewise structure (see e.g. [18, Lem. 8.4.1]). Given a symmetric monoidal category $\mathbb{C}$, the slice categories of $\int \mathsf{CI}(\mathbb{C})$ are the Kleisli categories for the (oplax) symmetric monoidal comonads of comonoid indexing. So the argument is the same as for Prop. B.6 above. In order to show that $\mathsf{s}_{\mathsf{CI}}(\mathbb{C}) : \int \mathsf{CI}(\mathbb{C}) \to \mathbf{Comon}(\mathbb{C})$ is fibrewise symmetric monoidal, we have to show that substitution functors are strong symmetric monoidal. Given a commonoid morphism $u : K \to L$ in $\mathbf{Comon}(\mathbb{C})$, the substitution functor u^* is the identity on objects, so the strength is made of identities. It remains to show that the required diagrams commute (see §J.1.1), which amounts to

$$u^*(\alpha^{\mathbf{Kl}(L)}) = \alpha^{\mathbf{Kl}(K)} \qquad u^*(\rho^{\mathbf{Kl}(L)}) = \rho^{\mathbf{Kl}(K)} \qquad u^*(\lambda^{\mathbf{Kl}(L)}) = \lambda^{\mathbf{Kl}(K)} \qquad u^*(\gamma^{\mathbf{Kl}(L)}) = \gamma^{\mathbf{Kl}(K)}$$

where $\alpha^{\mathbf{Kl}(-)}$, $\rho^{\mathbf{Kl}(-)}$, $\lambda^{\mathbf{Kl}(-)}$ and $\gamma^{\mathbf{Kl}(-)}$ are the symmetric monoidal structure maps of $\mathbf{Kl}(-)$. But by §J.3.8 each of these maps $f^{\mathbf{Kl}(-)}$ is $f \circ \lambda \circ (e \otimes \mathrm{id})$ (where f is the corresponding map of $\mathbb{C}$), so that in $\mathbb{C}$:

$$u^*(f^{\mathbf{Kl}(L)}) \quad = \quad f \circ \lambda \circ (e \otimes \mathrm{id}) \circ (u \otimes \mathrm{id})$$

and we are done since $e \circ u = e$ as u is a comonoid morphism (see §J.3.4).

The argument is the same for the fibrewise symmetric monoidal *closed* structure of DialZ, since the closed structure of the fibre DialZ_Σ is directly lifted by the comonad Σ from the closed structure of $\mathbf{DZ}_D$. $\square$

B.6 The Distributive Law of Comonoid over Monoid Indexing (§4.6)

Proposition B.9 (Prop. 4.12)**.** *In $\mathbf{DZ}$, the objects $\Sigma = (\Sigma, \mathbf{1})$ can be equipped with a commutative comonoid structure $(\tilde{e}_\Sigma, \tilde{d}_\Sigma)$ such that $e_\Sigma = \mathsf{F}_D(\tilde{e}_\Sigma)$ and $d_\Sigma = \mathsf{F}_D(\tilde{e}_\Sigma)$.*

Proof. This is the construction of Prop. J.9 applied to Prop. G.6.(ii). $\square$

The following is given by Prop. J.14.

Proposition B.10 (Prop. 4.13)**.** *The family of maps $\Phi^\Sigma_A : \Sigma \otimes (A \otimes D) \multimap (\Sigma \otimes A) \otimes D$ forms a distributive law.*

Note that the $\mathbf{DZ}$-monoids $\Sigma = (\Sigma, \mathbf{1})$ on which Prop. B.10 is based are directly obtained from Prop. G.6.(ii).

Remark B.11 (Rem. 4.14). The fact that the Kleisli category $\mathbf{Kl}(\Phi^\Sigma)$ of Φ^Σ is equivalent to the Kleisli category DialZ_Σ of the lift Σ_D of the comonad Σ is well-known and stated in Prop. J.5.

C. Proofs of §2, §3 and §5

C.1 Proofs of §5 (Fibrations of Tree Automata)

Proposition C.1 (Prop. 5.1)**.** *Given $L \in \mathbf{T}[\Sigma, \Gamma]$, L^* restricts to a functor $\mathsf{DialAut}^{\mathrm{W}}_\Gamma \to \mathsf{DialAut}^{\mathrm{W}}_\Sigma$.*

Proof. Assume given a $\mathsf{DialAut}^{\mathrm{W}}_\Gamma$-strategy $\sigma : A \to B$ where $A = ((Q_A, U, X, \alpha), \Omega_A)$ and $B = ((Q_B, V, Y, \beta), \Omega_B)$. First, note that $\sigma : (\Sigma \times U, X) \multimap (V, Y \times D)$ in $\mathbf{DZ}$, so that $\sigma^\uparrow : (\Sigma \times U, X \times D) \multimap (\Sigma \times V, Y \times D)$ (see Rem. 4.14).

Consider an infinite play ϖ of $(L^*(\sigma))^\uparrow$. Reasonning as in App. 7.2 & 8.2 of the long version of [35][8] the play ϖ can be mapped to an infinite play ϖ' of σ, where the sequences of characters $\bar{\mathsf{b}} \in \Gamma$ provided by O in the game $\partial(A) \multimap \partial(B)$ are the image under L of the input characters $\bar{\mathsf{a}} \in \Sigma$ (from corresponding tree positions $p \in D^*$) provided by O in the game $\partial(L^*(A)) \multimap \partial(L^*(B))$. It follows that the infinite sequences in $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ produced by ϖ and ϖ' are the same, so that ϖ is P-winning iff ϖ' is P-winning. Hence ϖ is P-winning since σ is winning. $\square$

Proposition C.2 (Prop. 5.2)**.** *The fibrations $\mathsf{da}^{(\mathrm{W})} : \mathsf{DialAut}^{(\mathrm{W})} \to \mathbf{T}$ are fibrewise symmetric monoidal closed.*

Proof. The case of $\mathsf{DialAut}$ follows from Prop. B.8 and fact that the operations $\alpha \sqcap \beta$ and $\alpha \sqsupset \beta$ are preserved by substitution. For $\mathsf{DialAut}^{\mathrm{W}}$, note in addition that all the symmetric monoidal structure maps as well as the evaluation map are (total) winning, and that the Currying map $\mathbf{\Lambda}(-)$ preserves (total) winning strategies. $\square$

C.2 Proofs of §2 (A Curry-Howard Approach to Tree Automata)

Proposition C.3 (Prop. 2.12)**.** *Given $\mathcal{A} : \Sigma$ and $\mathcal{B} : \Sigma$, if there is a winning P-strategy σ in $\mathcal{A} \multimap \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.*

Proof. Let $T : D^* \to \Sigma$. Since σ is a $\mathsf{DialAut}^{\mathrm{W}}_\Sigma$-map, it follows from Prop. C.1 that $T^*(\sigma)$ is a $\mathsf{DialAut}^{\mathrm{W}}_{\mathbf{1}}$-map from $\mathcal{A}(T)$ to $\mathcal{B}(T)$.

Now, if $T \in \mathcal{L}(\mathcal{A})$, then there is a $\mathsf{DialAut}^{\mathrm{W}}_{\mathbf{1}}$-strategy τ in $\mathbf{I_1} \multimap \mathcal{A}(T)$. It follows from App. A that $T^*(\sigma) \circ \tau$ is winning on $\mathbf{I_1} \multimap \mathcal{B}(T)$, hence that $T \in \mathcal{L}(\mathcal{B})$. $\square$

C.3 Proofs of §3 (A Dialectica-Like Approach to Automata)

Proposition C.4 (Prop. 2.20)**.** $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.

Proof. The inclusion $(\subseteq)$ follows using the projections $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ and $\mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$.

For the other direction, using Prop. C.2, tensor σ winning on $\mathbf{I_1} \multimap \mathcal{A}(T)$ with τ winning on $\mathbf{I_1} \multimap \mathcal{B}(T)$ and then precompose with a monoidal unit map. $\square$

Proposition C.5 (Prop. 3.3)**.** *If $\Omega_{\mathcal{A}}$ is Borel, then $T \in \mathcal{L}(\mathcal{A}^\perp)$ iff $T \notin \mathcal{L}(\mathcal{A})$.*

[8] Available at https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf.

Proof. The argument is an adaptation of [42]. By Martin's Theorem [26], it is equivalent to show that P wins the game $\mathcal{A}^{\perp}(T)$ iff O wins $\mathcal{A}(T)$, where, using the notions of §A, an O-strategy is just a P-strategy on the dual game.

For ($\Rightarrow$), assuming given a winning P-strat σ on $\mathcal{A}(T) \multimap \perp$, we build a winning O-strat τ in $\mathcal{A}(T)$. The strategy τ is build by induction on plays. To each play t of τ, we associate a play s of σ such that if t leads to state $q_{\mathcal{A}}$, then s leads to state $(q_{\mathcal{A}}, \mathbb{f})$. In the base case, both t and s are the empty plays, and the invariant is respected. For the induction step, assume that P plays u from t in $\mathcal{A}(T)$. Let (f, F) be the move of σ from s. We then let τ answer the pair $(F(u, f(u)), f(u))$ from $s.u$, and $\mathcal{A}$ goes to state $q'_{\mathcal{A}}$. In $\mathcal{A}(T) \multimap \perp$, we let O play the pair $(f(u), u)$. Then $\mathcal{A} \multimap \perp$ goes to state $(q'_{\mathcal{A}}, \mathbb{f})$ and the invariant is respected. Since σ is winning and $\mathcal{A} \multimap \perp$ stays in states of the form $(\_, \mathbb{f})$ the infinite sequence of states produced in $\mathcal{A}(T)$ is rejecting, as required.

For the conversion direction, assuming given a winning O-strat τ on $\mathcal{A}(T)$, we build a winning P-strat σ in $\mathcal{A}(T) \multimap \perp$. The strategy σ is build by induction on plays as long as $\mathcal{A} \multimap \perp$ stays in states of the form $(\_, \mathbb{f})$ (if it switches to $(\_, \mathbb{t})$ then P trivially wins). So to each play s of σ which leads to state $(q_{\mathcal{A}}, \mathbb{f})$, we associate a play t of τ which leads to state $q_{\mathcal{A}}$. The base case is trivial. For the induction step, we build (f, F) from σ as follows: to each u, σ associates (from t) a pair (x, d). We let $F(u, \_) := d$ and $f(u) := x$. Assume then that from $s.(f, F)$, O plays some (u, d). If $d \neq f(u)$ then we are done. Otherwise, $\mathcal{A} \multimap \perp$ switches to $(q'_{\mathcal{A}}, \mathbb{f})$. We then let P play u from t, so that by construction τ answers $(F(u, \_), d)$, and $\mathcal{A}$ goes to state $q'_{\mathcal{A}}$. But then, since τ is winning for O, the sequence of $\mathcal{A}$-states is rejecting, so that P wins in $\mathcal{A}(T) \multimap \perp$, as required. ☐

The following proposition contains an effective strengthening of the part of Ex. 3.4. It asserts the existence of a winning P-strategy on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ as soon as $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic regular automata over Σ such that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

Proposition C.6 (Ex. 3.4)**.** *Given non-deterministic Borel $\mathcal{A} : \Sigma$ and $\mathcal{B} : \Sigma$, if $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, then there are winning* P*-strategies in $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ and $\mathcal{A} \multimap \mathcal{B}^{\perp}$. If moreover $\mathcal{A}$ and $\mathcal{B}$ are regular then the* P*-strategies can be assumed to be regular.*

The effectiveness part of the statment can be seen to follow from Ex. 3.7.(ii). It is nevertheless interesting to note how the strategy can be effectivelly computed in this particular case.

Proof. Since $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, we have $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \emptyset$ by Prop. C.4. Since $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic, so is $\mathcal{A} \otimes \mathcal{B}$. It then follows from Prop. 2.15that $\mathcal{L}(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B})) = \emptyset$, hence, by Prop. C.5 that the automaton $(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp} : \mathbf{1}$ accepts the unique tree $\mathbf{1} : D^* \to \mathbf{1}$. But winning P-strategies in $(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp}(\mathbf{1})$ can be lifted to winning P-strategies in

$$\mathbf{I_1} \quad \multimap \quad (\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp}(\mathbf{1})$$

But note that since $(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp} : \mathbf{1}$, that game is actually the same as the game

$$\mathbf{I_1} \quad \multimap \quad (\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp}$$

It then follows from Prop. C.2 that there is a winning P-strategy in the game

$$\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}) \quad \multimap \quad \perp$$

and therefore by Prop. 5.5 that there is a winning P-strategy on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ and therefore also in $\mathcal{A} \multimap \mathcal{B}^{\perp}$.

If the automata $\mathcal{A}$ and $\mathcal{B}$ are regular, then the automaton $(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp}$ is regular. It is therefore effectivelly equivalent to a parity automaton (see Ex. 2.10.(ii)). It is then well-known (see e.g. [40, Thm. 6.18]) that there is effectivelly a regular winning P-strategy in the acceptance game $(\exists_{\Sigma}(\mathcal{A} \otimes \mathcal{B}))^{\perp}(\mathbf{1})$. It is easy to see that this strategy is lifted (as above) to regular winning P-strategies in $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ and $\mathcal{A} \multimap \mathcal{B}^{\perp}$. ☐

Proposition C.7 (Ex. 3.7.(iii))**.** $\mathcal{L}(\mathcal{B} \multimap \mathcal{A}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$.

Proof. The inclusion $\mathcal{L}(\mathcal{B} \multimap \mathcal{A}) \subseteq \mathcal{L}(\mathcal{C})$ follows from the fact that since the games $(\mathcal{B} \multimap \mathcal{A})(T)$ and $\mathcal{C}(T)$ have the same moves, any P-strategy σ in $(\mathcal{B} \multimap \mathcal{A})(T)$ is also a P-strategy in $\mathcal{C}(T)$. Moreover, if σ is winning on $(\mathcal{B} \multimap \mathcal{A})(T)$ then it is also winning on $\mathcal{C}(T)$, because the only possibility for a play to produce different state sequences on $(\mathcal{B} \multimap \mathcal{A})$ and $\mathcal{C}$ (modulo the projection π used in the definition of $\Omega_{\mathcal{C}}$) is that $\mathcal{C}$ switches to state $\mathbb{t}$, which is an accepting trap.

For the inclusion $\mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$, consider a winning P-strategy σ in $\mathcal{C}(T)$. Recall that the P-moves of $\mathcal{B}^{\perp}$ are D^V and that its O-moves are V, and that the P-moves of $\mathcal{C}$ are U^V and that its O-moves are V. Recall also from see Ex. 3.7.(iii), that there is a winning P-strategy τ on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ (whose P-moves are D and O-moves are $U \times V$). We define a P-strategy θ by combining σ and τ as follows: modulo Currying, θ plays from $v \in V$ the tree direction $d \in D$ proposed by $T^*(\tau)$ from v and the $u \in U$ given by σ on v. Hence the strategies σ and θ play the same moves in $\mathcal{B}$ (provided by O). So the sequences of $Q_{\mathcal{B}}$-states produced by σ and θ are the same, unless O plays in $\mathcal{B}^{\perp}$ a tree direction $d \in D$ different from the one proposed by θ, *i.e.* different from the one proposed by τ. In this case, the play on $\mathcal{B}^{\perp}(T)$ is P-winning and we are done. Assume now that the sequences of $Q_{\mathcal{B}}$-states agree. We claim that they can not be in $\Omega_{\mathcal{B}}$: The play respects σ, so the sequence of $Q_{\mathcal{A}}$-states must belong to $\Omega_{\mathcal{A}}$ since σ is winning. But the play also respects $T^*(\tau)$, which is winning in $\mathcal{A}(T) \otimes \mathcal{B}(T) \multimap \perp$, so the sequence of $Q_{\mathcal{A}}$-states can not belong to $\Omega_{\mathcal{A}}$. It follows that the sequence of $Q_{\mathcal{B}}$-states can not belong to $\Omega_{\mathcal{B}}$, and we are done since the play in $\mathcal{B}^{\perp}(T)$ is then P-winning. ☐

D. Proofs of §6 (A Deduction System for Automata)

We give here a justification to the fact that if

$$M \,;\, \mathcal{A}_1 \,,\, \ldots \,,\, \mathcal{A}_n \vdash \mathcal{A}$$

is derivable using the rules of Figs. 6, 7 and 8, then there is a winning P-strategy σ in

$$\mathcal{A}_1(M) \otimes \ldots \otimes \mathcal{A}_n(M) \vdash \mathcal{A}(M)$$

The justification for derivations involving the exponential rules of Fig. 9 is given in §7 and Appendix E.

The proof is as usual by induction on the derivations and by cases on the last applied rules.

- The rules of Fig. 6 follow from the facts that $\mathsf{SAG}^{\mathrm{W}}_{\Sigma}$ are categories, moreover equipped with a symmetric monoidal closed structure (Prop. 5.4) and from Ex. 2.19.
- The rules of Fig. 7 follow from the facts that $\mathsf{SAG}^{\mathrm{W}}$ is fibred over $\mathbf{T}$, and from the internalization of $\mathbf{A}$-maps in automata (Def. 5.3).
- The rules of Fig. 8 follow from the adjunctions $\exists_{(-)} \dashv \mathsf{p}^* \dashv \forall_{(-)}$ (Prop. 5.5) and from Rem. 5.6 (consequence of the adjunctions together with Beck-Chevalley).

E. Proofs of §7 (The Exponential Modality)

Proposition E.1 (Ex. 7.1)**.** *The law of Peirce $!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \vdash ?\mathcal{A}$, (where $?\mathcal{A} = (!\mathcal{A}^{\perp})^{\perp}$, see Ex 3.9. (i)) can be derived using the exponential rules.*

Proof. We can derive

$$!\mathcal{A}^{\perp}\,,\,?\mathcal{A} \vdash \perp$$

so that (since $?\mathcal{B} = (!\mathcal{B}^{\perp})^{\perp}$)

$$!\mathcal{A}^{\perp}\,,\,?\mathcal{A} \vdash ?\mathcal{B}$$

from which follows that

$$!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A})\,,\,!\mathcal{A}^{\perp} \vdash ?\mathcal{A}$$

and thus

$$!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A})\,,\,!\mathcal{A}^{\perp} \vdash \perp$$

and we are done since $?\mathcal{A} = (!\mathcal{A}^{\perp})^{\perp}$. □

Proposition E.2 (Prop. 7.2)**.** *If $\mathcal{N}$ is regular non-deterministic and $\mathcal{A}$ is regular, and if there is a winning P-strategy on $\mathcal{N}(L) \multimap \mathcal{A}(M)$ then there is a winning P-strategy on $\mathcal{N}(L) \multimap\, !\mathcal{A}(M)$.*

Proof. Note that we can assume $\mathcal{N}$ and $\mathcal{A}$ to be parity automata. Write G for the game graph of $\mathcal{N}(L) \multimap \mathcal{A}(M)$. Thanks to [20–22, 43], that there is a positional (w.r.t. G) winning P-strategy σ on $\mathcal{N}(L) \multimap \mathcal{A}(M)$.

We build a winning P-strategy τ on $\mathcal{N}(L) \multimap\, !\mathcal{A}(M)$ such that the following invariant is satisfied:

- To each play t of τ with $\mathsf{pos}(t) = ((p, \bar{\mathsf{a}}, q_{\mathcal{N}})\,,\,(p, \bar{\mathsf{a}}, S))$ where $S = \{(\_, q_1), \dots, (\_, q_n)\}$, we associate a set $E(t) = \{s_1, \dots, s_n\}$ of plays of σ, with $\mathsf{pos}(s_i) = ((p, \bar{\mathsf{a}}, q_{\mathcal{N}})\,,\,(p, \bar{\mathsf{a}}, q_i))$,
- and if moreover t' extends t and is such that $\mathsf{pos}(t') = ((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_{\mathcal{N}})\,,\,(p.d, \bar{\mathsf{a}}.\mathsf{a}, S'))$ then each for all $s' \in E(t')$ there is some $s \in E(t)$ such that s' extends s.

The strategy τ is build by induction on plays as follows:

- For the base case (initial position ε), we have by definition $S = \{(q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{A}})\}$ and $E(\varepsilon) = \{q^{\imath}_{\mathcal{A}}\}$.
- For the inductive step, let t with $\mathsf{pos}(t) = ((p, \bar{\mathsf{a}}, q_{\mathcal{N}})\,,\,(p, \bar{\mathsf{a}}, S))$ and let O play from t some (a, v) in component $\mathcal{N}(L)$ of $\mathcal{N}(L) \multimap\, !\mathcal{A}$. For $s_i \in E(t)$, let u_i be the move of σ from position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\,,\,(p, \bar{\mathsf{a}}, q_i))$ (thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\,,\,(p, \bar{\mathsf{a}}.\mathsf{a}, q_i, u_i))$). This defines a map $h_{t.(\mathsf{a},v)} : Q_{\mathcal{A}} \to U$ taking q_i to u_i (the definition of $h_{t.(\mathsf{a},v)}$ on irrelevant q's is arbitrary), and we let τ play $h_{t.(\mathsf{a},v)}$ in the component $!\mathcal{A}(M)$ of $\mathcal{N}(L) \multimap\, !\mathcal{A}(M)$, thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\,,\,(p, \bar{\mathsf{a}}.\mathsf{a}, S, h_{t.(\mathsf{a},v)}))$. Then if O answers some $d \in D$ in the component $!\mathcal{A}(M)$, and we let P play $\bullet$ in the component $\mathcal{N}(L)$ (recall that both $!\mathcal{A}$ and $\mathcal{N}$ are non-deterministic), the current position in $\mathcal{N}(L) \multimap\, !\mathcal{A}(M)$ becomes $((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_{\mathcal{N}})\,,\,(p.d, \bar{\mathsf{a}}.\mathsf{a}, S')))$ where

  $$q'_{\mathcal{N}} \quad := \quad \delta_{\mathcal{N}}(q_{\mathcal{N}}, L(\bar{\mathsf{a}}.\mathsf{a}, p), v, \bullet, d) \qquad \text{and} \qquad S' \quad := \quad \delta_{!\mathcal{A}}(S, M(\bar{\mathsf{a}}.\mathsf{a}, p), h_{t.(\mathsf{a},v)}, \bullet, d)$$

  Let

  $$t' \quad := \quad t.(\mathsf{a}, v).h_{t.(\mathsf{a},v)}.d.\bullet$$

  and write $S' = \{(\_, q'_1), \dots, (\_, q'_m)\}$. By definition of $!\mathcal{A}$, each q'_j is $\delta_{\mathcal{A}}(q_{i_j}, M(\bar{\mathsf{a}}.\mathsf{a}, p), u_{i_j}, x_j, d)$ for fome i_j and some x_j (note that there might be several such i_j and x_j, but we select one). For each j, we let O play (x_j, d) in the component $\mathcal{A}(M)$ of $\mathcal{N}(L) \multimap \mathcal{A}(M)$ from position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\,,\,(p, \bar{\mathsf{a}}.\mathsf{a}, q_{i_j}, u_{i_j}))$ thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\,,\,(p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_j))$. We then let P answer $\bullet$ in the component $\mathcal{N}(L)$, thus leading to position $((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_{\mathcal{N}})\,,\,(p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_j))$.

  We finally put

  $$E(t') \quad := \quad \{s_{i_0}.(\mathsf{a}, v).u_{i_0}.(x_0, d).\bullet\;,\;\cdots\;,\;s_{i_m}.(\mathsf{a}, v).u_{i_m}.(x_m, d).\bullet\}$$

This completes the definition of τ.

We now show that τ is winning. Consider an infinite play $(t_i)_{i \in \mathbb{N}}$ of τ, and let $(q_n, S_n)_{n \in \mathbb{N}}$ be the associated sequence of states in $(Q_{\mathcal{N}} \times Q_{!\mathcal{A}})^{\omega}$. Assume that $(q_n)_n \in \Omega_{\mathcal{N}}$. We show that $(S_n)_n \in \Omega_{!\mathcal{A}}$. Let $(q'_n)_n$ be a trace in $(S_n)_n$, so that $(q'_n, q'_{n+1}) \in S_{n+1}$. We have to show that $(q'_n)_n \in \Omega_{\mathcal{A}}$. Note that for all $n \in \mathbb{N}$,

$$\mathsf{pos}(t_{4n}) \quad = \quad ((p_n, \bar{\mathsf{a}}_n, q_n)\,,\,(p_n, \bar{\mathsf{a}}_n, S_n))$$

By construction, for each $n \in \mathbb{N}$ there are $s_n \in E(t_{4n})$ and $s'_n \in E(t_{4(n+1)})$ such that s'_n extends s_n:

$$s'_n \quad = \quad s_n.(\mathsf{a}_n, v_n).u_n.d_n.\bullet \qquad \text{where} \qquad \overline{\mathsf{a}}_{n+1} = \overline{\mathsf{a}}_n.\mathsf{a}_n \qquad \text{and} \qquad p_{n+1} = p_n.d_n$$

and such that moreover

$$\mathsf{pos}(s_n) \quad = \quad ((p_n, \overline{\mathsf{a}}_n, q_n)\,,\,(p_n, \overline{\mathsf{a}}_n, q'_n)) \qquad \text{and} \qquad \mathsf{pos}(s'_n) \quad = \quad ((p_{n+1}, \overline{\mathsf{a}}_{n+1}, q_{n+1})\,,\,(p_{n+1}, \overline{\mathsf{a}}_{n+1}, q'_{n+1}))$$

so that

$$\mathsf{pos}(s'_n) \quad = \quad \mathsf{pos}(s_{n+1})$$

Since σ is positional, it follows that the infinite sequence

$$\varpi \quad := \quad \varepsilon.(\mathsf{a}_0, v_0).u_0.d_0.\cdots.p_n.(\mathsf{a}_n, v_n).u_n.d_n.\cdots$$

is an infinite play of σ. Since ϖ produces the sequence of states $(q_n, q'_n)_n \in (Q_{\mathcal{N}} \times Q_{\mathcal{A}})^\omega$, we get $(q'_n)_n \in \Omega_{\mathcal{A}}$ since $(q_n)_n \in \Omega_{\mathcal{N}}$ by assumption. $\square$

Proposition E.3 (Prop. 7.3)**.** *If $\mathcal{A}$ is regular, there is a winning* P*-strategy ϵ on $!\mathcal{A}(N) \multimap \mathcal{A}(N)$.*

Proof. Note that we can assume $\mathcal{A}$ to be a parity automaton. We define $\mathrm{HS}(\epsilon)$ by induction on plays as follows, with the following invariant: for each $(s,t) \in \mathrm{HS}(\epsilon)$, with s,t of even length, writing q for the state of t and S for the state of s, we have $q \in S{\restriction}2$.

The base case is trivial. Let $(s,t) \in \mathrm{HS}(\epsilon)$ with s and t even-length, and with t in state q and s in state S. Given an O-move (a,h), we let $(s.(a,h), t.h(q)) \in \mathrm{HS}(\epsilon)$, and for all (x,d) we further let $(s.(a,h).(\bullet,d), t.h(q).(x,d)) \in \mathrm{HS}(\epsilon)$. Then the invariant is insured by definition of $!\mathcal{A}$.

The strategy τ is winning since the sequence of states produced in $\mathcal{A}$ is a trace in the sequence of states produced in $!\mathcal{A}$. $\square$

Proposition E.4 (Weak Completeness – Prop. 7.5)**.** *Given regular eutomata $\mathcal{A}$ and $\mathcal{B}$ on Σ, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then there is an effective winning* P*-strategy on $!\mathcal{A} \multimap (!(\mathcal{B}^\perp))^\perp$.*

Proof. By Cor. 7.4, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then $\mathcal{L}(!\mathcal{A}) \cap \mathcal{L}(!(\mathcal{B}^\perp)) = \emptyset$, and we conclude by Prop. C.6. $\square$

F. Simple Self Dualization

In this appendix, we present some aspects of the construction called *simple self dualization* in [17]. We begin by basic definitions and facts, and then give a general method to construct (lax) symmetric monoidal monads and oplax symmetric monoidal comonads in this setting, which will be used later on in §G to explain the monoidal structure of $\mathbf{DZ}$.

F.1 Some Basic Definitions and Facts

We recall here some basic material about Dialectica-like categories from [8, 17]. Given a category $\mathbb{C}$, its *simple self-dualization* is $\mathbf{G}(\mathbb{C}) := \mathbb{C} \times \mathbb{C}^{\mathrm{op}}$ (also written $\mathbb{C}^{\mathrm{d}}$ in [17]). Its objects are pairs U, X of objects of $\mathbb{C}$, and a morphism from (U,X) to (V,Y) is given by a pair of maps (f,F), denoted

$$(f,F) \quad : \quad (U,X) \quad \longrightarrow\!\!\!\!\!+ \quad (V,Y)$$

where $f : U \to V$ and $F : Y \to X$. If $\mathbb{C}$ is symmetric monoidal, then $\mathbf{G}(\mathbb{C})$ is an instance of a *Girard* category, in the sense of de Paiva [8, 17].

Assume that $\mathbb{C}$ is symmetric monoidal closed w.r.t. $(\otimes, \mathbf{I})$. Then $\mathbf{G}(\mathbb{C})$ is symmetric monoidal closed w.r.t.

$$(U,X) \otimes_{\mathbf{G}} (V,Y) \;:=\; (U \otimes V, X^V \otimes Y^U) \qquad \text{with unit } (\mathbf{I}, \mathbf{I})$$

The linear exponentials are given by

$$(U,X) \multimap_{\mathbf{G}} (V,Y) \quad := \quad (V^U \times X^Y\,,\, U \times Y)$$

Assume now that the monoidal structure $(\otimes, \mathbf{I}) = (\times, \mathbf{1})$ of $\mathbb{C}$ is Cartesian. Then $\mathbf{G}(\mathbb{C})$ can be equipped with a comonad (T, ϵ, δ) where the action on objects of T is

$$T(U,X) \quad := \quad (U, X^U)$$

and the maps ϵ and δ are given by

$$\begin{array}{lclcl} (f_\epsilon, F_\epsilon) & : & (U, X^U) & \longrightarrow\!\!\!\!\!+ & (U,X) \\ (f_\delta, F_\delta) & : & (U, X^U) & \longrightarrow\!\!\!\!\!+ & (U, X^{U \times U}) \end{array}$$

where $f_\epsilon = f_\delta = \mathrm{id}_U$, $F_\epsilon(u,x) = x$ and $F_\delta(h,u) = h(u,u)$ (see e.g. [8, Def. 15, §4.2]).

The co-Kleiseli category $\mathbf{D}(\mathbb{C}) := \mathbf{Kl}(T)$ is a Dialectica category in the sense of [8, 15] (see e.g. [8, Prop. 52, §4.3]). Explicitely, its objects are pairs $A = (U,X)$ of objects of $\mathbb{C}$, and a map from A to (V,Y) is a $\mathbf{G}(\mathbb{C})$-morphism (f,F) from TA to (V,Y), that is

$$(f,F) \quad : \quad (U, X^U) \quad \longrightarrow\!\!\!\!\!+ \quad (V,Y)$$

$\mathbf{D}(\mathbb{C})$ is symmetric monoidal closed w.r.t. the product

$$(U,X) \otimes (V,Y) \;:=\; (U \times V, X \times Y) \qquad \text{with unit } (\mathbf{1}, \mathbf{1})$$

Note that with $A = (U,X)$ and $B = (V,Y)$,

$$\begin{array}{lcl} T(A \otimes B) & = & (U \times V, (X \times Y)^{U \times V}) \\ & \simeq & (U \times V, X^{U^V} \times Y^{V^U}) \\ & = & TA \otimes_{\mathbf{G}} TB \end{array}$$

The linear exponentials of $\mathbf{D}(\mathbb{C})$ are given by

$$(U,X) \multimap (V,Y) \quad := \quad (V^U \times X^{U \times Y}\ ,\ U \times Y)$$

Note that $A \multimap B \simeq TA \multimap_{\mathbf{G}} B$, so the monoidal closure of $\mathbf{D}(\mathbb{C})$ actually follows from that of $\mathbf{G}(\mathbb{C})$:

$$\begin{array}{rcl} \mathbf{D}(\mathbb{C})[A \otimes B, C] & = & \mathbf{G}(\mathbb{C})[T(A \otimes B), C] \\ & \simeq & \mathbf{G}(\mathbb{C})[TA \otimes_{\mathbf{G}} TB, C] \\ & \simeq & \mathbf{G}(\mathbb{C})[TA, TB \multimap_{\mathbf{G}} C] \\ & \simeq & \mathbf{D}(\mathbb{C})[A, B \multimap C] \end{array}$$

F.2 Self Duality

The category $\mathbf{G}(\mathbb{C})$ is equipped with an isomorphism

$$(-)^{\perp} \quad : \quad \mathbf{G}(\mathbb{C}) \quad \xrightarrow{\simeq} \quad \mathbf{G}(\mathbb{C})^{\mathrm{op}}$$

mapping the $\mathbf{G}(\mathbb{C})$-object (U,X) to (X,U) and taking $(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!+\;\; (V,Y)$ to $(F,f) : (X,U) \longrightarrow_{\mathbf{G}(\mathbb{C})^{\mathrm{op}}} (Y,V)$ (that is $(F,f) : (Y,V) \longrightarrow\!\!\!\!\!\!\!+\;\; (X,U)$). Note that $(-)^{\perp}$ is a strict involution: $\mathbf{G}(\mathbb{C})^{\perp\perp} = \mathbf{G}(\mathbb{C})$.

F.3 Monoidal Structure

Consider an SMC $\mathbb{C}$. Note that $\mathbb{C}^{\mathrm{op}}$ is also an SMC, and recall from §F.1 the tensor product $\otimes$ of $\mathbf{G}(\mathbb{C})$ given by

$$(U,X) \otimes (V,Y) \quad = \quad (U \otimes V, X \otimes Y) \qquad \text{with unit} \quad \mathbf{I} = (\mathbf{I}, \mathbf{I})$$

Assuming the following structure maps of $\mathbb{C}$

$$\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \qquad \lambda : \mathbf{I} \otimes A \longrightarrow A \qquad \rho : A \otimes \mathbf{I} \longrightarrow A \qquad \gamma : A \otimes B \longrightarrow B \otimes A$$

the structure maps of $(\mathbf{G}(\mathbb{C}), \otimes, \mathbf{I})$ are given by:

$$\begin{array}{rclcl} \alpha & := & (\alpha, \alpha^{-1}) & : & ((U,X) \otimes (V,Y)) \otimes (W,Z) \longrightarrow (U,X) \otimes ((V,Y) \otimes (W,Z)) \\ \lambda & := & (\lambda, \lambda^{-1}) & : & (\mathbf{I}, \mathbf{I}) \otimes (U,X) \longrightarrow (U,X) \\ \rho & := & (\rho, \rho^{-1}) & : & (U,X) \otimes (\mathbf{I}, \mathbf{I}) \longrightarrow (U,X) \\ \gamma & := & (\gamma, \gamma^{-1}) & : & (U,X) \otimes (V,Y) \longrightarrow (V,Y) \otimes (U,X) \end{array}$$

Proposition F.1 ([17]). *Equipped with the above data, the category $\mathbf{G}(\mathbb{C})$ is symmetric monoidal.*

F.4 (Commutative) Monoids

Proposition F.2. *Consider an SMC $\mathbb{C}$. Given a comutative monoid (M,u,m) and a commutative comonoid (K,e,d) in $\mathbb{C}$, the $\mathbf{G}(\mathbb{C})$-object (M,K) is a commutative monoid in $\mathbf{G}(\mathbb{C})$ with structure maps*

$$\begin{array}{rclcl} u_{(M,K)} & := & (u,e) & : & (\mathbf{I}, \mathbf{I}) \longrightarrow\!\!\!\!\!\!\!+\;\; (M,K) \\ m_{(M,K)} & := & (m,d) & : & (M \otimes M, K \otimes K) \longrightarrow\!\!\!\!\!\!\!+\;\; (M,K) \end{array}$$

Proof. The proof is trivial since (1) commutation of the required diagrams amounts to componentwise commutation of the corresponding diagrams in $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$, and (2) the second components of commutative monoids diagrams in $\mathbf{G}(\mathbb{C})$ are commutative comonoids diagrams in $\mathbb{C}^{\mathrm{op}}$. □

F.5 (Commutative) Comonoids

Recall that a (commutative) comonoid in a category is a (commutative) monoid in the oppostive category. Since $\mathbf{G}(\mathbb{C})^{\mathrm{op}} \simeq \mathbf{G}(\mathbb{C})^{\perp}$, it follows that Prop. F.2 dualizes to:

Corollary F.3. *Consider an SMC $\mathbb{C}$. Given a comonoid (K,e,d) and a monoid (M,u,m) in $\mathbb{C}$, the $\mathbf{G}(\mathbb{C})$-object (K,M) is a commutative comonoid in $\mathbf{G}(\mathbb{C})$ with structure maps*

$$\begin{array}{rclcl} e_{(K,M)} & := & (e,u) & : & (K,M) \longrightarrow\!\!\!\!\!\!\!+\;\; (\mathbf{I}, \mathbf{I}) \\ d_{(M,K)} & := & (d,m) & : & (K,M) \longrightarrow\!\!\!\!\!\!\!+\;\; (K \otimes K, M \otimes M) \end{array}$$

F.6 A (Lax) Symmetric Monoidal Monad

Assume now that $\mathbb{C}$ is Cartesian closed, and fix a functor $H : \mathbb{C} \to \mathbb{C}$. Recall (from e.g. [27, §5.2]) that H lifts in a unique way to an oplax symmetric monoidal functor, with strength

$$\mathsf{t}^2_{A,B} \ := \ \langle H(\pi_1), H(\pi_2) \rangle \quad : \quad H(A \times B) \longrightarrow HA \times HB \qquad \text{and} \qquad \mathsf{t}^0 \ := \ \mathbf{1}_{H\mathbf{1}} \quad : \quad H\mathbf{1} \longrightarrow \mathbf{1}$$

Note that the naturality of $\mathsf{t}^2_{(-),(-)}$, that is

$$(H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2) \rangle \quad = \quad \langle H(\pi_1), H(\pi_2) \rangle \circ H(f \times g)$$

follows from the universality property of the Cartesian product since (say)

$$\pi_1 \circ (H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2) \rangle \quad = \quad H(f \circ \pi_1) \quad = \quad H(\pi_1 \circ (f \times g))$$

Consider now the functor

$$(-)^H \quad : \quad \mathbf{G}(\mathbb{C}) \quad \longrightarrow \quad \mathbf{G}(\mathbb{C})$$

defined as

$$(U,X)^H \quad := \quad (U^{HX},X) \qquad \text{and} \qquad (f,F)^H \quad := \quad (\boldsymbol{\lambda} h.f\circ h\circ H(F)\ ,\ F) \quad : \quad (U^{HX},X) \twoheadrightarrow (V^{HY},Y)$$

(where $(f,F):(U,X)\twoheadrightarrow(V,Y)$), and the maps

$$\begin{array}{lllllllll} \eta_{(U,X)} & = & (f_\eta,F_\eta) & := & (\boldsymbol{\lambda} u.\boldsymbol{\lambda} \_.u\ ,\ \mathrm{id}_X) & : & (U,X) & \twoheadrightarrow & (U^{HX},X) \\ \mu_{(U,X)} & = & (f_\mu,F_\mu) & := & (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)\ ,\ \mathrm{id}_X) & : & (U^{HX\times HX},X) & \twoheadrightarrow & (U^{HX},X) \end{array}$$

Proposition F.4. *$((-)^H,\eta,\mu)$ is a (lax) symmetric monoidal monad, with strength*

$$m^2_{A,B} \quad = \quad (f^2_{A,B},F^2_{A,B}) \quad := \quad (\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathrm{t}^2_{X,Y}\ ,\ \mathrm{id}_{X\times Y}) \quad : \quad (U^{HX}\times V^{HY}\ ,\ X\times Y) \quad \longrightarrow \quad ((U\times V)^{H(X\times Y)}\ ,\ X\times Y)$$

(where $A=(U,X)$ and $B=(V,Y)$), and

$$m^0 \quad := \quad (\mathbf{1},\mathbf{1}) \quad : \quad (\mathbf{1},\mathbf{1}) \quad \longrightarrow \quad (\mathbf{1}^{H\mathbf{1}},\mathbf{1})$$

The proof of Prop. F.4 is defered to §H.

F.7 An Oplax Symmetric Monoidal Comonad

Proposition F.4 can be dualized thanks to the self duality $\mathbf{G}(\mathbb{C})^{\mathrm{op}}=\mathbf{G}(\mathbb{C})^\perp$:

Corollary F.5. *Assume $\mathbb{C}$ is a CCC and $H:\mathbb{C}\to\mathbb{C}$ is a funtor. Then $((-)_H,\epsilon,\delta)$ is an oplax symmetric monoidal comonad on $\mathbb{C}$, where*

$$(U,X)_H \quad := \quad (U,X^{HU}) \qquad \textit{and} \qquad (f,F)_H \quad := \quad (f\ ,\ \boldsymbol{\lambda} h.F\circ h\circ H(f)) \quad : \quad (U,X^{HU}) \twoheadrightarrow (V,Y^{HV})$$

(for $(f,F):(U,X)\twoheadrightarrow(V,Y)$), and

$$\begin{array}{lllllllll} \epsilon_{(U,X)} & = & (f_\epsilon,F_\epsilon) & := & (\mathrm{id}_U\ ,\ \boldsymbol{\lambda} x.\boldsymbol{\lambda}\_.x) & : & (U,X^{HU}) & \twoheadrightarrow & (U,X) \\ \delta_{(U,X)} & = & (f_\delta,F_\delta) & := & (\mathrm{id}_U\ ,\ \boldsymbol{\lambda} h.\boldsymbol{\lambda} u.h(u,u)) & : & (U,X^{HU}) & \twoheadrightarrow & (U,X^{HU\times HU}) \end{array}$$

and where the oplax strength of $(-)_H$ is given by

$$n^2_{A,B} \quad = \quad (f^2_{A,B},F^2_{A,B}) \quad := \quad (\mathrm{id}_{U\times V}\ ,\ \boldsymbol{\lambda}(h,k).(h\times k)\circ \mathrm{t}^2_{U,V}) \quad : \quad (U\times V\ ,\ (X\times Y)^{H(U\times V)}) \quad \longrightarrow \quad (U\times V\ ,\ X^{HU}\times Y^{HV})$$

where $A=(U,X)$, $B=(V,Y)$ and $\mathrm{t}^2_{U,V}$ is defined as in F.6, and

$$n^0 \quad := \quad (\mathbf{1},\mathbf{1}) \quad : \quad (\mathbf{1},\mathbf{1}^{H\mathbf{1}}) \quad \longrightarrow \quad (\mathbf{1},\mathbf{1})$$

G. A Dialectica-Like Interpretation of Zig-Zag Strategies

We give here a Dialectica-like presentation of total zig-zag strategies $\sigma : A \multimap B$ for A and B positive full games. It relies on a distributive law ζ in an instance of Dialectica called *simple self-dualization* in [17]. We will perform it in the topos of trees $\mathscr{S}$.

We first instantiate the constructions and results of §H to the case of $\mathbf{G}(\mathscr{S})$. We then show in §G.4 that the category $\mathbf{DZ}$ of simple zig-zag games can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$. In §G.5 we present the distributive law ζ based on the constructions of §H. Finally, using the fact that $\mathbf{DZ}$ can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$ described as the Kleisli category of the distributive law ζ, we discuss the monoidal structure of $\mathbf{DZ}$ and $\mathbf{DZ}_D$.

G.1 The Topos of Trees

The *topos of trees* $\mathscr{S}$ is the presheaf category over the order $(\mathbb{N},\leq)$ seen as a category, see e.g. [6].

An object X of $\mathscr{S}$ is given by a family of sets $(X_n)_{n\in\mathbb{N}}$ equipped with *restriction maps* $r^X_n : X_{n+1}\to X_n$. A morphism f from X to Y is a family of functions $f_n : X_n \to Y_n$ compatible with restriction: $r^Y_n\circ f_{n+1} = f_n\circ r^X_n$.

As a topos, $\mathscr{S}$ is Cartesian closed w.r.t. to the Cartesian product of presheaves, which is given by $(X\times Y)_n := X_n\times Y_n$. Exponentials are defined as usual for presheaves (see e.g. [25]) by

$$(X^Y)_n \quad := \quad \mathrm{Nat}[\mathbb{N}[-,n]\times Y, X]$$

Explicitly, $(X^Y)_n$ consists of sequences $(\xi_k : Y_k \to X_k)_{k\leq n}$ which are compatible with r^X and r^Y. The restriction map of X^Y takes $(\xi_k)_{k\leq n+1}\in (X^Y)_{n+1}$ to $(\xi_k)_{k\leq n}\in(X^Y)_n$.

We will use the functor $\blacktriangleright : \mathscr{S}\to\mathscr{S}$ of [6]. On objects, it maps X to $((\blacktriangleright X)_n)_{n\in\mathbb{N}}$ where $(\blacktriangleright X)_{n+1} := X_n$ and $(\blacktriangleright X)_0 := \mathbf{1}$, with $r^{\blacktriangleright X}_{n+1} := r^X_n$ and $r^{\blacktriangleright X}_0 := \mathbf{1} : X_0\to\mathbf{1}$. On morphisms, $(\blacktriangleright f)_{n+1} := f_n$ and $(\blacktriangleright f)_0 := \mathbf{1} : \mathbf{1}\to\mathbf{1}$. Note that $\blacktriangleright(X\times Y)\simeq \blacktriangleright X\times\blacktriangleright Y$.

Define the family of maps $\mathrm{pred}^X : X\Rightarrow\blacktriangleright X$, natural in X, as $\mathrm{pred}^X_0 := \mathbf{1} : X_0\to\mathbf{1}$ and $\mathrm{pred}^X_{n+1} := r^X_n$.

The functor $\blacktriangleright$ allows $\mathscr{S}$ to be equipped with fixpoint operators $\mathrm{fix}^X : X^{\blacktriangleright X}\Rightarrow X$, defined as

$$\mathrm{fix}^X_n((f_m)_{m\leq n}) \quad := \quad (f_n\circ\ldots\circ f_0)(\bullet)$$

The maps fix^X are natural in X. Given $f : \blacktriangleright X\times Y\Rightarrow X$, writing $f^t : Y\Rightarrow X^{\blacktriangleright X}$ for the exponential transpose of f, $\mathrm{fix}^X\circ f^t$ is the unique map $h : Y\Rightarrow X$ satisfying $f\circ\langle \mathrm{pred}^X\circ h, \mathrm{id}_Y\rangle = h$ (see [6, Thm. 2.4]).

Given a sequence of sets $\overline{M}=(M_n)_n$, we also denote by $\overline{M}$ the $\mathscr{S}$-object with $\overline{M}_n := \prod_{i=0}^n M_i$ and restriction maps $r^{\overline{M}}_n(\overline{m}.m) := \overline{m}$. ($r^{\overline{M}}$ is an epi). Note that $\overline{M}\times\overline{N}\simeq\overline{M\times N}$, where $\overline{M\times N}_n := \prod_{i=0}^n M_i\times N_i$. If $M_n = M$ for all n, then we write $M^\star$ for the $\mathscr{S}$-object $\overline{M}$.

G.2 The Monoidal Structure of $\mathbf{G}(\mathscr{S})$

Following §G.1, we take for $\mathscr{S}$ the monoidal structure given by its Cartesian product (so that $\otimes := \times$ with $\mathbf{I} := \mathbf{1}$). Since $(A_n)_n \times (B_n)_n = (A_n \times B_n)_n$ the structure maps of $(\mathscr{S}, \otimes, \mathbf{I})$ (induced from its Cartesian structure) have as components the corresponding structure maps of $\mathbf{Set}$:

$$\alpha_n := \alpha : (A_n \times B_n) \times C_n \to A_n \times (B_n \times C_n) \qquad \lambda_n := \lambda : \mathbf{1} \times A_n \to A_n \qquad \rho_n := \rho : A_n \times \mathbf{1} \to A_n \qquad \gamma_n := \gamma : A_n \times B_n \to B_n \times A_n$$

The required diagrams follow as usual from the fact that Cartesian categories are monoidal (using the universal property of the Cartesian product).

G.3 Monoids and Comonoids in $\mathbf{G}(\mathscr{S})$

Prop. F.2 and Cor. F.3 (on monoid and comonoid objects in categories of the form $\mathbf{G}(\mathbb{C})$) specialize to:

Proposition G.1. *Let X be an object of $\mathscr{S}$.*

(i) The $\mathbf{G}(\mathscr{S})$-object $(\mathbf{1}, X)$ is a commutative monoid of $\mathbf{G}(\mathscr{S})$, with structure maps

$$\begin{array}{rcrcllcl} u & := & (\mathbf{1}, \mathbf{1}) & : & (\mathbf{1}, \mathbf{1}) & \to\!\!\!\!\!\to & (\mathbf{1}, X) \\ m & := & (\mathbf{1}, \langle \mathrm{id}, \mathrm{id} \rangle) & : & (\mathbf{1} \times \mathbf{1}, X \times X) & \to\!\!\!\!\!\to & (\mathbf{1}, X) \end{array}$$

(ii) The $\mathbf{G}(\mathscr{S})$-object $(X, \mathbf{1})$ is a commutative comonoid of $\mathbf{G}(\mathscr{S})$, with structure maps

$$\begin{array}{rcrcllcl} e & := & (\mathbf{1}, \mathbf{1}) & : & (X, \mathbf{1}) & \to\!\!\!\!\!\to & (\mathbf{1}, \mathbf{1}) \\ d & := & (\langle \mathrm{id}, \mathrm{id} \rangle, \mathbf{1}) & : & (X, \mathbf{1}) & \to\!\!\!\!\!\to & (X \times X, \mathbf{1} \times \mathbf{1}) \end{array}$$

Proof. By Prop. F.2 and Cor. F.3, since the terminal object $\mathbf{1}$ of a Cartesian category is a commutative monoid, and since any object of a Cartesian category is a commutative comonoid. $\square$

G.4 A Dialectica-Like Interpretation of Zig-Zag Strategies

We now show that $\mathbf{DZ}$ is equivalent to a category obtained from a distributive law in $\mathbf{G}(\mathscr{S})$. We first show (Prop. G.2) that total zig-zag strategies are in 1-1 correspondence with $\mathbf{G}(\mathscr{S})$ morphisms

$$(f, F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \to\!\!\!\!\!\to \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

We then describe a composition of these morphisms respecting composition of strategies. The distributive law ζ is presented in §G.5.

G.4.1 Total Zig-Zag Strategies in $\mathbf{G}(\mathscr{S})$

Consider a positive full game $A = (U, X)$. Recall from App. A the bijection

$$\partial \quad = \quad \langle \partial_U, \partial_X \rangle \quad : \quad \wp_A^{\mathsf{even}} \quad \longrightarrow \quad \cup_{n \in \mathbb{N}} (U^n \times X^n)$$

with $\partial(\varepsilon) = (\bullet, \bullet)$ and $\partial(s.u.x) = (\partial_U(s).u, \partial_X(s).x)$.

Consider now another positive full game $B = (V, X)$ and let $\sigma : A \multimap B$ be a total zig-zag strategy. By induction on $n \in \mathbb{N}$, it is easy to see that for all $(\overline{u}, \overline{y}) \in U^n \times Y^n$, there is a unique $(s, t) \in \mathrm{HS}(\sigma)$ such that $\overline{u} = \partial_U(s)$ and $\overline{y} = \partial_Y(t)$.

The property vacuously holds for $n = 0$. Assuming it for n, given $(\overline{u}.u, \overline{y}.y) \in U^{n+1} \times Y^{n+1}$, by induction hypothesis, there is a unique $(s, t) \in \mathrm{HS}(\sigma)$ such that $\overline{u} = \partial_U(s)$ and $\overline{y} = \partial_Y(t)$. Now, since σ is total and zig-zag, there is a unique $v \in V$ such that $(s.u, t.v) \in \mathrm{HS}(\sigma)$. Similarly, there is a unique $x \in X$ such that $(s.u.x, t.v.y) \in \mathrm{HS}(\sigma)$, and the property follows.

Furthermore, since $\overline{u}.u$ and $\overline{y}$ uniquely determine $\overline{v} = \partial_V(t)$ and v, and since $\overline{u}.u$ and $\overline{y}.y$ uniquely determine $\overline{x} = \partial_X(s)$ and x, we obtain functions

$$\begin{array}{lcllcl} f_{n+1} & : & U^{n+1} \times Y^n & \longrightarrow & V^{n+1} \\ F_{n+1} & : & U^{n+1} \times Y^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

It follows that σ uniquely determine a $\mathbf{G}(\mathscr{S})$-morphism

$$\sigma_{\mathbf{G}(\mathscr{S})} \quad = \quad (f, F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \to\!\!\!\!\!\to \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

Conversely, each (f, F) uniquely determine a total zig-zag strategy σ, with, for all $\overline{u}.u \in U^{n+1}$, and all $\overline{y} \in Y^n$,

$$(\partial^{-1}(\overline{u}, \overline{x}).u \ , \ \partial^{-1}(\overline{v}, \overline{y}).v) \in \mathrm{HS}(\sigma)$$

where $\overline{v}.v = f_{n+1}(\overline{u}.u, \overline{y})$ and $\overline{x} = F_n(\overline{u}, \overline{y})$; and for all y,

$$(\partial^{-1}(\overline{u}, \overline{x}).u.x \ , \ \partial^{-1}(\overline{v}, \overline{y}).v.y) \in \mathrm{HS}(\sigma)$$

where $\overline{x}.x = F_{n+1}(\overline{u}.u, \overline{y}.y)$.

We therefore have shown:

Proposition G.2. *Given positive full games $A = (U, X)$ and $B = (V, Y)$, the map $(-)_{\mathbf{G}(\mathscr{S})}$ is a bijection from total zig-zag strategies $\sigma : A \multimap B$ to $\mathbf{G}(\mathscr{S})$-morphisms*

$$(f, F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \to\!\!\!\!\!\to \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

G.4.2 Composition of Total Zig-Zag Strategies in $\mathbf{G}(\mathscr{S})$

Note that given $(\overline{u},\overline{x},\overline{v},\overline{y}) \in (U \times X \times V \times Y)^n$, we have $((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$ if and only if $\overline{v} = f_n(\overline{u}, \blacktriangleright(\overline{y}))$ and $\overline{x} = F_n(\overline{u},\overline{y})$. Here, we have written $((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$ for $(\partial^{-1}(\overline{u},\overline{x}),\partial^{-1}(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$. We adopt the same convention in the following.

Consider positive full games $A = (U,X)$, $B = (V,Y)$ and $C = (W,Z)$, and $\mathbf{G}(\mathscr{S})$-morphisms

$$\begin{array}{lcrcl} (f,F) & : & (U^\star, X^{\star U^\star}) & \longrightarrow\!\!\!\!\!\!\!+ & (V^{\star \blacktriangleright Y^\star}, Y^\star) \\ (g,G) & : & (V^\star, Y^{\star V^\star}) & \longrightarrow\!\!\!\!\!\!\!+ & (W^{\star \blacktriangleright Z^\star}, Z^\star) \end{array}$$

We want to define their composite

$$(h,H) \quad : \quad (U^\star, X^{\star U^\star}) \quad \longrightarrow\!\!\!\!\!\!\!+ \quad (W^{\star \blacktriangleright Z^\star}, Z^\star)$$

Write σ and τ for the total zig-zag strategies corresponding to resp. (f,F) and (g,G). Then the relational composite

$$\mathrm{HS}(\tau \circ \sigma) \quad = \quad \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$$

must be such that $((\overline{u},\overline{x}),(\overline{w},\overline{z})) \in \mathrm{HS}(\tau)\circ\mathrm{HS}(\sigma)$ if and only if there are $(\overline{v},\overline{y})$ such that

$$((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma) \qquad \text{and} \qquad ((\overline{v},\overline{y}),(\overline{w},\overline{z})) \in \mathrm{HS}(\tau)$$

But this is possible iff the following equations are satisfied:

$$\begin{array}{rcl@{\qquad\qquad}rcl} \overline{v} & = & f_n(\overline{u}, \blacktriangleright(\overline{y})) & \overline{w} & = & g_n(\overline{v}, \blacktriangleright(\overline{z})) \\ \overline{x} & = & F_n(\overline{u},\overline{y}) & \overline{y} & = & G_n(\overline{v},\overline{z}) \end{array}$$

The derived equation

$$\overline{y} \quad = \quad G_n(f_n(\overline{u}, \blacktriangleright(\overline{y})), \overline{z})$$

uniquely defines $\overline{y}$ from $\overline{u}$ and $\overline{z}$ as

$$\overline{y} \quad = \quad y(\overline{u},\overline{z}) \quad = \quad \mathrm{fix}^Y_n(\boldsymbol{\lambda} y. G_n(f_n(\overline{u},y),\overline{z}))$$

(We have here tacitly used the fact that $\xi \in (M^{\star \blacktriangleright M^\star})_n$ is completely determined by its last component ξ_n.) Now, since $\blacktriangleright(y(\overline{u},\overline{z})) = y(\blacktriangleright\overline{u}, \blacktriangleright\overline{z})$, we can define

$$\begin{array}{rcl} h_{n+1}(\overline{u}u, \overline{z}) & := & g_{n+1}(f_{n+1}(\overline{u}u, y(\overline{u},\overline{z})))\ ,\ \overline{z}) \\ H_{n+1}(\overline{u}u, \overline{z}z) & := & F_{n+1}(\overline{u}u, y(\overline{u}u, \overline{z}z)) \end{array}$$

More generally, given $\mathbf{G}(\mathscr{S})$-objects (U,X), (V,Y), (W,Z), and $\mathbf{G}(\mathscr{S})$-morphisms

$$\begin{array}{lcrcl} (f,F) & : & (U, X^U) & \longrightarrow\!\!\!\!\!\!\!+ & (V^{\blacktriangleright Y}, Y) \\ (g,G) & : & (V, Y^V) & \longrightarrow\!\!\!\!\!\!\!+ & (W^{\blacktriangleright Z}, Z) \end{array}$$

we can define their composite

$$(g,G)\circ(f,F) \ = \ (h,H) \quad : \quad (U,X^U) \quad \longrightarrow\!\!\!\!\!\!\!+ \quad (W^{\blacktriangleright Z}, Z)$$

as, modulo exponential transpose and again using the internal $\boldsymbol{\lambda}$-calculus of $\mathscr{S}$:

$$\begin{array}{rrcl} & h(u,z) & := & g(f(u, y(\blacktriangleright u, z)), z) \\ & H(z,u) & := & F(u, y(u,z)) \\ \text{where} \qquad & y(u,z) & := & \mathrm{fix}^Y(\boldsymbol{\lambda} y. G(f(u,y),z)) \end{array}$$

G.5 The Distributive Law ζ

It is possible to directly check that the composition described in the previous paragraph is associative and preserves identities. We can actually do better: The category $\mathbf{DZ}$ of simple zig-zag games can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$ described as the Kleisli category of a distributive law ζ.

The law ζ is based on the constructions of §H. It distributes an oplax symmetric monoidal comonad obtained from Cor. F.5 over a (lax) symmetric monoidal monad obtained from Prop. F.4:

- The oplax symmetric monoidal comonad, denoted $T = (T,\epsilon,\delta)$, is obtained from Cor. F.5 by taking $H := \mathrm{Id}_{\mathscr{S}}$.

  Explicitely, $T(U,X) := (U, X^U)$ and the action of T on morphisms is given by:

$$(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!+ (V,Y) \qquad \overset{T}{\longmapsto} \qquad (f\ ,\ \boldsymbol{\lambda} h. F \circ h \circ f)\ :\ (U,X^U)\ \longrightarrow\!\!\!\!\!\!\!+\ (V,Y^V)$$

  The maps ϵ and δ are given by:

$$\begin{array}{rclcrcl} (f_\epsilon, F_\epsilon) & := & (\mathrm{id}_U\ ,\ \boldsymbol{\lambda} x. \boldsymbol{\lambda}\_. x) & : & (U, X^U) & \longrightarrow\!\!\!\!\!\!\!+ & (U, X) \\ (f_\delta, F_\delta) & := & (\mathrm{id}_U\ ,\ \boldsymbol{\lambda} h. \boldsymbol{\lambda} u. h(u,u)) & : & (U, X^U) & \longrightarrow\!\!\!\!\!\!\!+ & (U, X^{U\times U}) \end{array}$$

- The (lax) symmetric monoidal monad, denoted $(-)^\blacktriangleright = ((-)^\blacktriangleright, \epsilon, \delta)$, is obtained from Prop. F.4 by taking $H(-) := \blacktriangleright(-)$ (see §G.1 and [6]).

  Explicitely, $(U,X)^\blacktriangleright := (U^{\blacktriangleright X}, X)$ and the action of $(-)^\blacktriangleright$ on morphisms is given by:

$$(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!+ (V,Y) \qquad \overset{(-)^\blacktriangleright}{\longmapsto} \qquad (\boldsymbol{\lambda} h. f \circ h \circ \blacktriangleright F\ ,\ F)\ :\ (U^{\blacktriangleright X}, X)\ \longrightarrow\!\!\!\!\!\!\!+\ (V^{\blacktriangleright Y}, Y)$$

The maps η and μ are given by:

$$\begin{array}{lcrcrcl} (f_\eta, F_\eta) & := & (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u \ , \ \mathrm{id}_X) & : & (U, X) & \longrightarrow\!\!\!\!\!\!\!+ & (U^{\blacktriangleright X}, X) \\ (f_\mu, F_\mu) & := & (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x) \ , \ \mathrm{id}_X) & : & (U^{\blacktriangleright X \times \blacktriangleright X}, X) & \longrightarrow\!\!\!\!\!\!\!+ & (U^{\blacktriangleright X}, X) \end{array}$$

The distributive law

$$\zeta \quad : \quad T((-)^{\blacktriangleright}) \quad \Longrightarrow \quad (T(-))^{\blacktriangleright}$$

is given by

$$\zeta_A \quad = \quad (f^\zeta, F^\zeta) \quad : \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \quad \longrightarrow\!\!\!\!\!\!\!+ \quad (U^{\blacktriangleright (X^U)}, X^U)$$

where the maps

$$f^\zeta \quad : \quad U^{\blacktriangleright X} \times \blacktriangleright(X^U) \quad \longrightarrow \quad U \qquad \text{and} \qquad F^\zeta \quad : \quad U^{\blacktriangleright X} \times X^U \quad \longrightarrow \quad X$$

are defined as follows. Let $f_0^\zeta(\theta_0, \bullet) := \theta_0$. Given $\xi \in (X^U)_n$, $\theta \in (U^{\blacktriangleright X})_n$ and $\theta' \in (U^{\blacktriangleright X})_{n+1}$,

$$\begin{array}{rcl} F_n^\zeta(\theta, \xi) & := & \mathrm{fix}_n^X(\xi \circ \theta) \\ f_{n+1}^\zeta(\theta', \xi) & := & \theta'_{n+1}(\mathrm{fix}_n^X(\xi \circ r_n(\theta'))) \\ & = & \theta'_{n+1}(F_n(r_n(\theta'), \xi)) \end{array}$$

The maps ζ_A form a distributive law of T over $(-)^{\blacktriangleright}$, which is moreover monoidal in the sense of Prop. J.6. These facts are summarized in the following Proposition whose proof is defered to §I.

Proposition G.3.

(i) The family of maps $\zeta_A : T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\!\!\!+ (TA)^{\blacktriangleright}$ forms a distributive law.

(ii) Moreover, $\zeta_{(-)}$ is monoidal in the sense of Prop. J.6, that is:

$$\begin{tikzcd} T(A^{\blacktriangleright} \otimes B^{\blacktriangleright}) \arrow[r, "T(m^2_{A,B})"] \arrow[d, "g^2_{A^{\blacktriangleright}, B^{\blacktriangleright}}"'] & T((A \otimes B)^{\blacktriangleright}) \arrow[d, "\zeta_{A \otimes B}"] \\ T(A^{\blacktriangleright}) \otimes T(B^{\blacktriangleright}) \arrow[d, "\zeta_A \otimes \zeta_B"'] & (T(A \otimes B))^{\blacktriangleright} \arrow[d, "(g^2_{A,B})^{\blacktriangleright}"] \\ (TA)^{\blacktriangleright} \otimes (TB)^{\blacktriangleright} \arrow[r, "m^2_{TA,TB}"'] & (TA \otimes TB)^{\blacktriangleright} \end{tikzcd} \tag{8}$$

where (m^2, m^0) is the (lax) strength of $(-)^{\blacktriangleright}$ defined as in Prop. F.4, and (g^2, g^0) is the oplax strength of T defined as in Cor. F.5, so that:

- *For $(-)^{\blacktriangleright}$:*

$$m^2_{A,B} \quad := \quad (\boldsymbol{\lambda}(h,k).(h \times k) \circ \langle \blacktriangleright(\pi_1), \blacktriangleright(\pi_2) \rangle \ , \ \mathrm{id}_{X \times Y}) \quad : \quad (U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \ , \ X \times Y) \quad \longrightarrow \quad ((U \times V)^{\blacktriangleright (X \times Y)} \ , \ X \times Y)$$

(where $A = (U, X)$ and $B = (V, Y)$), and $m^0 := (\mathbf{1}, \mathbf{1}) : (\mathbf{1}, \mathbf{1}) \longrightarrow (\mathbf{1}^{\blacktriangleright \mathbf{1}}, \mathbf{1})$.

- *For T:*

$$g^2_{A,B} \quad := \quad (\mathrm{id}_{U \times V} \ , \ \boldsymbol{\lambda}(h,k).(h \times k)) \quad : \quad (U \times V \ , \ (X \times Y)^{U \times V}) \quad \longrightarrow \quad (U \times V \ , \ X^U \times Y^V)$$

(where $A = (U, X)$ and $B = (V, Y)$), and $g^0 := (\mathbf{1}, \mathbf{1}) : (\mathbf{1}, \mathbf{1}^{\mathbf{1}}) \longrightarrow (\mathbf{1}, \mathbf{1})$.

It then follows from Prop. G.3 and Cor. J.7 that $\mathbf{Kl}(\zeta)$ is symmetric monoidal.

- Its monoidal product is that of $\mathbf{G}(\mathscr{S})$ on objects, so that

$$(U, X) \otimes_{\mathbf{Kl}(\zeta)} (V, Y) = (U, X) \otimes (V, Y) = (U \times V, X \times Y) \qquad \text{and} \qquad \mathbf{I} = (\mathbf{1}, \mathbf{1})$$

  and on maps, given $(f, F) \in \mathbf{Kl}(\zeta)[A_0, B_0]$ and $(g, G) \in \mathbf{Kl}(\zeta)[A_1, B_1]$, we let

$$(f, F) \otimes_{\mathbf{Kl}(\zeta)} (g, G) \quad := \quad m^2_{B_0, B_1} \circ ((f, F) \otimes (g, G)) \circ g^2_{A_0, A_1}$$

- The structure maps are the image under $\boldsymbol{\lambda} h^{A \to B}.\eta_B \circ h \circ \epsilon_A$ of the structure maps of $\mathbf{G}(\mathscr{S})$.

From now on, if no ambiguity arises, we write $\otimes$ for the monoidal product of $\mathbf{Kl}(\zeta)$.

We write $\mathbf{Kl}(\zeta^\star)$ for the full subcategory of $\mathbf{Kl}(\zeta)$ whose objects are of the form $(U^\star, X^\star)$. Together with §G.4, Prop. G.3 gives:

Proposition G.4. *The category $\mathbf{DZ}$ is equivalent to $\mathbf{Kl}(\zeta^\star)$.*

G.6 The Symmetric Monoidal Structure of DZ

Recall from Prop. G.4 that $\mathbf{DZ}$ is isomorphic to $\mathbf{Kl}(\zeta^\star)$ the full subcategory of $\mathbf{Kl}(\zeta)$ whose objects are of the form $(U^\star, X^\star)$.

Note that $\mathbf{I}$ is an object of $\mathbf{Kl}(\zeta^\star)$, as well as $A \otimes B$ as soon as A and B are objects of $\mathbf{Kl}(\zeta^\star)$. It thus follows from Prop. G.4, Prop. G.3 and Cor. J.7 that:

Proposition G.5. *Equipped with the above data, the category $\mathbf{Kl}(\zeta^\star)$ (and thus $\mathbf{DZ}$) is symmetric monoidal.*

G.7 Monoids and Comonoids in $\mathbf{DZ}$

Thanks to Prop. J.9, we therefore get from Prop. G.3 and Prop. G.1:

Proposition G.6 (Prop. B.3)**.**

(i) Objects of the form $M = (\mathbf{1}, M)$ equipped with structure maps

$$\begin{array}{c|ccc|c} & \mathbf{I} & \overset{u}{\multimap} & M & \\ \hline \mathsf{O} & \bullet & & & \\ & & & \bullet & \mathsf{P} \\ & & & m & \mathsf{O} \\ \mathsf{P} & \bullet & & & \end{array} \qquad\qquad \begin{array}{c|ccc|c} & M \otimes M & \overset{m}{\multimap} & M & \\ \hline \mathsf{O} & (\bullet, \bullet) & & & \\ & & & \bullet & \mathsf{P} \\ & & & m & \mathsf{O} \\ \mathsf{P} & (m, m) & & & \end{array}$$

are monoids in $\mathbf{DZ}$.

(ii) Objects of the form $\Sigma = (\Sigma, \mathbf{1})$ equipped with structure maps

$$\begin{array}{c|ccc|c} & \Sigma & \overset{e}{\multimap} & \mathbf{I} & \\ \hline \mathsf{O} & \mathsf{a} & & & \\ & & & \bullet & \mathsf{P} \\ & & & \bullet & \mathsf{O} \\ \mathsf{P} & \bullet & & & \end{array} \qquad\qquad \begin{array}{c|ccc|c} & \Sigma & \overset{d}{\multimap} & \Sigma \otimes \Sigma & \\ \hline \mathsf{O} & \mathsf{a} & & & \\ & & & (\mathsf{a}, \mathsf{a}) & \mathsf{P} \\ & & & (\bullet, \bullet) & \mathsf{O} \\ \mathsf{P} & \bullet & & & \end{array}$$

are comonoids in $\mathbf{DZ}$.

G.8 Comonoids in $\mathbf{DZ}_D$

From Prop. G.6.(ii) together with Prop. J.9 applied to Prop. J.10 and Prop. G.6.(i) we get:

Proposition G.7 (Prop. B.5)**.** *Objects of the form $\Sigma = (\Sigma, \mathbf{1})$ are comonoids in $\mathbf{DZ}_D$, with structure maps*

$$\begin{array}{c|ccc|c} & \Sigma & \overset{e}{\multimap} & \mathbf{I} & \\ \hline \mathsf{O} & \mathsf{a} & & & \\ & & & \bullet & \mathsf{P} \\ & & & (\bullet, d) & \mathsf{O} \\ \mathsf{P} & \bullet & & & \end{array} \qquad\qquad \begin{array}{c|ccc|c} & \Sigma & \overset{d}{\multimap} & \Sigma \otimes \Sigma & \\ \hline \mathsf{O} & \mathsf{a} & & & \\ & & & (\mathsf{a}, \mathsf{a}) & \mathsf{P} \\ & & & ((\bullet, \bullet), d) & \mathsf{O} \\ \mathsf{P} & \bullet & & & \end{array}$$

G.9 The Base Category $\mathbf{T}$

Proposition G.8 (Prop. B.7)**.** *The category $\mathbf{T}$ embeds to $\mathbf{Comon}(\mathbf{DZ}_D)$ via the functor $\mathsf{E}_{\mathbf{T}}$ mapping an object Σ of $\mathbf{T}$ to the comonoid $(\Sigma, e_\Sigma, d_\Sigma)$ and a morphism $M : \mathbf{T}[\Gamma, \Sigma]$ to itself.*

Proof of Proposition G.8. Fix $M \in \mathbf{T}[\Sigma, \Gamma]$, so that

$$M \quad\simeq\quad (f_M, \mathbf{1}) \quad:\quad (\Sigma, \mathbf{1}^\Sigma) \quad\rightarrow\!\!\!\!\!+\quad (\Gamma^{\blacktriangleright(\mathbf{1}\times D)}, \mathbf{1} \times D)$$

The comonoid structure mpas can be explicitelly defined as

$$e_\Sigma \quad\simeq\quad (\mathbf{1}, \mathbf{1}) \quad:\quad (\Sigma, \mathbf{1}^\Sigma) \quad\rightarrow\!\!\!\!\!+\quad (\mathbf{1}^{\blacktriangleright(\mathbf{1}\times D)}, \mathbf{1} \times D)$$

and

$$d_\Sigma \quad\simeq\quad (\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a}, \mathsf{a})}\,,\, \mathbf{1}) \quad:\quad (\Sigma, \mathbf{1}^\Sigma) \quad\rightarrow\!\!\!\!\!+\quad ((\Sigma \times \Sigma)^{\blacktriangleright(\mathbf{1}\times\mathbf{1}\times D)}\,,\, \mathbf{1} \times \mathbf{1} \times D)$$

We check the required diagrams:

- First,

$$\begin{tikzcd} \Sigma \arrow[r, "M"] \arrow[d, "d_\Sigma"'] & \Gamma \arrow[d, "d_\Gamma"] \\ \Sigma \otimes \Sigma \arrow[r, "M \otimes M"] & \Gamma \otimes \Gamma \end{tikzcd}$$

  Note that all maps involved are $\mathbf{1}$ on the second component, so we only check the first one.
  We then compute (leaving implicit the monad maps used for composition in $\mathbf{DZ}_D$):

$$(f_M \times f_M) \circ (\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a}, \mathsf{a})}) \quad=\quad \boldsymbol{\lambda}\blacktriangleright(p).\boldsymbol{\lambda}\bar{\mathsf{a}}.\langle f_M(\blacktriangleright(p), \bar{\mathsf{a}})\,,\ f_M(\blacktriangleright(p), \bar{\mathsf{a}})\rangle$$

  and we are done since on the other hand

$$(\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a}, \mathsf{a})}) \circ f_M \quad=\quad \boldsymbol{\lambda}\blacktriangleright(p).\boldsymbol{\lambda}\bar{\mathsf{a}}.(f_M(\blacktriangleright(p), \bar{\mathsf{a}}), f_M(\blacktriangleright(p), \bar{\mathsf{a}}))$$

- Second, the coherence diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "M"] \arrow[dr, "e_\Sigma"'] & & \Gamma \arrow[dl, "e_\Gamma"] \\ & \mathbf{I} & \end{tikzcd}$$

trivially holds since all involved maps are in the second component are $\mathbf{1}$, and, for the first component, since $\mathbf{1}$ is terminal in $\mathscr{S}$.

□

H. Proof of Proposition F.4

In this appendix we give a proof of Prop. F.4. We first recall its statment.

Assume that $\mathbb{C}$ is Cartesian closed, and fix a functor $H : \mathbb{C} \to \mathbb{C}$. Recall (from e.g. [27, §5.2]) that H lifts in a unique way to an oplax symmetric monoidal functor, with strength

$$\mathsf{t}^2_{A,B} \;:=\; \langle H(\pi_1), H(\pi_2)\rangle \;:\; H(A\times B) \longrightarrow HA\times HB \qquad \text{and} \qquad \mathsf{t}^0 \;:=\; \mathbf{1}_{H\mathbf{1}} \;:\; H\mathbf{1} \longrightarrow \mathbf{1}$$

Note that the naturality of $\mathsf{t}^2_{(-),(-)}$, that is

$$(H(f)\times H(g))\circ\langle H(\pi_1), H(\pi_2)\rangle \;=\; \langle H(\pi_1), H(\pi_2)\rangle\circ H(f\times g)$$

follows from the universality property of the Cartesian product since (say)

$$\pi_1\circ(H(f)\times H(g))\circ\langle H(\pi_1), H(\pi_2)\rangle \;=\; H(f\circ\pi_1) \;=\; H(\pi_1\circ(f\times g))$$

Consider now the functor

$$(-)^H \;:\; \mathbf{G}(\mathbb{C}) \longrightarrow \mathbf{G}(\mathbb{C})$$

defined as

$$(U,X)^H \;:=\; (U^{HX}, X) \qquad \text{and} \qquad (f,F)^H \;:=\; (\boldsymbol{\lambda} h.f\circ h\circ H(F)\;,\;F) \;:\; (U^{HX},X) \longrightarrow\!\!\!\!\!\!\!\;+\; (V^{HY},Y)$$

(where $(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!\;+\; (V,Y)$), and the maps

$$\begin{array}{rclcrcrcl} \eta_{(U,X)} & = & (f_\eta, F_\eta) & := & (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u\;,\;\mathrm{id}_X) & : & (U,X) & \longrightarrow\!\!\!\!\!\!\!\;+ & (U^{HX},X) \\ \mu_{(U,X)} & = & (f_\mu, F_\mu) & := & (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)\;,\;\mathrm{id}_X) & : & (U^{HX\times HX},X) & \longrightarrow\!\!\!\!\!\!\!\;+ & (U^{HX},X) \end{array}$$

Proposition H.1 (Prop. F.4). *$((-)^H, \eta, \mu)$ is a (lax) symmetric monoidal monad, with strength*

$$m^2_{A,B} \;=\; (f^2_{A,B}, F^2_{A,B}) \;:=\; (\boldsymbol{\lambda}(h,k).(h\times k)\circ\mathsf{t}^2_{X,Y}\;,\;\mathrm{id}_{X\times Y}) \;:\; (U^{HX}\times V^{HY}\;,\;X\times Y) \longrightarrow ((U\times V)^{H(X\times Y)}\;,\;X\times Y)$$

(where $A = (U,X)$ and $B = (V,Y)$), and

$$m^0 \;:=\; (\mathbf{1},\mathbf{1}) \;:\; (\mathbf{1},\mathbf{1}) \longrightarrow (\mathbf{1}^{H\mathbf{1}},\mathbf{1})$$

H.1 $(-)^H$ is a lax symmetric monoidal functor

$(-)^H$ is a functor. First, given $A = (U,X)$ we have

$$(\mathrm{id}_A)^H \;=\; (\boldsymbol{\lambda} h.\mathrm{id}_U\circ h\circ H(\mathrm{id}_X)\;,\;\mathrm{id}_X) \;=\; (\boldsymbol{\lambda} h.h\;,\;\mathrm{id}_X) \;=\; \mathrm{id}_{A^H}$$

Moreover, given $(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!\;+\; (V,Y)$ and $(g,G) : (V,Y) \longrightarrow\!\!\!\!\!\!\!\;+\; (W,Z)$, we have

$$((g,G)\circ(f,F))^H \;=\; (g\circ f\;,\;F\circ G)^H \;=\; (\boldsymbol{\lambda} h.g\circ f\circ h\circ H(F\circ G)\;,\;F\circ G) \;=\; (\boldsymbol{\lambda} h.g\circ h\circ HG, G)\circ(\boldsymbol{\lambda} h.f\circ h\circ HF, F)$$

since

$$\boldsymbol{\lambda} h.g\circ f\circ h\circ H(F\circ G) \;=\; \boldsymbol{\lambda} h.g\circ f\circ h\circ H(F)\circ H(G) \;=\; \boldsymbol{\lambda} h.(\boldsymbol{\lambda} k.g\circ k\circ H(G))(f\circ h\circ H(F))$$

The maps $m^2_{(-),(-)}$ are natural. We have to check that given $(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!\!\;+\; (V,Y)$ and $(g,G) : (U',X') \longrightarrow\!\!\!\!\!\!\!\;+\; (V',Y')$ we have

$$m^2_{B,B'}\circ((f,F)^H\otimes(g,G)^H) \;=\; ((f,F)\otimes(g,G))^H\circ m^2_{A,A'}$$

(where $A = (U, X)$, $B = (V, Y)$, $A' = (U', X')$ and $B' = (V', Y')$). We compute

$$\begin{array}{rcl}
m^2_{B,B'} \circ ((f, F)^H \otimes (g, G)^H) & = & m^2_{B,B'} \circ ((\boldsymbol{\lambda} h.f \circ h \circ H(F), F) \otimes (\boldsymbol{\lambda} k.g \circ k \circ H(G), G)) \\
& = & m^2_{B,B'} \circ ((\boldsymbol{\lambda} h.f \circ h \circ H(F)) \times (\boldsymbol{\lambda} k.g \circ k \circ H(G))\ ,\ F \times G) \\
& = & ((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{Y,Y'}) \circ ((\boldsymbol{\lambda} h.f \circ h \circ H(F)) \times (\boldsymbol{\lambda} k.g \circ k \circ H(G)))\ ,\ F \times G) \\
& = & ((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{Y,Y'}) \circ (\boldsymbol{\lambda}(h,k).\langle f \circ h \circ H(F)\ ,\ g \circ k \circ H(G)\rangle)\ ,\ F \times G) \\
& = & (\boldsymbol{\lambda}(h,k).((f \circ h \circ H(F)) \times (g \circ k \circ H(G))) \circ \mathsf{t}^2_{Y,Y'}\ ,\ F \times G) \\
& = & (\boldsymbol{\lambda}(h,k).(f \times g) \circ (h \times k) \circ (H(F) \times H(G)) \circ \mathsf{t}^2_{Y,Y'}\ ,\ F \times G) \\
& = & (\boldsymbol{\lambda}(h,k).(f \times g) \circ (h \times k) \circ \mathsf{t}^2_{X,X'} \circ H(F \times G)\ ,\ F \times G) \\
& = & (\boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda} p.(f \times g) \circ p \circ H(F \times G)) \circ ((h \times k) \circ \mathsf{t}^2_{X,X'})\ ,\ F \times G) \\
& = & ((f, F) \otimes (g, G))^H \circ m^2_{A,A'}
\end{array}$$

$(-)^H$ ***is lax symmetric monoidal.*** Note that $(-)^H$ is the identity on the second components, so we only have to check diagrams for the first components.

- The associativity diagram leads to check

$$\begin{tikzcd}
(U^{HX} \times V^{HY}) \times W^{HZ} \arrow[r, "\alpha_{U^{HX},V^{HY},W^{HZ}}"] \arrow[d, "(\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y})\times \mathrm{id}_{W^{HZ}}"'] & U^{HX} \times (V^{HY} \times W^{HZ}) \arrow[d, "\mathrm{id}_{U^{HX}} \times (\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{Y,Z})"] \\
(U \times V)^{H(X\times Y)} \times W^{HZ} \arrow[d, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X\times Y,Z}"'] & U^{HX} \times (V \times W)^{H(Y\times Z)} \arrow[d, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y\times Z}"] \\
((U \times V) \times W)^{H((X\times Y)\times Z)} \arrow[r, "\boldsymbol{\lambda} h.\alpha_{U,V,W}\circ h \circ H(\alpha^{-1}_{X,Y,Z})"'] & (U \times (V \times W))^{H(X\times(Y\times Z))}
\end{tikzcd}$$

(where $A = (U, X)$, $B = (V, Y)$ and $C = (W, Z)$). Note that since $\mathbb{C}$ is Cartesian closed:

$$\alpha \quad = \quad \langle \pi_1 \circ \pi_1\ ,\ \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle \quad = \quad \boldsymbol{\lambda}((u, v), w).(u, (v, w))$$

We have to check

$$\boldsymbol{\lambda}((h,k),l).\alpha_{U,V,W} \circ (((h \times k) \circ \mathsf{t}^2_{X,Y}) \times l) \circ \mathsf{t}^2_{X\times Y,Z} \circ H(\alpha^{-1}_{X,Y,Z}) \quad = \quad \boldsymbol{\lambda}((h,k),l).(h \times ((k \times l) \circ \mathsf{t}^2_{Y,Z})) \circ \mathsf{t}^2_{X,Y\times Z}$$

But we are done since it follows from the universal property of the Cartesian product of $\mathbb{C}$ that we have

$$\begin{tikzcd}
(HX \times HY) \times HZ \arrow[r, "\alpha_{HX,HY,HZ}"] & HX \times (HY \times HZ) \\
H(X \times Y) \times HZ \arrow[u, "\mathsf{t}^2_{X,Y}\times \mathrm{id}_{HZ}"] & HX \times H(Y \times Z) \arrow[u, "\mathrm{id}_{HX}\times \mathsf{t}^2_{Y,Z}"'] \\
H((X \times Y) \times Z) \arrow[u, "\mathsf{t}^2_{X\times Y,Z}"] & H(X \times (Y \times Z)) \arrow[l, "H(\alpha^{-1}_{X,Y,Z})"] \arrow[u, "\mathsf{t}^2_{X,Y\times Z}"']
\end{tikzcd}$$

- The unit diagrams are dealt-with similarly. We only check the diagram for the unit $\lambda_{(-)}$, which lead to check

$$\begin{tikzcd}
\mathbf{1} \times U^{HX} \arrow[r, "\lambda_{U^{HX}}"] \arrow[d, "\mathbf{1}\times \mathrm{id}_{U^{HX}}"'] & U^{HX} \\
\mathbf{1}^{\mathbf{1}} \times U^{HX} \arrow[r, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{\mathbf{1},X}"'] & (\mathbf{1} \times U)^{H(\mathbf{1}\times X)} \arrow[u, "\boldsymbol{\lambda} h.\lambda_U \circ h \circ H(\lambda^{-1}_X)"']
\end{tikzcd}$$

Since $\lambda_{(-)} = \pi_2$, we have to show

$$\boldsymbol{\lambda}(\bullet, h).h \quad = \quad \boldsymbol{\lambda}(\bullet, h).\lambda_U \circ (\bullet \times h) \circ \mathsf{t}^2_{\mathbf{1},X} \circ H(\lambda^{-1}_X)$$

It follows from the unversal property of the Cartesian product of $\mathbb{C}$ that we have have

$$\begin{tikzcd}
\mathbf{1}\times HX & HX \arrow[l, "\lambda^{-1}_{HX}"'] \arrow[d, "H(\lambda^{-1}_{X})"] \\
H\mathbf{1}\times HX \arrow[u, "\mathbf{1}\times\mathrm{id}_{HX}"] & H(\mathbf{1}\times X) \arrow[l, "\mathrm{t}^2_{\mathbf{1},X}"]
\end{tikzcd}$$

We are therefore lead to check

$$\boldsymbol{\lambda}(\bullet, h).h \quad = \quad \boldsymbol{\lambda}(\bullet, h).\lambda_U \circ (\bullet \times h) \circ \lambda^{-1}_{HX}$$

and we are done since $\lambda^{-1}_{(-)} = \langle \mathbf{1}, \mathrm{id}_{(-)}\rangle$.

- The symmetry diagram is dealt-with similarly.

H.2 $((-)^H, \eta, \mu)$ is a monad

The maps $\eta_{(-)}$ are natural. Let $(f, F) : (U, X) \multimap (V, Y)$. We have to check

$$\eta_{(V,Y)} \circ (f, F) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)\ ,\ F) \circ \eta_{(U,X)}$$

which amounts to

$$(\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u) \circ f \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)) \circ (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u)$$

that is

$$\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.f(u) \quad = \quad \boldsymbol{\lambda} u.f \circ (\boldsymbol{\lambda}\_.u) \circ H(F)$$

and we are done.

The maps $\mu_{(-)}$ are natural. Let $(f, F) : (U, X) \multimap (V, Y)$. We have to check

$$\mu_{(V,Y)} \circ (\boldsymbol{\lambda} h.(\boldsymbol{\lambda} k.f \circ k \circ H(F)) \circ h \circ H(F)\ ,\ F) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)\ ,\ F) \circ \mu_{(U,X)}$$

which amounts to

$$(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x, x)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.f \circ (h(H(F)(x))) \circ H(F)) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x, x))$$

that is

$$(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x, x)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.\boldsymbol{\lambda} y.f(h(H(F)(x)), H(F)(y))) \quad = \quad \boldsymbol{\lambda} h.f \circ (\boldsymbol{\lambda} x.h(x, x)) \circ H(F)$$

which reduces to

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} x.\boldsymbol{\lambda} y.f(h(H(F)(x)), H(F)(y)))(x, x) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.f(h(H(F)(x), H(F)(x)))$$

and we are done.

Associativity Law. Since $\mu_{(-)}$ is the identity on the second component, we only have to check

$$\begin{tikzcd}
U^{HX\times HX\times HX} \arrow[r, "\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)"] \arrow[d, "\boldsymbol{\lambda} h.(\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y))\circ h"'] & U^{HX\times HX} \arrow[d, "\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)"] \\
U^{HX\times HX} \arrow[r, "\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)"'] & U^{HX}
\end{tikzcd}$$

that is

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} y.(\boldsymbol{\lambda} x.h(x, x))(y, y) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.((\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y, y)) \circ h)(x, x)$$

We compute

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} y.(\boldsymbol{\lambda} x.h(x, x))(y, y) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} y.h(y, y, y)$$

and we are done since

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.((\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y, y)) \circ h)(x, x) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} z.\boldsymbol{\lambda} y.h(z)(y, y))(x, x) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} y.h(x)(y, y))x \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x, x, x)$$

Unit Laws. Since $\eta_{(-)}$ and $\mu_{(-)}$ are the identity on the second component, we only have to check

$$\begin{tikzcd}
U^{HX} \arrow[r, "\boldsymbol{\lambda} u.\boldsymbol{\lambda}_.u"] \arrow[dr, equal] & U^{HX\times HX} \arrow[d, "\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)"] & U^{HX} \arrow[l, "\boldsymbol{\lambda} h.(\boldsymbol{\lambda} u.\boldsymbol{\lambda}_.u)\circ h"'] \arrow[dl, equal] \\
 & U^{HX} &
\end{tikzcd}$$

We are done since

$$(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x, x)) \circ (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u) \quad = \quad \boldsymbol{\lambda} u.\boldsymbol{\lambda} x.(\boldsymbol{\lambda}\_.u)(x, x) \quad = \quad \boldsymbol{\lambda} u.\boldsymbol{\lambda} x.ux \quad = \quad \mathrm{id}_{U^{HX}}$$

and

$$
\begin{array}{rcl}
(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \circ (\boldsymbol{\lambda} h.(\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u) \circ h) & = & \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.((\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u) \circ h)(x,x) \\
& = & \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} y.\boldsymbol{\lambda}\_.h(y))(x,x) \\
& = & \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda}\_.h(x))x \\
& = & \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.hx \\
& = & \mathrm{id}_{U^{HX}}
\end{array}
$$

H.3 $((-)^H, \eta, \mu)$ is lax symmetric monoidal

It remains to show that η and μ are lax monoidal natural transformations. Once again, we only check the second components, which amount to the following.

$\eta_{(-)}$ ***is lax monoidal.*** We have to check

$$
\begin{tikzcd}
U \times V \arrow[r, "(\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u)\times(\boldsymbol{\lambda} v.\boldsymbol{\lambda}\_.v)"] \arrow[d, equal] & U^{HX} \times V^{HY} \arrow[d, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y}"] \\
U \times V \arrow[r, "\boldsymbol{\lambda} p.\boldsymbol{\lambda}\_.p"'] & (U \times V)^{H(X\times Y)}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbf{1} \arrow[dl, equal] \arrow[dr, "\mathbf{1}"] & \\
\mathbf{1} \arrow[rr, "\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u"'] & & \mathbf{1}^{H\mathbf{1}}
\end{tikzcd}
$$

The second diagram is obvious. The first one amounts to

$$
\boldsymbol{\lambda} p.\boldsymbol{\lambda}\_.p \quad = \quad \boldsymbol{\lambda}(u,v).((\boldsymbol{\lambda}\_.u) \times (\boldsymbol{\lambda}\_.v)) \circ \langle H(\pi_1), H(\pi_2)\rangle
$$

and we are done since

$$
\boldsymbol{\lambda}(u,v).((\boldsymbol{\lambda}\_.u) \times (\boldsymbol{\lambda}\_.v)) \circ \langle H(\pi_1), H(\pi_2)\rangle \quad = \quad \boldsymbol{\lambda}(u,v).\langle \boldsymbol{\lambda}\_.u, \boldsymbol{\lambda}\_.v\rangle \quad = \quad \boldsymbol{\lambda}(u,v).\boldsymbol{\lambda}\_.\langle u,v\rangle \quad = \quad \boldsymbol{\lambda} p.\boldsymbol{\lambda}\_.p
$$

$\mu_{(-)}$ ***is lax monoidal.***

- Preservation of the binary strength amounts to

$$
\begin{tikzcd}
U^{HX\times HX} \times V^{HY\times HY} \arrow[r, "(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x))\times(\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y))"] \arrow[d, "n"'] & U^{HX} \times V^{HY} \arrow[d, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y}"] \\
(U \times V)^{H(X\times Y)\times H(X\times Y)} \arrow[r, "\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)"'] & (U \times V)^{H(X\times Y)}
\end{tikzcd}
$$

  where n is the first component of $(m^2_{A,B})^H \circ m^2_{A^H,B^H}$ (for $A = (U,X)$ and $B = (V,Y)$), so that

$$
\begin{array}{rcl}
n & = & (\boldsymbol{\lambda} l.((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \circ l) \circ ((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \\
& = & \boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda}(h',k').(h' \times k') \circ \mathsf{t}^2_{X,Y}) \circ ((h \times k) \circ \mathsf{t}^2_{X,Y}) \\
& = & \boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda}(h',k').(h' \times k') \circ \mathsf{t}^2_{X,Y}) \circ \langle h \circ H(\pi_1)\ ,\ k \circ H(\pi_2)\rangle \\
& = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.((h(H(\pi_1)p)) \times (k(H(\pi_2)p))) \circ \mathsf{t}^2_{X,Y} \\
& = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.((h(H(\pi_1)p)) \times (k(H(\pi_2)p))) \circ \langle H(\pi_1)\ ,\ H(\pi_2)\rangle \\
& = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda}(p,q).\langle h(H(\pi_1)p), H(\pi_1)q)\ ,\ k(H(\pi_2)p, H(\pi_2)q)\rangle
\end{array}
$$

  and therefore

$$
\begin{array}{rcl}
(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \circ n & = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} x.n(h,k)(x,x) \\
& = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} x.\langle h(H(\pi_1)x, H(\pi_1)x)\ ,\ k(H(\pi_2)x, H(\pi_2)x)\rangle
\end{array}
$$

  But now we are done since on the other hand,

$$
\begin{array}{rcl}
(\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \circ ((\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y))) & = & \boldsymbol{\lambda}(h,k).((\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} y.k(y,y))) \circ \mathsf{t}^2_{X,Y} \\
& = & \boldsymbol{\lambda}(h,k).((\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} y.k(y,y))) \circ \langle H(\pi_1), H(\pi_2)\rangle \\
& = & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.\langle h(H(\pi_1)p, H(\pi_1)p)\ ,\ k(H(\pi_2)p, H(\pi_2)p)\rangle
\end{array}
$$

- Preservation of the unit strength amounts to

$$
\begin{tikzcd}
& \mathbf{1} \arrow[dl, "\mathbf{1}"'] \arrow[dr, "n^0"] & \\
\mathbf{1}^{H\mathbf{1}\times H\mathbf{1}} \arrow[rr, "\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{\mathbf{1},\mathbf{1}}"'] & & \mathbf{1}^{H\mathbf{1}}
\end{tikzcd}
$$

where n^0 is the first component of $(m^0)^H \circ m^0$, so that $n^0 = (\boldsymbol{\lambda} h.\mathbf{1} \circ h) \circ \mathbf{1} = \mathbf{1}$ and we are done since

$$(\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{\mathbf{1},\mathbf{1}}) \circ \mathbf{1} \quad = \quad \mathbf{1}$$

I. Proof of Proposition G.3

This appendix is devoted to the proof of Prop. G.3. We first recall its statment.

Proposition I.1 (Prop. G.3)**.**

(i) The family of maps $\zeta_A : T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\!+\; (TA)^{\blacktriangleright}$ forms a distributive law.
(ii) Moreover, $\zeta_{(-)}$ is monoidal in the sense of Prop. J.6, that is:

$$\begin{tikzcd}
T(A^{\blacktriangleright} \otimes B^{\blacktriangleright}) \arrow[r, "T(m^2_{A,B})"] \arrow[d, "g^2_{A^{\blacktriangleright},B^{\blacktriangleright}}"'] & T((A \otimes B)^{\blacktriangleright}) \arrow[d, "\zeta_{A\otimes B}"] \\
T(A^{\blacktriangleright}) \otimes T(B^{\blacktriangleright}) \arrow[d, "\zeta_A \otimes \zeta_B"'] & (T(A \otimes B))^{\blacktriangleright} \arrow[d, "(g^2_{A,B})^{\blacktriangleright}"] \\
(TA)^{\blacktriangleright} \otimes (TB)^{\blacktriangleright} \arrow[r, "m^2_{TA,TB}"'] & (TA \otimes TB)^{\blacktriangleright}
\end{tikzcd} \tag{9}$$

where (m^2, m^0) is the (lax) strength of $(-)^{\blacktriangleright}$ defined as in Prop. F.4, and (g^2, g^0) is the oplax strength of T defined as in Cor. F.5, so that:

- *For $(-)^{\blacktriangleright}$:*

$$m^2_{A,B} \quad := \quad (\boldsymbol{\lambda}(h,k).(h \times k) \circ \langle \blacktriangleright(\pi_1), \blacktriangleright(\pi_2)\rangle \;,\; \mathrm{id}_{X\times Y}) \quad : \quad (U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \;,\; X \times Y) \quad \longrightarrow \quad ((U \times V)^{\blacktriangleright(X\times Y)} \;,\; X \times Y)$$

  (where $A = (U, X)$ and $B = (V, Y)$), and $m^0 := (\mathbf{1}, \mathbf{1}) : (\mathbf{1}, \mathbf{1}) \longrightarrow (\mathbf{1}^{\blacktriangleright \mathbf{1}}, \mathbf{1})$.

- *For T:*

$$g^2_{A,B} \quad := \quad (\mathrm{id}_{U\times V} \;,\; \boldsymbol{\lambda}(h,k).(h \times k)) \quad : \quad (U \times V \;,\; (X \times Y)^{U\times V}) \quad \longrightarrow \quad (U \times V \;,\; X^U \times Y^V)$$

  (where $A = (U, X)$ and $B = (V, Y)$), and $g^0 := (\mathbf{1}, \mathbf{1}) : (\mathbf{1}, \mathbf{1}^{\mathbf{1}}) \longrightarrow (\mathbf{1}, \mathbf{1})$.

I.1 Proof of Proposition I.1.(i)

We have to check that $\zeta : T((-)^{\blacktriangleright}) \to (T-)^{\blacktriangleright}$ is natural and that the following four coherence diagrams commute (see e.g. [11]):

$$\begin{tikzcd}
 & & (TA)^{\blacktriangleright} \arrow[drr, "(\delta_A)^{\blacktriangleright}"] & & \\
T(A^{\blacktriangleright}) \arrow[urr, "\zeta_A"] \arrow[dr, "\delta_{A^{\blacktriangleright}}"'] & & & & (TTA)^{\blacktriangleright} \\
 & TT(A^{\blacktriangleright}) \arrow[rr, "T\zeta_A"'] & & T((TA)^{\blacktriangleright}) \arrow[ur, "\zeta_{TA}"'] &
\end{tikzcd} \tag{10}$$

$$\begin{tikzcd}
 & & T(A^{\blacktriangleright}) \arrow[drr, "\zeta_A"] & & \\
T(A^{\blacktriangleright\blacktriangleright}) \arrow[urr, "T(\mu_A)"] \arrow[dr, "\zeta_{A^{\blacktriangleright}}"'] & & & & (TA)^{\blacktriangleright} \\
 & (T(A^{\blacktriangleright}))^{\blacktriangleright} \arrow[rr, "(\zeta_A)^{\blacktriangleright}"'] & & (TA)^{\blacktriangleright\blacktriangleright} \arrow[ur, "\mu_{TA}"'] &
\end{tikzcd} \tag{11}$$

$$\begin{tikzcd}
 & (TA)^{\blacktriangleright} \arrow[dr, "(\epsilon_A)^{\blacktriangleright}"] & \\
T(A^{\blacktriangleright}) \arrow[ur, "\zeta_A"] \arrow[rr, "\epsilon_{A^{\blacktriangleright}}"'] & & A^{\blacktriangleright}
\end{tikzcd} \tag{12}$$

$$\begin{tikzcd} & T(A^{\blacktriangleright}) \arrow[dr, "\zeta_A"] & \\ TA \arrow[ur, "T(\eta_A)"] \arrow[rr, "\eta_{TA}"'] & & (TA)^{\blacktriangleright} \end{tikzcd} \tag{13}$$

Recall that T is the comonad $T = (T, \epsilon, \delta)$ and that $(-)^{\blacktriangleright}$ is the monad $((-)^{\blacktriangleright}, \eta, \mu)$ on $\mathbf{G}(\mathscr{S})$. We repeat the definitions of the functors T and $(-)^{\blacktriangleright}$:

$$\begin{array}{rclcrcl} (f,F):(U,X) \twoheadrightarrow (V,Y) & \overset{T}{\longmapsto} & (f\ ,\ \boldsymbol{\lambda} h.F \circ h \circ f) & : & (U,X^U) & \twoheadrightarrow & (V,Y^V) \\ (f,F):(U,X) \twoheadrightarrow (V,Y) & \overset{(-)^{\blacktriangleright}}{\longmapsto} & (\boldsymbol{\lambda} h.f \circ h \circ {\blacktriangleright} F\ ,\ F) & : & (U^{\blacktriangleright X},X) & \twoheadrightarrow & (V^{\blacktriangleright Y},Y) \end{array}$$

and of the natural maps η and μ:

$$\begin{array}{rcrcl} (f_\eta, F_\eta) & : & (U,X) & \twoheadrightarrow & (U^{\blacktriangleright X}, X) \\ (f_\mu, F_\mu) & : & (U^{\blacktriangleright X \times \blacktriangleright X}, X) & \twoheadrightarrow & (U^{\blacktriangleright X}, X) \end{array}$$

where $F_\eta = F_\mu = \mathrm{id}_X$, $f_\eta(u,x) = u$ and $f_\mu(h,x) = h(x,x)$.
Moreover, the natural maps ϵ and δ are given by

$$\begin{array}{rcrcl} (f_\epsilon, F_\epsilon) & : & (U,X^U) & \twoheadrightarrow & (U,X) \\ (f_\delta, F_\delta) & : & (U,X^U) & \twoheadrightarrow & (U,X^{U\times U}) \end{array}$$

where $f_\epsilon = f_\delta = \mathrm{id}_U$, $F_\epsilon(u,x) = x$ and $F_\delta(h,u) = h(u,u)$.

We check in turn the required diagrams.

Lemma I.2. *ζ is natural, that is, given $(g,G) : A \twoheadrightarrow B$, we have*

$$\begin{tikzcd} T(A^{\blacktriangleright}) \arrow[r, "T((g,G)^{\blacktriangleright})"] \arrow[d, "\zeta_A"'] & T(B^{\blacktriangleright}) \arrow[d, "\zeta_B"] \\ (TA)^{\blacktriangleright} \arrow[r, "(T(g,G))^{\blacktriangleright}"'] & (TB)^{\blacktriangleright} \end{tikzcd}$$

Proof. Let $A = (U,X)$ and $B = (V,Y)$, and consider $(g,G) : (U,X) \twoheadrightarrow (V,Y)$. Note that

$$\begin{array}{rcrcrcl} (g,G)^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}G\ ,\ G) & : & (U^{\blacktriangleright X}, X) & \twoheadrightarrow & (V^{\blacktriangleright Y}, Y) \\ T((g,G)^{\blacktriangleright}) & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}G\ ,\ \boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G)) & : & (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) & \twoheadrightarrow & (V^{\blacktriangleright Y}, Y^{V^{\blacktriangleright Y}}) \\ T(g,G) & = & (g\ ,\ \boldsymbol{\lambda} h.Ghg) & : & (U,X^U) & \twoheadrightarrow & (V,Y^V) \\ (T(g,G))^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}(\boldsymbol{\lambda} h.Ghg)\ ,\ \boldsymbol{\lambda} h.Ghg) & : & (U^{\blacktriangleright(X^U)}, X^U) & \twoheadrightarrow & (V^{\blacktriangleright(Y^V)}, Y^V) \end{array}$$

We have to show that

$$(T(g,G))^{\blacktriangleright} \circ \zeta_A \quad = \quad \zeta_B \circ T((g,G)^{\blacktriangleright})$$

that is

$$(\boldsymbol{\lambda} h.gh{\blacktriangleright}(\boldsymbol{\lambda} h.Ghg)) \circ f^{\zeta_A} \quad = \quad f^{\zeta_B} \circ (\boldsymbol{\lambda} h.gh{\blacktriangleright}G) \qquad \text{and} \qquad F^{\zeta_A} \circ (\boldsymbol{\lambda} h.Ghg) \quad = \quad \boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G) \circ F^{\zeta_B}$$

For the first equation, which has type $U^{\blacktriangleright X} \to V^{\blacktriangleright(Y^V)}$, given $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$ and $\xi_n \in (Y^V)_n$, one has to show the following (where some $\circ$ are replaced by juxtaposition)

$$((\boldsymbol{\lambda} h.g_{n+1}h{\blacktriangleright}(\boldsymbol{\lambda} h.G_{n+1}hg_{n+1})) \circ f^{\zeta_A}_{n+1})(\theta_{n+1})(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1} \circ (\boldsymbol{\lambda} h.g_{n+1}h{\blacktriangleright}G_{n+1}))(\theta_{n+1})(\xi_n)$$

that is

$$((\boldsymbol{\lambda} h.g_{n+1} \circ h \circ (\boldsymbol{\lambda} h.G_nhg_n))(f^{\zeta_A}_{n+1}(\theta_{n+1})))(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1}((\boldsymbol{\lambda} h.g_{n+1} \circ h \circ G_n)(\theta_{n+1})))(\xi_n)$$

that is

$$(g_{n+1} \circ (f^{\zeta_A}_{n+1}(\theta_{n+1})) \circ (\boldsymbol{\lambda} h.G_nhg_n))(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n))(\xi_n)$$

that is

$$g_{n+1}(f^{\zeta_A}_{n+1}(\theta_{n+1})((\boldsymbol{\lambda} h.G_nhg_n)\xi_n)) \quad = \quad f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n\ ,\ \xi_n)$$

that is

$$g_{n+1}(f^{\zeta_A}_{n+1}(\theta_{n+1}\ ,\ G_n\xi_ng_n)) \quad = \quad f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n\ ,\ \xi_n)$$

that is

$$g_{n+1} \circ \theta_{n+1} \circ \mathrm{fix}_n(G_n\xi_n\theta_n) \quad = \quad g_{n+1} \circ \theta_{n+1} \circ G_n \circ \mathrm{fix}_n(\xi_ng_n\theta_nG_{n-1})$$

which is easily seens to hold, when unfolding the fixpoints, thanks to associativity of composition.

The second equation, of type $Y^V \to X^{U^{\blacktriangleright X}}$, amounts, for $\xi_n \in (Y^V)_n$ and $\theta_n \in (U^{\blacktriangleright X})_n$, to the following (where some $\circ$ are replaced by juxtaposition)

$$F^{\zeta_A}_n(G_n\xi_ng_n\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G))(F^{\zeta_B}(\xi_n)))(\theta_n)$$

that is

$$F_n^{\zeta_A}(G_n\xi_n g_n\ ,\ \theta_n) \quad = \quad (G_n \circ (F_n^{\zeta_B}(\xi_n)) \circ (\boldsymbol{\lambda} h.g_n h \blacktriangleright G_n))(\theta_n)$$

that is

$$F_n^{\zeta_A}(G_n\xi_n g_n\ ,\ \theta_n) \quad = \quad G_n(F_n^{\zeta_B}(\xi_n)((\boldsymbol{\lambda} h.g_n h \blacktriangleright G_n)(\theta_n)))$$

that is

$$F_n^{\zeta_A}(G_n\xi_n g_n\ ,\ \theta_n) \quad = \quad G_n(F_n^{\zeta_B}(\xi_n\ ,\ g_n\theta_n \blacktriangleright G_n))$$

which also holds thanks to associativity of composition (when unfolding the fixpoints). □

Lemma I.3. *Diagram (10) commutes.*

Proof. Let $A = (U, X)$, so that

$$T(A^{\blacktriangleright}) \quad = \quad T(U^{\blacktriangleright X}, X) \quad = \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \qquad \text{and} \qquad (TA)^{\blacktriangleright} \quad = \quad (U, X^U)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright (X^U)}, X^U)$$

The diagram has type

$$T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\to (TTA)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X}\ ,\ X^{U^{\blacktriangleright X}}) \longrightarrow\!\!\!\!\to (U^{\blacktriangleright (X^{U\times U})}\ ,\ X^{U\times U})$$

Moreover,

$$\begin{array}{rcccl} (\delta_A)^{\blacktriangleright} & = & (\mathrm{id}_U\ ,\ \boldsymbol{\lambda} hu.h(u,u))^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.h \blacktriangleright (\boldsymbol{\lambda} hu.h(u,u))\ ,\ \boldsymbol{\lambda} hu.h(u,u)) \\ T\zeta_A & = & T(f^{\zeta_A}, F^{\zeta_A}) & = & (f^{\zeta_A}\ ,\ \boldsymbol{\lambda} h.F^{\zeta_A} h f^{\zeta_A}) \end{array}$$

We have to check the following two equations:

$$f_{\delta_A{}^{\blacktriangleright}} \circ f^{\zeta_A} \quad = \quad f^{\zeta_{TA}} \circ f_{T\zeta_A} \circ f_{\delta_A{}^{\blacktriangleright}} \qquad \text{and} \qquad F^{\zeta_A} \circ F_{\delta_A{}^{\blacktriangleright}} \quad = \quad F_{\delta_A{}^{\blacktriangleright}} \circ F_{T\zeta_A} \circ F^{\zeta_{TA}}$$

The first one, of type $U^{\blacktriangleright X} \to U^{\blacktriangleright (X^{U\times U})}$, amounts, for $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$ and $\xi_{n+1} \in X_{n+1}^{U\times U}$, to the following

$$((\boldsymbol{\lambda} h.h \blacktriangleright (\boldsymbol{\lambda} hu.h(u,u))) \circ f_{n+1}^{\zeta_A})(\theta_{n+1})(\xi_{n+1}) \quad = \quad (f_{n+1}^{\zeta_{TA}} f_{n+1}^{\zeta_A})(\theta_{n+1})(\blacktriangleright \xi_{n+1})$$

that is

$$(f_{n+1}^{\zeta_A}(\theta_{n+1}) \circ \blacktriangleright(\boldsymbol{\lambda} hu.h(u,u)))(\xi_{n+1}) \quad = \quad f_{n+1}^{\zeta_{TA}}(f_{n+1}^{\zeta_A}(\theta_{n+1})\ ,\ \xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \boldsymbol{\lambda} u.\xi_n(u,u)) \quad = \quad f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n)))$$

Write

$$l_n \quad := \quad f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \boldsymbol{\lambda} u.\xi_n(u,u)) \qquad \text{and} \qquad r_n \quad := \quad f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n)))$$

The proof is then by induction on n. In the base case $n = 0$, both sides unfold to $\theta_1(\bullet)$. For the induction step, assuming the property for $r_n = l_n$, we show $l_{n+1} = r_{n+1}$.

First, note that Note that

$$\begin{array}{rcl} \mathrm{fix}_{n+1}^U(\boldsymbol{\lambda} u.\xi_{n+1}(u,u) \circ \theta_{n+1}) & = & \mathrm{fix}_{n+1}^U(\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x), \theta_{n+1}(x))) \\ & = & (\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x)\ ,\ \theta_{n+1}(x)))(\mathrm{fix}_n^U(\boldsymbol{\lambda} x.\xi_n(\theta_n(x), \theta_n(x)))) \\ & = & (\boldsymbol{\lambda} u.\xi_{n+1}(u,u))(\theta_{n+1}(\mathrm{fix}_n^U((\boldsymbol{\lambda} u.\xi_n(u,u)) \circ \theta_n))) \\ & = & \xi_{n+1}(l_n, l_n) \end{array}$$

so that

$$l_{n+1} \quad = \quad \theta_{n+2}(\xi_{n+1}(l_n, l_n))$$

On the other hand, note that

$$\begin{array}{rcl} \mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) & = & \xi_{n+1}(f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n)))) \\ & = & \xi_{n+1}(r_n) \end{array}$$

and so in particular

$$\begin{array}{rcl} r_n & = & \theta_{n+1}(\mathrm{fix}_n(\mathrm{fix}_n^{X^U}(\xi_n \circ f^{\zeta_A}(\theta_n)) \circ \theta_n)) \\ & = & \theta_{n+1}(\mathrm{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)) \end{array}$$

We thus have

$$\begin{array}{rcl} r_{n+1} & = & \theta_{n+2}(\mathrm{fix}_{n+1}(\mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) \circ \theta_{n+1})) \\ & = & \theta_{n+2}(\mathrm{fix}_{n+1}(\xi_{n+1}(r_n) \circ \theta_{n+1}) \\ & = & \theta_{n+2}(\xi_{n+1}(r_n)(\theta_{n+1}(\mathrm{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)))) \\ & = & \theta_{n+2}(\xi_{n+1}(r_n)(r_n)) \end{array}$$

and we conclude by induction hypothesis.

The second equation, of type $X^{U\times U} \to X^{U^{\blacktriangleright X}}$, amounts, for $\xi_n \in (X^{U\times U})_n$ and $\theta_n \in (U^{\blacktriangleright X})_n$, to the following:

$$F_n^{\zeta_A} \circ (\boldsymbol{\lambda} hu.h(u,u))(\xi_n)(\theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k)) \circ (\boldsymbol{\lambda} h.F_n^{\zeta_A} h f_n^{\zeta_A}) \circ F_n^{\zeta_{TA}})(\xi_n)(\theta_n)$$

that is

$$F_n^{\zeta_A}((\boldsymbol{\lambda} hu.h(u,u))\xi_n\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k))((\boldsymbol{\lambda} h.F_n^{\zeta_A} h f_n^{\zeta_A})(F_n^{\zeta_{TA}}(\xi_n))))(\theta_n)$$

that is

$$F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k))((F_n^{\zeta_A} \circ F_n^{\zeta_{TA}}(\xi_n) \circ f_n^{\zeta_A})))(\theta_n)$$

that is

$$F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad (\boldsymbol{\lambda} k.(F_N^{\zeta_A} \circ F_n^{\zeta_{TA}}(\xi_n) \circ f_n^{\zeta_A})(k,k))\theta_n$$

that is

$$F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad (F_n^{\zeta_A} \circ F_n^{\zeta_{TA}}(\xi_n) \circ f_n^{\zeta_A})(\theta_n)(\theta_n)$$

that is

$$F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n\ ,\ f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)$$

Reasonning as for the first equation, write

$$l_n \quad := \quad F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \qquad \text{and} \qquad r_n \quad := \quad F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n\ ,\ f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)$$

with

$$\begin{array}{rcl} l_{n+1} & = & \mathrm{fix}_{n+1}((\boldsymbol{\lambda} u.\xi_{n+1}(u,u)) \circ \theta_{n+1}) \\ & = & \xi_{n+1}(\theta_{n+1}(l_n)\ ,\ \theta_{n+1}(l_n)) \end{array}$$

and on the other hand

$$\begin{array}{rcl} F_{n+1}^{\zeta_{TA}}(\xi_{n+1}\ ,\ f_{n+1}^{\zeta_A}(\theta_{n+1})) & = & \mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) \\ & = & \xi_{n+1}(f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n))) \\ & = & \xi_{n+1}(\theta_{n+1}(F_n^{\zeta_A}(\mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)) \\ & = & \xi_{n+1}(\theta_{n+1}(F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n, f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)) \end{array}$$

We thus have

$$\begin{array}{rcl} r_{n+1} & = & \mathrm{fix}_{n+1}(\mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) \circ \theta_{n+1}) \\ & = & \mathrm{fix}_{n+1}(\xi_{n+1}(\theta_{n+1}(r_n)) \circ \theta_{n+1}) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)\ ,\ \theta_{n+1}(\mathrm{fix}_n(\xi_n(\theta_n(r_{n-1})) \circ \theta_n))) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)\ ,\ \theta_{n+1}(r_n)) \end{array}$$

and we conclude by induction hypothesis. □

Lemma I.4. *Diagram (11) commutes.*

Proof. Let $A = (U, X)$ so that the diagram has type

$$T(A^{\blacktriangleright\blacktriangleright}) \longrightarrow\!\!\!\!\!\rightarrow (TA)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X \times \blacktriangleright X}, X^{U^{\blacktriangleright X \times \blacktriangleright X}}) \longrightarrow\!\!\!\!\!\rightarrow (U^{\blacktriangleright (X^U)}, X^U)$$

Note that

$$\begin{array}{rcccl} T(\mu_A) & = & T(\boldsymbol{\lambda} hx.h(x,x)\ ,\ \mathrm{id}_X) & = & (\boldsymbol{\lambda} hx.h(x,x)\ ,\ \boldsymbol{\lambda} k.(k \circ \boldsymbol{\lambda} hx.h(x,x))) \\ (\zeta_A)^{\blacktriangleright} & = & (f^{\zeta_A}, F^{\zeta_A})^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.f^{\zeta_A} \circ h \circ \blacktriangleright F^{\zeta_A}\ ,\ F^{\zeta_A}) \end{array}$$

We have to check the following two equations:

$$f^{\zeta_A} \circ f_{T\mu_A} \quad = \quad f_{\mu_{TA}} \circ f_{(\zeta_A)^{\blacktriangleright}} \circ f^{\zeta_A \blacktriangleright} \qquad \text{and} \qquad F_{T\mu_A} \circ F^{\zeta_A} \quad = \quad F^{\zeta_A \blacktriangleright} \circ F_{(\zeta_A)^{\blacktriangleright}} \circ F_{\mu_{TA}}$$

The first equation, of type $U^{\blacktriangleright X \times \blacktriangleright X} \to U^{\blacktriangleright (X^U)}$, amounts, for $\theta_{n+1} \in (U^{\blacktriangleright X \times \blacktriangleright X})_{n+1}$ and $\xi_n \in (X^U)_n$, to the following:

$$(f_{n+1}^{\zeta_A} \circ (\boldsymbol{\lambda} hx.h(x,x)))(\theta_{n+1})(\xi_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k)) \circ (\boldsymbol{\lambda} h.f_{n+1}^{\zeta_A} h \blacktriangleright F_{n+1}^{\zeta_A}) \circ f_{n+1}^{\zeta_A \blacktriangleright})(\theta_{n+1})(\xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k)) \circ (\boldsymbol{\lambda} h.f_{n+1}^{\zeta_A} h F_n^{\zeta_A}) \circ f_{n+1}^{\zeta_A \blacktriangleright})(\theta_{n+1})(\xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad (\boldsymbol{\lambda} hk.h(k,k))(f_{n+1}^{\zeta_A} \circ f_{n+1}^{\zeta_A \blacktriangleright}(\theta_{n+1}) \circ F_n^{\zeta_A})(\xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad (f_{n+1}^{\zeta_A} \circ f_{n+1}^{\zeta_A \blacktriangleright}(\theta_{n+1}) \circ F_n^{\zeta_A})(\xi_n)(\xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad f_{n+1}^{\zeta_A}(f_{n+1}^{\zeta_A \blacktriangleright}(\theta_{n+1}\ ,\ F_n^{\zeta_A}(\xi_n))\ ,\ \xi_n)$$

Let

$$l_n \quad := \quad f_{n+1}^{\zeta_A}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \qquad \text{and} \qquad r_n \quad := \quad f_{n+1}^{\zeta_A}(f_{n+1}^{\zeta_A \blacktriangleright}(\theta_{n+1}\ ,\ F_n^{\zeta_A}(\xi_n))\ ,\ \xi_n)$$

Note that for all n we have

$$\begin{array}{rcl} l_{n+1} & = & (\boldsymbol{\lambda} x.\theta_{n+2}(x,x))\mathrm{fix}_{n+1}(\xi_{n+1} \circ \boldsymbol{\lambda} x.\theta_{n+1}(x,x)) \\ & = & (\boldsymbol{\lambda} x.\theta_{n+2}(x,x))((\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x,x)))\mathrm{fix}_n(\xi_n \circ \boldsymbol{\lambda} x.\theta_n(x,x)))) \\ & = & \theta_{n+2}(\xi_{n+1}(l_n), \xi_{n+1}(l_n)) \end{array}$$

On the other hand,

$$\begin{array}{rcl} r_{n+1} & = & f^{\zeta_A}_{n+2}(f^{\zeta_A\blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))\ ,\ \xi_{n+1}) \\ & = & f^{\zeta_A\blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))(\mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A\blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \\ & = & \theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1})\ ,\ \mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A\blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \end{array}$$

So we show by induction on n that

$$\xi_{n+1}(r_n) \quad = \quad \mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1}) \quad = \quad \mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A\blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))$$

The base case is trivial. For the induction step, on the one hand we have

$$\begin{array}{rcl} \mathrm{fix}_{n+2}(F^{\zeta_A}_{n+2}(\xi_{n+2})\circ\theta_{n+2}) & = & F^{\zeta_A}_{n+2}(\xi_{n+2}\ ,\ \theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1}))) \\ & = & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1})\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}\ ,\ \theta_{n+1}(\mathrm{fix}_n(F^{\zeta_A}_n(\xi_n)\circ\theta_n)))) \\ & = & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1})\ ,\ \mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1})) \end{array}$$

and we conclude by induction hypothesis, and on the other hand

$$\begin{array}{rcl} \mathrm{fix}^X_{n+2}(\xi_{n+2}\circ f^{\zeta_A\blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))) & = & \xi_{n+2}\circ f^{\zeta_A\blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))(\mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A\blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \\ & = & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_n(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1})\ ,\ \mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A\blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n))))) \end{array}$$

and we also conclude by induction hypothesis.

The second equation, of type $X^U \to X^{U^{\blacktriangleright X\times\blacktriangleright X}}$, amounts, for $\xi_n \in (X^U)_n$ and $\theta_n \in (U^{\blacktriangleright X\times\blacktriangleright X})_n$, to the following

$$((\boldsymbol{\lambda} k.(k\circ\boldsymbol{\lambda} hx.h(x,x)))\circ F^{\zeta_A}_n)(\xi_n)(\theta_n) \quad = \quad (F^{\zeta_A\blacktriangleright}_n\circ F^{\zeta_A}_n)(\xi_n)(\theta_n)$$

that is

$$(F^{\zeta_A}_n(\xi_n)\circ\boldsymbol{\lambda} hx.h(x,x))(\theta_n) \quad = \quad F^{\zeta_A\blacktriangleright}_n(F^{\zeta_A}_n(\xi_n)\ ,\ \theta_n)$$

that is

$$F^{\zeta_A}_n(\xi_n\ ,\ \boldsymbol{\lambda} x.\theta_n(x,x)) \quad = \quad F^{\zeta_A\blacktriangleright}_n(F^{\zeta_A}_n(\xi_n)\ ,\ \theta_n)$$

This is dealt-with similarly to (but in a much simpler way than) the first equation. □

Lemma I.5. *Diagram (12) commutes.*

Proof. Let $A = (U, X)$, so that the diagram has type

$$T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\!\rightarrow A^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \longrightarrow\!\!\!\!\!\rightarrow (U^{\blacktriangleright X}, X)$$

Note that

$$(\epsilon_A)^{\blacktriangleright} \quad = \quad (\mathrm{id}_U, \boldsymbol{\lambda} xu.x)^{\blacktriangleright} \quad = \quad (\boldsymbol{\lambda} h.(h\circ\blacktriangleright(\boldsymbol{\lambda} xu.x)), \boldsymbol{\lambda} xu.x)$$

We have to show

$$\boldsymbol{\lambda} h.(h\circ\blacktriangleright(\boldsymbol{\lambda} xu.x))\circ f^{\zeta_A} \quad = \quad \mathrm{id}_{U^{\blacktriangleright X}} \qquad \text{and} \qquad F^{\zeta_A}\circ\boldsymbol{\lambda} xu.x \quad = \quad \boldsymbol{\lambda} xu.x$$

For the first equation, given $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$, we have to show

$$f^{\zeta_A}_{n+1}(\theta_{n+1})\circ\blacktriangleright(\boldsymbol{\lambda} xu.x) \quad = \quad \theta_{n+1}$$

The result is trivial since the left-hand side unfolds to

$$\boldsymbol{\lambda}\blacktriangleright x.f^{\zeta_A}_{n+1}(\theta_{n+1}, \boldsymbol{\lambda}\_.x) \quad = \quad \boldsymbol{\lambda}\blacktriangleright x.\theta_{n+1}(\mathrm{fix}_n(\boldsymbol{\lambda}\_.x)) \quad = \quad \boldsymbol{\lambda}\blacktriangleright x.\theta_{n+1}(x)$$

The second equation is simpler and omitted. □

Lemma I.6. *Diagram (13) commutes.*

Proof. Let $A = (U, X)$, so that the diragram has type

$$TA \longrightarrow\!\!\!\!\!\rightarrow (TA)^{\blacktriangleright} \quad = \quad (U, X^U) \longrightarrow\!\!\!\!\!\rightarrow (U^{\blacktriangleright(X^U)}, X^U)$$

Note that

$$T(\eta_A) \quad = \quad T(\boldsymbol{\lambda} ux.u, \mathrm{id}_X) \quad = \quad (\boldsymbol{\lambda} ux.u, \boldsymbol{\lambda} h.h\circ(\boldsymbol{\lambda} ux.u))$$

We have to show

$$f^{\zeta_A}\circ(\boldsymbol{\lambda} ux.u) \quad = \quad \boldsymbol{\lambda} ux.u \qquad \text{and} \qquad (\boldsymbol{\lambda} h.h\circ(\boldsymbol{\lambda} ux.u))\circ F^{\zeta_A} \quad = \quad \mathrm{id}_{X^U}$$

For the first equation, given $u \in U_{n+1}$ and $\xi_n \in (X^U)_n$, we have to show

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.u\ ,\ \xi_n) \quad = \quad u$$

which is trivial. For the second equation, given $\xi_n \in X_n$ and $u \in U_n$ we have to show

$$F^{\zeta_A}(\xi_n\ ,\ \boldsymbol{\lambda} x.u) \quad = \quad \xi_n(u)$$

which is also trivial. □

I.2 Proof of Proposition I.1.(ii)

Fix $\mathbf{G}(\mathscr{S})$-objects $A = (U, X)$ and $B = (V, Y)$. Diagram (9) amounts, to the following two diagrams, for resp. the first and second component of $\mathbf{G}(\mathscr{S})$:

$$
\begin{tikzcd}
U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \arrow[rr, "\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle\blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle"] \arrow[d, "\mathrm{id}_{U^{\blacktriangleright X}\times V^{\blacktriangleright Y}}"'] & & (U \times V)^{\blacktriangleright(X\times Y)} \arrow[d, "f^{\zeta_{A\otimes B}}"] \\
U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \arrow[d, "f^{\zeta_A}\times f^{\zeta_B}"'] & & (U \times V)^{\blacktriangleright((X\times Y)^{U\times V})} \arrow[d, "\boldsymbol{\lambda} h.h\circ\blacktriangleright(\boldsymbol{\lambda}(h,k).h\times k)"] \\
U^{\blacktriangleright(X^U)} \times V^{\blacktriangleright(Y^V)} \arrow[rr, "\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle\blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle"'] & & (U \times V)^{\blacktriangleright(X^U\times Y^V)}
\end{tikzcd}
\tag{14}
$$

$$
\begin{tikzcd}
(X \times Y)^{U^{\blacktriangleright X}\times V^{\blacktriangleright Y}} & & (X \times Y)^{(U\times V)^{\blacktriangleright(X\times Y)}} \arrow[ll, "\boldsymbol{\lambda} h.h\circ(\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle\blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle)"'] \\
X^{U^{\blacktriangleright X}} \times Y^{V^{\blacktriangleright Y}} \arrow[u, "\boldsymbol{\lambda}(h,k).h\times k"] & & (X \times Y)^{U\times V} \arrow[u, "F^{\zeta_{A\otimes B}}"'] \\
X^U \times Y^V \arrow[u, "F^{\zeta_A}\times F^{\zeta_B}"] & & X^U \times Y^V \arrow[ll, "\mathrm{id}_{X^U\times Y^V}"] \arrow[u, "\boldsymbol{\lambda}(h,k).h\times k"']
\end{tikzcd}
\tag{15}
$$

I.2.1 Commutation of (15).

We reason modulo $((-)\times(-))_n \simeq (-)_n \times (-)_n$. Consider $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$, $\theta'_{n+1} \in (V^{\blacktriangleright Y})$, and $\xi_{n+1} \in (X^U)_{n+1}$, $\xi'_{n+1} \in (Y^V)_{n+1}$.

We have to show that

$$\langle F^{\zeta_A}_{n+1}(\xi_{n+1}, \theta_{n+1})\ ,\ F^{\zeta_B}_{n+1}(\xi'_{n+1}, \theta'_{n+1})\rangle \quad = \quad F^{\zeta_{A\otimes B}}_{n+1}(\xi_{n+1} \times \xi'_{n+1}, \boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle)$$

which amounts to

$$\langle \mathrm{fix}_{n+1}(\xi_{n+1} \circ \theta_{n+1})\ ,\ \mathrm{fix}_{n+1}(\xi'_{n+1} \circ \theta'_{n+1})\rangle \quad = \quad \mathrm{fix}_{n+1}((\xi_{n+1} \times \xi'_{n+1}) \circ (\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle))$$

that is

$$\begin{aligned}
\langle(\xi_{n+1} \circ \theta_{n+1} \circ \xi_n \circ \theta_n \circ \ldots \circ \xi_0 \circ \theta_0)(\bullet)\ ,\ (\xi'_{n+1} \circ \theta'_{n+1} \circ \xi'_n \circ \theta'_n \circ \ldots \circ \xi'_0 \circ \theta'_0)(\bullet)\rangle \quad = \\
((\xi_{n+1} \times \xi'_{n+1}) \circ (\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle) \circ \ldots \circ (\xi_0 \times \xi'_0) \circ (\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_0(x), \theta'_0(y)\rangle))(\bullet, \bullet)
\end{aligned}$$

which follows from an easy induction on $n \in \mathbb{N}$.

I.2.2 Commutation of (14).

We reason modulo $((-) \times (-))_n \simeq (-)_n \times (-)_n$. Consider $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$, $\theta'_{n+1} \in (V^{\blacktriangleright Y})$, and $\xi_n \in (X^U)_n$, $\xi'_n \in (Y^V)_n$.

We have to show that

$$\langle f^{\zeta_A}_{n+1}(\theta_{n+1}, \xi_n)\ ,\ f^{\zeta_B}_{n+1}(\theta'_{n+1}, \xi'_n)\rangle \quad = \quad f^{\zeta_{A\otimes B}}_{n+1}(\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle\ ,\ \xi_n \times \xi'_n)$$

which amounts to (leaving implicit the restriction map r_n):

$$\begin{aligned}
&\langle\theta_{n+1}(F^{\zeta_A}_{n+1}(\theta_n, \xi_n))\ ,\ \theta'_{n+1}(F^{\zeta_B}_{n+1}(\theta'_n, \xi'_n))\rangle \\
&\qquad = \quad (\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle)(F^{\zeta_{A\otimes B}}_n(\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_n(x), \theta'_n(y)\rangle\ ,\ \xi_n \times \xi'_n))
\end{aligned}$$

that is

$$\begin{aligned}
&(\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle)\langle F^{\zeta_A}_{n+1}(\theta_n, \xi_n)\ ,\ F^{\zeta_B}_{n+1}(\theta'_n, \xi'_n)\rangle \\
&\qquad = \quad (\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle)(F^{\zeta_{A\otimes B}}_n(\boldsymbol{\lambda}\blacktriangleright(x, y).\langle\theta_n(x), \theta'_n(y)\rangle\ ,\ \xi_n \times \xi'_n))
\end{aligned}$$

and we are done by (15).

J. Monoids, Monads and Monoidal Categories

This appendix gathers easy and possibly well-known facts about monoidal categories, to be used in the proofs of §4. We refer to [24, 27] for missing details.

J.1 Monads and Comonads

Monads. A monad on a category $\mathbb{C}$ is a triple $T = (T, \eta, \mu)$ consisting of a functor $T : \mathbb{C} \to \mathbb{C}$ and two natural transformations $\eta_A : A \to TA$ and $\mu_A : TTA \to TA$ satisfying:

$$\begin{tikzcd} TTTA \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & TTA \arrow[d, "\mu_A"] \\ TTA \arrow[r, "\mu_A"'] & TA \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, equal] & TTA \arrow[d, "\mu_A"] & TA \arrow[l, "T\eta_A"'] \arrow[dl, equal] \\ & TA & \end{tikzcd}$$

The Kleisli category $\mathbf{Kl}(T) = \mathbb{C}_T$ of T has the same objects as $\mathbb{C}$ and $\mathbf{Kl}(T)[A, B] := \mathbb{C}[A, TB]$. The categories $\mathbb{C}$ and $\mathbf{Kl}(T) = \mathbb{C}_T$ are related by an adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "\mathsf{F}_T"'] \arrow[r, phantom, "\top"] & \mathbf{Kl}(T) = \mathbb{C}_T \arrow[l, bend right, "\mathsf{U}_T"'] \end{tikzcd}$$

where:

- The right adjoint $\mathsf{U}_T : \mathbf{Kl}(T) \to \mathbb{C}$ maps objects A of $\mathbf{Kl}(T)$ to TA and takes $f \in \mathbf{Kl}(T)[A, B] = \mathbb{C}[A, TB]$ to
$$\mu_B \circ T(f) \in \mathbb{C}[\mathsf{U}_T A, \mathsf{U}_T B] = \mathbb{C}[TA, TB]$$
- The left adjoint $\mathsf{F}_T : \mathbb{C} \to \mathbf{Kl}(T)$ is the identity on objects and takes $f \in \mathbb{C}[A, B]$ to $\mathsf{F}_T(f) := \eta_B \circ f \in \mathbf{Kl}(T)[A, B] = \mathbb{C}[A, TB]$.

Comonads. Dually a comonad on $\mathbb{C}$ is a monad on $\mathbb{C}^{\mathrm{op}}$. It is therefore given by a triple $G = (G, \epsilon, \delta)$ where the functor $G : \mathbb{C} \to \mathbb{C}$ and the natural transformations $\epsilon_A : GA \to A$ and $\delta_A : GA \to GGA$ satisfy:

$$\begin{tikzcd} GA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & GGA \arrow[d, "\delta_{GA}"] \\ GGA \arrow[r, "G\delta_A"'] & GGGA \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} GA & GGA \arrow[l, "\epsilon_{GA}"'] \arrow[r, "G\epsilon_A"] & GA \\ & GA \arrow[ul, equal] \arrow[u, "\delta_A"] \arrow[ur, equal] & \end{tikzcd}$$

The coKleisli category $\mathbf{Kl}(G) = \mathbb{C}_G$ of G has the same objects as $\mathbb{C}$ and $\mathbf{Kl}(G)[A, B] := \mathbb{C}[GA, B]$. The categories $\mathbb{C}$ and $\mathbf{Kl}(G) = \mathbb{C}_G$ are related by an adjunction

$$\begin{tikzcd} \mathbf{Kl}(G) = \mathbb{C}_G \arrow[r, bend right, "\mathsf{U}_G"'] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, bend right, "\mathsf{F}_G"'] \end{tikzcd}$$

where:

- The left adjoint $\mathsf{U}_G : \mathbf{Kl}(G) \to \mathbb{C}$ maps objects A of $\mathbf{Kl}(G)$ to GA and takes $f \in \mathbf{Kl}(G)[A, B] = \mathbb{C}[GA, B]$ to
$$G(f) \circ \delta_A \in \mathbb{C}[\mathsf{U}_G A, \mathsf{U}_G B] = \mathbb{C}[GA, GB]$$
- The right adjoint $\mathsf{F}_G : \mathbb{C} \to \mathbf{Kl}(G)$ is the identity on objects and takes $f \in \mathbb{C}[A, B]$ to $\mathsf{F}_G(f) := f \circ \epsilon_A \in \mathbf{Kl}(G)[A, B] = \mathbb{C}[GA, B]$.

J.1.1 (Lax) (Symmetric) Monoidal Monads

There are different notions of monoidal functor (see e.g. [27]). Here we use *lax* monoidal functors (as the functor part of *lax* monoidal *monads*), and the dual notion of *oplax* monoidal functor (as the functor part of *oplax* monoidal *comonads*).

(Lax) Symmetric Monoidal Functors. A (lax) symmetric monoidal functor on a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is a functor F equipped with natural transformations

$$m^2_{A,B} : FA \otimes FB \to F(A \otimes B) \qquad \text{and} \qquad m^0 : \mathbf{I} \to F(\mathbf{I})$$

making the following diagrams commute:

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[rr, "\alpha_{FA,FB,FC}"] \arrow[d, "m^2_{A,B} \otimes \mathrm{id}_{FC}"'] & & FA \otimes (FB \otimes FC) \arrow[d, "\mathrm{id}_{FA} \otimes m^2_{B,C}"] \\
F(A \otimes B) \otimes FC \arrow[d, "m^2_{A \otimes B, C}"'] & & FA \otimes F(B \otimes C) \arrow[d, "m^2_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[rr, "F(\alpha_{A,B,C})"'] & & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I} \otimes FA \arrow[r, "\lambda_{FA}"] \arrow[d, "m^0 \otimes \mathrm{id}_{FA}"'] & FA \\
F\mathbf{I} \otimes FA \arrow[r, "m^2_{\mathbf{I},A}"'] & F(\mathbf{I} \otimes A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes \mathbf{I} \arrow[r, "\rho_{FA}"] \arrow[d, "\mathrm{id}_{FA} \otimes m^0"'] & FA \\
FA \otimes F\mathbf{I} \arrow[r, "m^2_{A,\mathbf{I}}"'] & F(A \otimes \mathbf{I}) \arrow[u, "F(\rho_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes FB \arrow[r, "\gamma_{FA,FB}"] \arrow[d, "m^2_{A,B}"'] & FB \otimes FA \arrow[d, "m^2_{B,A}"] \\
F(A \otimes B) \arrow[r, "F(\gamma_{A,B})"'] & F(B \otimes A)
\end{tikzcd}$$

Monoidal Natural Transformations. A monoidal natural transformation between (lax) monoidal functors $\theta : (F, m^2, m^0) \Longrightarrow (G, n^2, n^0)$ is a natural transformation $\theta : F \Longrightarrow G$ making the following diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\theta_A \otimes \theta_B"] \arrow[d, "m^2_{A,B}"'] & GA \otimes GB \arrow[d, "n^2_{A,B}"] \\
F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbf{I} \arrow[dl, "m^0"'] \arrow[dr, "n^0"] & \\
F\mathbf{I} \arrow[rr, "\theta_{\mathbf{I}}"'] & & G\mathbf{I}
\end{tikzcd}$$

The following is [27, Prop. 10]:

Proposition J.1. *Symmetric monoidal categories, (lax) symmetric monoidal functors, and monoidal natural translformations form a 2-category* $\mathbf{SymMonCat}$.

Proof.

- The identity functor $\mathrm{Id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is monoidal (actually strict monoidal), with $m^2_{A,B} = \mathrm{id}_{A \otimes B}$ and $m^0 = \mathrm{id}_{\mathbf{I}}$.
- If (F, m^2, m^0) and (G, n^2, n^0) are strong monoidal, then so is FG, with structure maps

$$F(n^2_{A,B}) \circ m^2_{GA,GB} \quad : \quad FGA \otimes FGB \to F(GA \otimes GB) \to FG(A \otimes B) \qquad \text{and} \qquad F(n^0) \circ m^0 \quad : \quad \mathbf{I} \to F\mathbf{I} \to FG\mathbf{I}$$

□

(Lax) (Symmetric) Monoidal Monads. A (lax) syemmtric monoidal monad on a monoidal category $\mathbb{C}$ is a monad (T, η, μ) such that T is a (lax) symmetric monoidal functor and the transformations η, μ are monoidal (see e.g. [27]). It then follows from [27, §6.10] that:

Proposition J.2. *If $T = (T, \eta, \mu)$ is a (lax) symmetric monoidal monad on $(\mathbb{C}, \otimes, \mathbf{I})$ then its Kleisely category **Kl**$(T) = \mathbb{C}_T$ is symmetric monoidal. Moreover, the functor $\mathsf{F}_T : \mathbb{C} \to$ **Kl**$(T) = \mathbb{C}_T$ is strict and the adjunction*

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, bend right, "\mathsf{F}_T"'] \arrow[rr, phantom, "\top"] & & \mathbf{Kl}(T) = \mathbb{C}_T \arrow[ll, bend right, "\mathsf{U}_T"']
\end{tikzcd}$$

is (lax) symmetric monoidal (i.e. is an adjunction in $\mathbf{SymMonCat}$*).*

Proof.

- The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(T)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$. On morphisms, given $f \in \mathbf{Kl}(T)[A_0, B_0] = \mathbb{C}[A_0, TB_0]$ and $g \in \mathbf{Kl}(T)[A_1, B_1] = \mathbb{C}[A_1, TB_1]$, we let $f \otimes_{\mathbf{Kl}} g$ be the composite

$$A_0 \otimes A_1 \quad \xrightarrow{f \otimes g} \quad TB_0 \otimes TB_1 \quad \xrightarrow{m^2_{B_0,B_1}} \quad T(B_0 \otimes B_1)$$

where m^2 is the binary strength of T.

- The functor F_T is strict, since its strength is given by:

$$f^2_{A,B} \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{A\otimes B} \quad = \quad \eta_{A\otimes B} \quad \in \quad \mathbf{Kl}(T)[A \otimes_{\mathbf{Kl}} B\ ,\ A \otimes_{\mathbf{Kl}} B] \quad = \quad \mathbb{C}[A \otimes B\ ,\ T(A \otimes B)]$$

  and

$$f^0 \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{\mathbf{I}} \quad = \quad \eta_{\mathbf{I}} \quad \in \quad \mathbf{Kl}(T)[\mathbf{I}\ ,\ \mathbf{I}] \quad = \quad \mathbb{C}[\mathbf{I}\ ,\ T\mathbf{I}]$$

- The functor U_T is lax symmetric monoidal. Its strength is given by:

$$u^2_{A,B} \quad := \quad m^2_{A,B} \quad \in \quad \mathbb{C}[\mathsf{U}_T A \otimes \mathsf{U}_T B\ ,\ \mathsf{U}_T(A \otimes B)] \quad = \quad \mathbb{C}[TA \otimes TB\ ,\ T(A \otimes B)]$$

  and

$$u^0 \quad := \quad m^0 \quad \in \quad \mathbb{C}[\mathbf{I}\ ,\ \mathsf{U}_T\mathbf{I}] \quad = \quad \mathbb{C}[\mathbf{I}\ ,\ T\mathbf{I}]$$

  where m^2, m^0 is the strength of T.
- The structure maps of $\mathbf{Kl}(T)$ are taken to be the image under F_T of the structure maps of $\mathbb{C}$. It thus directly follows that the coherence conditions are met on $\mathbb{C}$.
- It remains to check the naturality of the structural maps of $\mathbf{Kl}(T)$, which amounts to the following diagrams:
  - For the associativity structure map $\alpha_{(-),(-),(-)}$:

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "\eta_{A\otimes(B\otimes C)}\circ\alpha_{A,B,C}"'] & (A' \otimes B') \otimes C' \arrow[d, "\eta_{A'\otimes(B'\otimes C')}\circ\alpha_{A',B',C'}"] \\ T(A \otimes (B \otimes C)) \arrow[r, "T(f\otimes(g\otimes h))"] & T(A' \otimes (B' \otimes C')) \end{tikzcd}$$

    Proof. By naturality of η and α, we have

$$\eta_{A'\otimes(B'\otimes C')} \circ \alpha_{A',B',C'} \circ ((f \otimes g) \otimes h) \quad = \quad T(f \otimes (g \otimes h)) \circ \eta_{A\otimes(B\otimes C)} \circ \alpha_{A,B,C}$$

    and we are done. □

  - For the unit structure maps $\lambda_{(-)}$ and $\rho_{(-)}$:

$$\begin{tikzcd} \mathbf{I} \otimes A \arrow[r, "\mathrm{id}_{\mathbf{I}}\otimes f"] \arrow[d, "\eta_A\circ\lambda_A"'] & \mathbf{I} \otimes A' \arrow[d, "\eta_{A'}\circ\lambda_{A'}"] \\ TA \arrow[r, "T(f)"] & TA' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \otimes \mathbf{I} \arrow[r, "f\otimes\mathrm{id}_{\mathbf{I}}"] \arrow[d, "\eta_A\circ\rho_A"'] & A' \otimes \mathbf{I} \arrow[d, "\eta_{A'}\circ\rho_{A'}"] \\ TA \arrow[r, "T(f)"] & TA' \end{tikzcd}$$

    Proof. By naturality of η, λ and ρ we have

$$\eta_{A'} \circ \lambda_{A'} \circ (\mathrm{id}_{\mathbf{I}} \otimes f) \quad = \quad T(f) \circ \eta_A \circ \lambda_A \qquad \text{and} \qquad \eta_{A'} \circ \lambda_{A'} \circ (f \otimes \mathrm{id}_{\mathbf{I}}) \quad = \quad T(f) \circ \eta_A \circ \lambda_A$$

    and we are done. □

  - For the symmetry structure map $\gamma_{(-),(-)}$:

$$\begin{tikzcd} A \otimes B \arrow[r, "f\otimes g"] \arrow[d, "\eta_{B\otimes A}\circ\gamma_{A,B}"'] & A' \otimes B' \arrow[d, "\eta_{B'\otimes A'}\circ\gamma_{A',B'}"] \\ T(B \otimes A) \arrow[r, "T(g\otimes f)"] & T(B' \otimes A') \end{tikzcd}$$

    Proof. By naturality of η and γ, we have

$$\eta_{B'\otimes A'} \circ \gamma_{A',B'} \circ (f \otimes g) \quad = \quad T(g \otimes f) \circ \eta_{B\otimes A} \circ \gamma_{A,B}$$

    and we are done. □

□

J.1.2 Oplax (Symmetric) Monoidal Comonads

We sketch the dual notion of *oplax* (symmetric) monoidal *comonad*. All constructions and results follow by duality from the case of lax monads.

Oplax Monoidal Functors. An oplax symmetric monoidal functor F on a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is equipped with natural transformations

$$m^2_{A,B} : F(A \otimes B) \to FA \otimes FB \qquad \text{and} \qquad m^0 : F(\mathbf{I}) \to \mathbf{I}$$

making the following diagrams commute:

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[rr, "\alpha_{FA,FB,FC}"] & & FA \otimes (FB \otimes FC) \\
F(A \otimes B) \otimes FC \arrow[u, "m^2_{A,B} \otimes \mathrm{id}_{FC}"] & & FA \otimes F(B \otimes C) \arrow[u, "\mathrm{id}_{FA} \otimes m^2_{B,C}"'] \\
F((A \otimes B) \otimes C) \arrow[u, "m^2_{A \otimes B,C}"] \arrow[rr, "F(\alpha_{A,B,C})"'] & & F(A \otimes (B \otimes C)) \arrow[u, "m^2_{A,B \otimes C}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I} \otimes FA \arrow[r, "\lambda_{FA}"] \arrow[d, "m^0 \otimes \mathrm{id}_{FA}"'] & FA \\
F\mathbf{I} \otimes FA & F(\mathbf{I} \otimes A) \arrow[l, "m^2_{\mathbf{I},A}"] \arrow[u, "F(\lambda_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes \mathbf{I} \arrow[r, "\rho_{FA}"] \arrow[d, "\mathrm{id}_{FA} \otimes m^0"'] & FA \\
FA \otimes F\mathbf{I} & F(A \otimes \mathbf{I}) \arrow[l, "m^2_{A,\mathbf{I}}"] \arrow[u, "F(\rho_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes FB \arrow[r, "\gamma_{FA,FB}"] & FB \otimes FA \\
F(A \otimes B) \arrow[u, "m^2_{A,B}"] \arrow[r, "F(\gamma_{A,B})"'] & F(B \otimes A) \arrow[u, "m^2_{B,A}"']
\end{tikzcd}$$

Monoidal Natural Transformations. A monoidal natural transformation between oplax monoidal functors $\theta : (F, m^2, m^0) \implies (G, n^2, n^0)$ is a natural transformation $\theta : F \implies G$ making the following diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\theta_A \otimes \theta_B"] & GA \otimes GB \\
F(A \otimes B) \arrow[u, "m^2_{A,B}"] \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B) \arrow[u, "n^2_{A,B}"']
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& \mathbf{I} & \\
F\mathbf{I} \arrow[ur, "m^0"] \arrow[rr, "\theta_{\mathbf{I}}"'] & & G\mathbf{I} \arrow[ul, "n^0"']
\end{tikzcd}$$

The following is [27, Prop. 11]:

Proposition J.3. *Symmetric monoidal categories, oplax symmetric monoidal functors, and monoidal natural translformations form a* 2*-category* $\mathbf{SymOplaxMonCat}$.

Oplax Monoidal Comonads. An oplax monoidal comonad on a monoidal category $\mathbb{C}$ is a comonad (G, ϵ, δ) such that G is an oplax monoidal functor and the transformations ϵ, δ are monoidal (see e.g. [27]). It then follows from [27, §6.10] that:

Proposition J.4. *If* $G = (G, \epsilon, \delta)$ *is an oplax symmetric monoidal comonad on* $\mathbb{C}$ *then the coKleisely category* ***Kl***$(G) = \mathbb{C}_G$ *is symmetric monoidal. Moreover, the functor* $\mathsf{F}_G : \mathbb{C} \to$ ***Kl***$(G) = \mathbb{C}_G$ *is strict and and the adjunction*

$$\begin{tikzcd}
\textbf{\textit{Kl}}(G) = \mathbb{C}_G \arrow[rr, bend right, "\mathsf{U}_G"'] \arrow[rr, phantom, "\top"] & & \mathbb{C} \arrow[ll, bend right, "\mathsf{F}_T"']
\end{tikzcd}$$

is oplax symmetric monoidal (i.e. is an adjunction in $\mathbf{SymOplaxMonCat}$*).*

Proof. By Prop. J.2, since an oplax comonad on $\mathbb{C}$ is a lax monad on $\mathbb{C}^{\mathrm{op}}$, and since $\mathbb{C}^{\mathrm{op}}$ is symmetric monoidal iff $\mathbb{C}$ is symmetric monoidal.

We record for future use the monoidal structure of $\mathbf{Kl}(G)$:

- The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(G)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$. On morphisms, given $f \in \mathbf{Kl}(G)[A_0, B_0] = \mathbb{C}[GA_0, B_0]$ and $g \in \mathbf{Kl}(G)[A_1, B_1] = \mathbb{C}[GA_1, B_1]$, we let $f \otimes_{\mathbf{Kl}} g$ be the composite

$$G(A_0 \otimes A_1) \quad \xrightarrow{g^2_{A_0,A_1}} \quad GA_0 \otimes GA_1 \quad \xrightarrow{f \otimes g} \quad B_0 \otimes B_1$$

  where g^2 is the binary strength of G.
- The functor F_G is strict, since its strength is given by:

$$f^2_{A,B} \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{A \otimes B} \quad = \quad \epsilon_{A \otimes B} \quad \in \quad \mathbf{Kl}(G)[A \otimes_{\mathbf{Kl}} B\ ,\ A \otimes_{\mathbf{Kl}} B] \quad = \quad \mathbb{C}[G(A \otimes B)\ ,\ A \otimes B]$$

  and

$$f^0 \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{\mathbf{I}} \quad = \quad \epsilon_{\mathbf{I}} \quad \in \quad \mathbf{Kl}(G)[\mathbf{I}\ ,\ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I}\ ,\ \mathbf{I}]$$

- The functor U_G is oplax symmetric monoidal. Its strength is given by:
$$u^2_{A,B} \quad := \quad g^2_{A,B} \quad \in \quad \mathbb{C}[\mathsf{U}_G(A \otimes B)\ ,\ \mathsf{U}_G A \otimes \mathsf{U}_G B)] \quad = \quad \mathbb{C}[G(A \otimes B)\ ,\ GA \otimes GB]$$
and
$$u^0 \quad := \quad g^0 \quad \in \quad \mathbb{C}[\mathsf{U}_G \mathbf{I}\ ,\ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I}\ ,\ \mathbf{I}]$$
where g^2, g^0 is the oplax strength of G.
- The structure maps of **Kl**(G) are taken to be the image under F_G of the structure maps of $\mathbb{C}$.

$\square$

J.2 Distributive Laws of a Comonad over a Monad

Consider a category $\mathbb{C}$ equipped with a comonad (G, ϵ, δ) and monad (T, η, μ).

A *distributive law* of G over T is a natural tranformation
$$\Lambda \quad : \quad G \circ T \quad \Longrightarrow \quad T \circ G$$
such that the following diagrams commute (see e.g. [11]):

$$\begin{tikzcd}
& & TGA \arrow[drr, "T\delta_A"] & & \\
GTA \arrow[urr, "\Lambda_A"] \arrow[dr, "\delta_{TA}"'] & & & & TGGA \\
& GGTA \arrow[rr, "G\Lambda_A"'] & & GTGA \arrow[ur, "\Lambda_{GA}"'] &
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
& & GTA \arrow[drr, "\Lambda_A"] & & \\
GTTA \arrow[urr, "G\mu_A"] \arrow[dr, "\Lambda_{TA}"'] & & & & TGA \\
& TGTA \arrow[rr, "T\Lambda_A"'] & & TTGA \arrow[ur, "\mu_{GA}"'] &
\end{tikzcd} \tag{17}$$

$$\begin{tikzcd}
& TGA \arrow[dr, "T\epsilon_A"] & \\
GTA \arrow[ur, "\Lambda_A"] \arrow[rr, "\epsilon_{TA}"'] & & TA
\end{tikzcd} \tag{18}$$

$$\begin{tikzcd}
& GTA \arrow[dr, "\Lambda_A"] & \\
GA \arrow[ur, "G\eta_A"] \arrow[rr, "\eta_{GA}"'] & & TGA
\end{tikzcd} \tag{19}$$

J.2.1 The Kleisli Category Kl(Λ)

The category **Kl**(Λ) has the same objects as $\mathbb{C}$, and its morphisms are given by **Kl**$(\Lambda)[A, B] := \mathbb{C}[GA, TB]$. Identitiy and composition laws follow from that of $\mathbb{C}$ using the monad T and comonad G and the coherence properties of $\Lambda : GT \Rightarrow TG$.

J.2.2 Lifting of a Comonad to the Kleiseli Category of a Monad

Given a distributive law $\Lambda : GT \Rightarrow TG$ as above, the comonad (G, ϵ, δ) on $\mathbb{C}$ lifts to a comonad $(G_T, \epsilon_T, \delta_T)$ on $\mathbb{C}_T = \mathbf{Kl}(T)$, where:

- $G_T(A) := G(A)$ and given $f \in \mathbf{Kl}(T)[A, B] = \mathbb{C}[A, TB]$,
$$G_T(f) \quad := \quad \Lambda_B \circ G(f) \quad \in \mathbf{Kl}(T)[G_T A, G_T B] \quad = \quad \mathbb{C}[GA, TGB]$$
- $\epsilon_{T,A} := \mathsf{F}_T(\epsilon_A) \in \mathbf{Kl}(T)[GA, A] = \mathbb{C}[GA, TA]$ is explicitely given by
$$\epsilon_{T,A} \quad := \quad \eta_A \circ \epsilon_A$$
- $\delta_{T,A} := \mathsf{F}_T(\delta_A) \in \mathbf{Kl}(T)[GA, GGA] = \mathbb{C}[GA, TGGA]$ is explictitely given by
$$\delta_{T,A} \quad := \quad \eta_{GGA} \circ \delta_A$$

Proposition J.5. *The category* $\mathbf{Kl}(\Lambda)$ *is equivalent to the coKleisli category* $\mathbf{Kl}(G_T)$.

Of course, one may alternatively consider the equivalent dual operation of lifting the monad T to the coKleiseli category $\mathbf{Kl}(G)$.

J.2.3 (Oplax) Monoidal Lifting

Assume now that G is an oplax (symmetric) monoidal comonad and that T is a (lax) (symmetric) monoidal monad on a symmetric monoidal category $\mathbb{C}$. It follows from Prop. J.2 that the Kleiseli category $\mathbf{Kl}(T)$ is symmetric monoidal. Moreover,

Proposition J.6. *If* $\Lambda : GT \Rightarrow TG$ *is monoidal, in the sense that*

$$\begin{tikzcd}
G(TA \otimes TB) \arrow[rr, "G(m^2_{A,B})"] \arrow[d, "g^2_{TA,TB}"'] & & GT(A \otimes B) \arrow[d, "\Lambda_{A\otimes B}"] \\
GTA \otimes GTB \arrow[d, "\Lambda_A \otimes \Lambda_B"'] & & TG(A \otimes B) \arrow[d, "T(g^2_{A,B})"] \\
TGA \otimes TGB \arrow[rr, "m^2_{GA,GB}"'] & & T(GA \otimes GB)
\end{tikzcd} \tag{20}$$

then $(G_T, \epsilon_T, \delta_T)$ *is an oplax (symmetric) monoidal comonad on* $\mathbf{Kl}(T)$*, where* (m^2, m^0) *is the strength of* T *and* (g^2, g^0) *is the strength of* G*. The oplax monoidal strength of* G_T *is given by*

$$g^2_{T,A,B} \quad := \quad \mathsf{F}_T(g^2_{A,B}) \quad = \quad \eta_{GA\otimes GB} \circ g^2_{A,B} \quad \in \quad \mathbf{Kl}(T)[G_T(A \otimes_{\mathbf{Kl}} B)\ ,\ G_T A \otimes_{\mathbf{Kl}} G_T B] \quad = \quad \mathbb{C}[G(A \otimes B)\ ,\ T(GA \otimes GB)]$$

and

$$g^0_T \quad := \quad \mathsf{F}_T(g^0) \quad = \quad \eta_{\mathbf{I}} \circ g^0 \quad \in \quad \mathbf{Kl}(T)[G_T\mathbf{I}\ ,\ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I}\ ,\ T\mathbf{I}]$$

where

$$g^2_{A,B} : G(A \otimes B) \to GA \otimes GB \qquad \textit{and} \qquad g^0 : G\mathbf{I} \to \mathbf{I}$$

since g^2*,* g^0 *is an* oplax *monoidal strength.*

By applying now Prop. J.4 together with Prop. J.6, we thus get:

Corollary J.7. *With the same assumptions,* $\mathbf{Kl}(\Lambda)$ *is symmetric monoidal.*

Proof. We record for future use the monoidal structure of $\mathbf{Kl}(\Lambda) = \mathbf{Kl}(G_T)$:

- The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(\Lambda)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$.
  On morphisms, given
  $$f \in \mathbf{Kl}(\Lambda)[A_0, B_0] = \mathbf{Kl}(G_T)[A_0, B_0] = \mathbf{Kl}(T)[GA_0, B_0] = \mathbb{C}[GA_0, TB_0] \qquad \text{and} \qquad g \in \mathbf{Kl}(\Lambda)[A_1, B_1] = \mathbb{C}[GA_1, TB_1]$$
  we let $f \otimes_{\mathbf{Kl}} g$ be the composite
  $$G(A_0 \otimes A_1) \quad \xrightarrow{g^2_{A_0,A_1}} \quad GA_0 \otimes GA_1 \quad \xrightarrow{f\otimes g} \quad TB_0 \otimes TB_1 \quad \xrightarrow{m^2_{B_0,B_1}} \quad T(B_0 \otimes B_1)$$
  where g^2 is the binary strength of G and m^2 that of T. Note that we could equivalently have taken the following composite (corresponding to composition in $\mathbf{Kl}(T)$):
  $$G(A_0 \otimes A_1) \quad \xrightarrow{g^2_{T,A_0,A_1}} \quad T(GA_0 \otimes GA_1) \quad \xrightarrow{T(f \otimes_{\mathbf{Kl}(T)} g)} \quad TT(B_0 \otimes B_1) \quad \xrightarrow{\mu_{B_0 \otimes B_1}} \quad T(B_0 \otimes B_1)$$
  since $g^2_{T,A_0,A_1} = \eta_{GA_0,GA_1} \circ g^2_{A_0,A_1}$ and by the monad laws:
  $$\mu_B \circ T(h) \circ \eta_A \quad = \quad \mu_B \circ \eta_B \circ h \quad = \quad h$$
- The structure maps of $\mathbf{Kl}(\Lambda)$ are taken to be the image under F_{G_T} of the structure maps of $\mathbf{Kl}(T)$, itself beeing the image under F_T of the structure maps of $\mathbb{C}$. Note that on maps,
  $$\mathsf{F}_{G_T}(\mathsf{F}_T(h)) \quad = \quad \eta_B \circ h \circ \epsilon_A \qquad \text{for } h : A \to B$$

$\square$

J.2.4 Proof of Proposition J.6

Naturality of $g^2_{T,A,B}$**.** The naturality of $g^2_{T,A,B}$, that is, in $\mathbf{Kl}(T)$:

$$\begin{tikzcd}
G_T(A \otimes_{\mathbf{Kl}} B) \arrow[r, "G_T(f \otimes_{\mathbf{Kl}} g)"] \arrow[d, "g^2_{T,A,B}"'] & G_T(A' \otimes_{\mathbf{Kl}} B') \arrow[d, "g^2_{T,A',B'}"] \\
G_T A \otimes_{\mathbf{Kl}} G_T B \arrow[r, "G_T(f) \otimes_{\mathbf{Kl}} G_T(g)"'] & G_T A' \otimes_{\mathbf{Kl}} G_T B'
\end{tikzcd}$$

(where $f \in \mathbf{Kl}(T)[A,B] = \mathbb{C}[A,TB]$ and $g \in \mathbf{Kl}(T)[A',B'] = \mathbb{C}[A',TB']$), amounts to, in $\mathbb{C}$:

$$\begin{tikzcd}
G(A\otimes B) \arrow[rrr, "\Lambda_{A'\otimes B'}\circ G(m^2_{A',B'}\circ(f\otimes g))"] \arrow[d, "\eta_{GA\otimes GB}\circ g^2_{A,B}"'] & & & TG(A'\otimes B') \arrow[d, "\mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'}\circ g^2_{A',B'})"] \\
T(GA\otimes GB) \arrow[rrr, "\mu_{GA'\otimes GB'}\circ T(m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))"'] & & & T(GA'\otimes GB')
\end{tikzcd}$$

By naturality of η, we have

$$\mu_{GA'\otimes GB'}\circ T(m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB} \quad=\quad \mu_{GA'\otimes GB'}\circ\eta_{T(GA'\otimes GB')}\circ m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g)))$$

and by the unit monad law, we get:

$$\mu_{GA'\otimes GB'}\circ T(m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB} \quad=\quad m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g)))$$

and therefore (by bifunctoriality of $\otimes$):

$$\mu_{GA'\otimes GB'}\circ T(m^2_{A',B'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB} \quad=\quad m^2_{GA',GB'}\circ(\Lambda_{A'}\otimes\Lambda_{B'})\circ(G(f)\otimes G(g))$$

From which it follows (by naturality of g^2) that

$$\mu_{GA'\otimes GB'}\circ T(m^2_{A',B'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB}\circ g^2_{A,B} \quad=\quad m^2_{GA',GB'}\circ(\Lambda_{A'}\otimes\Lambda_{B'})\circ g^2_{TA',TB'}\circ G(f\otimes g)$$

On the other hand, also using the unit monad law we get:

$$\mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'}\circ g^2_{A',B'}) \quad=\quad \mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'})\circ T(g^2_{A',B'}) \quad=\quad T(g^2_{A',B'})$$

We are therefore finally left with

$$m^2_{GA',GB'}\circ(\Lambda_{A'}\otimes\Lambda_{B'})\circ g^2_{TA',TB'} \quad=\quad T(g^2_{A',B'})\circ\Lambda_{A'\otimes B'}\circ G(m^2_{A',B'})$$

which follows from (20).
Note that

$$G(TA'\otimes TB') \quad\overset{g^2_{TA',TB'}}{\longrightarrow}\quad GTA'\otimes GTB' \quad\overset{\Lambda_{A'}\otimes\Lambda_{B'}}{\longrightarrow}\quad TGA'\otimes TGB' \quad\overset{m^2_{GA',GB'}}{\longrightarrow}\quad T(GA'\otimes GB')$$

and

$$G(TA'\otimes TB') \quad\overset{G(m^2_{A',B'})}{\longrightarrow}\quad GT(A'\otimes B') \quad\overset{\Lambda_{A'\otimes B'}}{\longrightarrow}\quad TG(A'\otimes B') \quad\overset{T(g^2_{A',B'})}{\longrightarrow}\quad T(GA'\otimes GB')$$

Oplax Symmetric Monoidal Coherence of g^2_T and g^0_T. The coherence of g^2_T and g^0_T amount to the following diagrams.

- The associativity diagram:

$$\begin{tikzcd}
(G_TA\otimes_{\mathbf{Kl}} G_TB)\otimes_{\mathbf{Kl}} G_TC \arrow[r, "\alpha^{\mathbf{Kl}}_{G_TA,G_TB,G_TC}"] & G_TA\otimes_{\mathbf{Kl}}(G_TB\otimes_{\mathbf{Kl}} G_TC) \\
G_T(A\otimes_{\mathbf{Kl}} B)\otimes_{\mathbf{Kl}} G_TC \arrow[u, "g^2_{T,A,B}\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_{G_TC}"] & G_TA\otimes_{\mathbf{Kl}} G_T(B\otimes_{\mathbf{Kl}} C) \arrow[u, "\mathrm{id}^{\mathbf{Kl}}_{G_TA}\otimes_{\mathbf{Kl}} g^2_{T,B,C}"'] \\
G_T((A\otimes_{\mathbf{Kl}} B)\otimes_{\mathbf{Kl}} C) \arrow[u, "g^2_{T,A\otimes_{\mathbf{Kl}}B,C}"] \arrow[r, "G_T(\alpha^{\mathbf{Kl}}_{A,B,C})"'] & G_T(A\otimes_{\mathbf{Kl}}(B\otimes_{\mathbf{Kl}} C)) \arrow[u, "g^2_{T,A,B\otimes_{\mathbf{Kl}}C}"']
\end{tikzcd} \tag{21}$$

  First, recall that $g^2_{T,A,B} = \mathsf{F}_T(g^2_{A,B})$ by definition and that on objects $G_TA = GA$, and also $\mathsf{F}_T(A) = A$ and $A\otimes_{\mathbf{Kl}} B = A\otimes B$. Moreover, $\alpha^{\mathbf{Kl}}_{A,B,C} = \mathsf{F}_T(\alpha_{A,B,C})$ and $\mathrm{id}^{\mathbf{Kl}}_A = \eta_A = \mathsf{F}_T(\mathrm{id}_A)$. Also, since $\eta_{(-)}$ is monoidal, given $\mathbb{C}$-maps f and g we have

$$(\eta_A\circ f)\otimes_{\mathbf{Kl}}(\eta_B\circ g) \quad=\quad m^2_{A,B}\circ((\eta_A\circ f)\otimes(\eta_B\circ g)) \quad=\quad \eta_{A\otimes B}\circ(f\otimes g) \quad=\quad \mathsf{F}_T(f\otimes g)$$

  Finally, thanks to the coherence diagram (19) of distributive laws, for the bottom horizontal map we have

$$G_T(\alpha^{\mathbf{Kl}}_{A,B,C}) \quad=\quad \Lambda_{A\otimes(B\otimes C)}\circ G(\eta_{A\otimes(B\otimes C)})\circ G(\alpha_{A,B,C}) \quad=\quad \eta_{G(A\otimes(B\otimes C))}\circ G(\alpha_{A,B,C}) \quad=\quad \mathsf{F}_T(G(\alpha_{A,B,C}))$$

It follows that (21) amounts to the following diagram in $\mathbf{Kl}(T)$:

$$\begin{tikzcd}
(GA \otimes GB) \otimes GC \arrow[rr, "\mathsf{F}_T(\alpha_{GA,GB,GC})"] & & GA \otimes (GB \otimes GC) \\
G(A \otimes B) \otimes GC \arrow[u, "\mathsf{F}_T(g^2_{A,B} \otimes \mathrm{id}_{GC})"] & & GA \otimes G(B \otimes C) \arrow[u, "\mathsf{F}_T(\mathrm{id}_{GA} \otimes g^2_{B,C})"'] \\
G((A \otimes B) \otimes C) \arrow[u, "\mathsf{F}_T(g^2_{A \otimes B,C})"] \arrow[rr, "\mathsf{F}_T(G(\alpha_{A,B,C}))"'] & & G(A \otimes (B \otimes C)) \arrow[u, "\mathsf{F}_T(g^2_{A,B \otimes C})"']
\end{tikzcd}$$

Now we are done since the above diagram is the image under the functor F_T of the associativity coherence diagram of oplax the monoidal functor G.

- The coherence diagrams for units and symmetry are:

$$\begin{tikzcd}
\mathbf{I} \otimes_{\mathbf{Kl}} G_T A \arrow[r, "\lambda^{\mathbf{Kl}}_{G_T A}"] \arrow[d, "g^0_T \otimes_{\mathbf{Kl}} \mathrm{id}_{G_T A}"'] & G_T A \\
G_T \mathbf{I} \otimes_{\mathbf{Kl}} G_T A & G_T(\mathbf{I} \otimes_{\mathbf{Kl}} A) \arrow[l, "g^2_{T,\mathbf{I},A}"] \arrow[u, "G_T(\lambda^{\mathbf{Kl}}_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G_T A \otimes_{\mathbf{Kl}} \mathbf{I} \arrow[r, "\rho^{\mathbf{Kl}}_{G_T A}"] \arrow[d, "\mathrm{id}_{G_T A} \otimes_{\mathbf{Kl}} g^0_T"'] & G_T A \\
G_T A \otimes_{\mathbf{Kl}} G_T \mathbf{I} & G_T(A \otimes_{\mathbf{Kl}} \mathbf{I}) \arrow[l, "g^2_{T,A,\mathbf{I}}"] \arrow[u, "G_T(\rho^{\mathbf{Kl}}_A)"']
\end{tikzcd}$$

$$\begin{tikzcd}
G_T A \otimes_{\mathbf{Kl}} G_T B \arrow[r, "\gamma^{\mathbf{Kl}}_{G_T A, G_T B}"] & G_T B \otimes_{\mathbf{Kl}} G_T A \\
G_T(A \otimes_{\mathbf{Kl}} B) \arrow[u, "g^2_{T,A,B}"] \arrow[r, "G_T(\gamma^{\mathbf{Kl}}_{A,B})"'] & G_T(B \otimes_{\mathbf{Kl}} A) \arrow[u, "g^2_{T,B,A}"']
\end{tikzcd}$$

They are dealt-with similarly. We only detail the case of the unit $\lambda^{\mathbf{Kl}}$. First, as above, we have $g^2_{T,\mathbf{I},A} = \mathsf{F}_T(g^2_{A,B})$ and $g^0_T = \mathsf{F}_T(g^0)$, and on objects: $G_T(A) = A$, $\mathsf{F}_T(A) = A$ and $A \otimes_{\mathbf{Kl}} B = A \otimes B$. Moreover, $\lambda^{\mathbf{Kl}}_A = \mathsf{F}_T(\lambda_A)$ and $\mathrm{id}^{\mathbf{Kl}}_A = \mathsf{F}_T(\mathrm{id}_A)$. Again by monoidality of $\eta_{(-)}$ we have

$$g^0_T \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_{G_T A} \quad = \quad m^2_{\mathbf{I},A} \circ (\mathsf{F}_T(g^0) \otimes \mathsf{F}_T(\mathrm{id}_{GA})) \quad = \quad m^2_{\mathbf{I},A} \circ ((\eta_{\mathbf{I}} \circ g^0) \otimes (\eta_{GA} \circ \mathrm{id}_{GA})) \quad = \quad \eta_{\mathbf{I} \otimes GA} \circ (g^0 \otimes \mathrm{id}_{GA}) \quad = \quad \mathsf{F}_T(g^0 \otimes \mathrm{id}_{GA})$$

Again by the coherence diagram (19) of distributive laws, we have

$$G_T(\lambda^{\mathbf{Kl}}_A) \quad = \quad \Lambda_A \circ G(\eta_A) \circ G(\lambda_A) \quad = \quad \eta_{GA} \circ G(\lambda_A) \quad = \quad \mathsf{F}_T(\lambda_A)$$

Then, as for the associativity coherence law above, we are done since we get the image under the functor F_T of the corresponding unit coherence diagram for the oplax strength of G in $\mathbb{C}$.

The natural maps $\epsilon_{T,A}$ are monoidal. The corresponding diagrams are:

$$\begin{tikzcd}
G_T A \otimes_{\mathbf{Kl}} G_T B \arrow[r, "\epsilon_{T,A} \otimes_{\mathbf{Kl}} \epsilon_{T,B}"] \arrow[d, "g^2_{T,A,B}"'] & A \otimes_{\mathbf{Kl}} B \arrow[d, equal] \\
G_T(A \otimes_{\mathbf{Kl}} B) \arrow[r, "\epsilon_{T,A \otimes_{\mathbf{Kl}} B}"'] & A \otimes_{\mathbf{Kl}} B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbf{I} \arrow[dl, "g^0_T"'] \arrow[dr, equal] & \\
G_T \mathbf{I} \arrow[rr, "\epsilon_{T,\mathbf{I}}"'] & & \mathbf{I}
\end{tikzcd}$$

Reasonning as above (and in part. using the lax monoidality of $\eta_{(-)}$), these diagrams are quivalent to

$$\begin{tikzcd}
GA \otimes GB \arrow[r, "\mathsf{F}_T(\epsilon_A \otimes \epsilon_B)"] \arrow[d, "\mathsf{F}_T(g^2_{A,B})"'] & A \otimes B \arrow[d, equal] \\
G(A \otimes B) \arrow[r, "\mathsf{F}_T(\epsilon_{A \otimes B})"'] & A \otimes_{\mathbf{Kl}} B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbf{I} \arrow[dl, "\mathsf{F}_T(g^0)"'] \arrow[dr, equal] & \\
G\mathbf{I} \arrow[rr, "\mathsf{F}(\epsilon_{\mathbf{I}})"'] & & \mathbf{I}
\end{tikzcd}$$

Now we are done since recalling that F_T is the identity on objects, the above diagrams are the image under F_T of the oplax monoidal coherence digrams of $\epsilon_{(-)}$.

The natural maps $\delta_{T,A}$ are monoidal.

$$\begin{tikzcd} G_T A \otimes_{\mathbf{Kl}} G_T B \arrow[r, "\delta_{T,A} \otimes_{\mathbf{Kl}} \delta_{T,B}"] \arrow[d, "g^2_{T,A,B}"'] & G_T G_T A \otimes_{\mathbf{Kl}} G_T G_T B \arrow[d, "G_T(g^2_{T,A,B}) \circ g^2_{T,G_T A,G_T B}"] \\ G_T(A \otimes_{\mathbf{Kl}} B) \arrow[r, "\delta_{T,A \otimes_{\mathbf{Kl}} B}"'] & G_T G_T(A \otimes_{\mathbf{Kl}} B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbf{I} \arrow[dl, "g^0_T"'] \arrow[dr, "G_T(g^0_T) \circ g^0_T"] & \\ G_T \mathbf{I} \arrow[rr, "\theta_{\mathbf{I}}"'] & & G_T G_T \mathbf{I} \end{tikzcd}$$

Reasonning as above, using coherence diagram (19) of distributive laws, we have

$$G_T(g^2_{T,A,B}) \quad = \quad \Lambda_{GA \otimes GB} \circ G(\eta_{GA \otimes GB}) \circ g^2_{A,B} \quad = \quad \eta_{G(GA \otimes GB)} \circ G(g^2_{A,B}) \quad = \quad \mathsf{F}_T(g^2_{A,B})$$

and we then conclude as in the case of $\epsilon_{(-)}$ above.

J.3 Monoids and Comonoids

J.3.1 Monoids

Recall from e.g. [27] that a commutative monoid in an SMC $(\mathbb{C}, \otimes, \mathbf{I})$ is a triple $M = (M, u, m)$ where M is an object of $\mathbb{C}$ and u and m are morphisms

$$\mathbf{I} \quad \xrightarrow{u} \quad M \quad \xleftarrow{m} \quad M \otimes M$$

subject to the following coherence diagrams:

$$\begin{tikzcd} (M \otimes M) \otimes M \arrow[r, "\alpha"] \arrow[d, "m \otimes \mathrm{id}_M"'] & M \otimes (M \otimes M) \arrow[r, "\mathrm{id}_M \otimes m"] & M \otimes M \arrow[d, "m"] \\ M \otimes M \arrow[rr, "m"'] & & M \end{tikzcd} \tag{22}$$

$$\begin{tikzcd} \mathbf{I} \otimes M \arrow[r, "u \otimes \mathrm{id}_M"] \arrow[dr, "\lambda"'] & M \otimes M \arrow[d, "m"] & M \otimes \mathbf{I} \arrow[l, "\mathrm{id}_M \otimes u"'] \arrow[dl, "\rho"] \\ & M & \end{tikzcd} \tag{23}$$

$$\begin{tikzcd} M \otimes M \arrow[rr, "\gamma"] \arrow[dr, "m"] & & M \otimes M \arrow[dl, "m"'] \\ & M & \end{tikzcd} \tag{24}$$

It is well-known (see e.g. [27, Prop. 2]) that we always have $\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}$ in a monoidal category.

Proposition J.8. *If $M = (M, u, m)$ is a monoid object in $\mathbb{C}$, then*

$$\begin{tikzcd} \mathbf{I} \otimes \mathbf{I} \arrow[r, "u \otimes u"] \arrow[d, "\rho_{\mathbf{I}} = \lambda_{\mathbf{I}}"'] & M \otimes M \arrow[d, "m"] \\ \mathbf{I} \arrow[r, "u"'] & M \end{tikzcd}$$

Proof. By bifunctoriality of $\otimes$, it is equivalent to show

$$\begin{tikzcd} \mathbf{I} \otimes \mathbf{I} \arrow[r, "\mathrm{id}_{\mathbf{I}} \otimes u"] \arrow[d, "\lambda_{\mathbf{I}}"'] & I \otimes M \arrow[r, "u \otimes \mathrm{id}_M"] & M \otimes M \arrow[d, "m"] \\ \mathbf{I} \arrow[rr, "u"'] & & M \end{tikzcd}$$

But $m \circ (u \otimes \mathrm{id}_M) = \lambda_M$ by the unit law (23), and we are done since by naturality of λ we have

$$\lambda_M \circ (\mathrm{id}_{\mathbf{I}} \otimes u) \quad = \quad u \circ \lambda_{\mathbf{I}}$$

□

J.3.2 The Category $\mathrm{Mon}(\mathbb{C})$ of Commutative Monoids

The category $\mathbf{Mon}(\mathbb{C})$ of commutative monoids of $\mathbb{C}$ has monoids as objects, and as morphisms from (M, u, m) to (M', u', m'), $\mathbb{C}$-morphisms $f : M \to M'$ making the following two diagrams commute:

$$
\begin{tikzcd}
M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "m"'] & M' \otimes M' \arrow[d, "m'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
 & \mathbf{I} \arrow[dl, "u"'] \arrow[dr, "u'"] & \\
M \arrow[rr, "f"'] & & M'
\end{tikzcd}
$$

J.3.3 Comonoids

Dually, a commutative monoid in $\mathbb{C}$ is a triple $K = (K, e, d)$ where

$$
\mathbf{I} \quad \stackrel{e}{\longleftarrow} \quad K \quad \stackrel{d}{\longrightarrow} \quad M \otimes M
$$

subject to the following coherence diagrams:

$$
\begin{tikzcd}
K \arrow[rr, "d"] \arrow[d, "d"'] & & K \otimes K \arrow[d, "d \otimes \mathrm{id}_K"] \\
K \otimes K \arrow[r, "\mathrm{id}_K \otimes d"] & K \otimes (K \otimes K) & (K \otimes K) \otimes K \arrow[l, "\alpha"']
\end{tikzcd}
\tag{25}
$$

$$
\begin{tikzcd}
\mathbf{I} \otimes K \arrow[dr, "\lambda"'] & K \otimes K \arrow[l, "(e \otimes \mathrm{id}_K)"'] \arrow[r, "(\mathrm{id}_K \otimes e)"] & K \otimes \mathbf{I} \arrow[dl, "\rho"] \\
 & K \arrow[u, "d"] &
\end{tikzcd}
\tag{26}
$$

$$
\begin{tikzcd}
 & K \arrow[dl, "d"'] \arrow[dr, "d"] & \\
K \otimes K \arrow[rr, "\gamma"] & & K \otimes K
\end{tikzcd}
\tag{27}
$$

J.3.4 The Category $\mathrm{Comon}(\mathbb{C})$ of Commutative Comonoids

The category $\mathbf{Comon}(\mathbb{C})$ of commutative comonoids of $\mathbb{C}$ has comonoids as objects, and as morphisms from (K, e, d) to (K', e', d'), $\mathbb{C}$-morphisms $f : K \to K'$ making the following two diagrams commute:

$$
\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "d"'] & K' \arrow[d, "d'"] \\
K \otimes K \arrow[r, "f \otimes f"'] & K' \otimes K'
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
K \arrow[rr, "f"] \arrow[dr, "e"'] & & K' \arrow[dl, "e'"] \\
 & \mathbf{I} &
\end{tikzcd}
$$

J.3.5 Lifting of Monoids and Comonoids to Kleiseli Categories

We note here the following proposition, to be used in §G (together with Prop. I.1).

Proposition J.9. *Let $\mathbb{C}$ be a symmetric monoidal category.*

(a) Let $T = (T, \eta, \mu)$ be a (lax) symmetric monoidal monad on $\mathbb{C}$.
- *(i) If (M, u, m) is a commutative monoid in $\mathbb{C}$, then $(M, \mathsf{F}_T(u), \mathsf{F}_T(m))$ is a commutative monoid in **Kl**(T).*
- *(ii) If (K, e, d) is a commutative comonoid in $\mathbb{C}$, then $(K, \mathsf{F}_T(e), \mathsf{F}_T(d))$ is a commutative comonoid in **Kl**(T).*

(b) Let $G = (G, \epsilon, \delta)$ be an oplax symmetric monoidal comonad on $\mathbb{C}$.
- *(i) If (M, u, m) is a commutative monoid in $\mathbb{C}$, then $(M, \mathsf{F}_G(u), \mathsf{F}_G(m))$ is a commutative monoid in **Kl**(G).*
- *(ii) If (K, e, d) is a commutative comonoid in $\mathbb{C}$, then $(K, \mathsf{F}_G(e), \mathsf{F}_G(d))$ is a commutative comonoid in **Kl**(G).*

We only prove Prop. J.9.(a) since the case J.9.(b) follows by duality.

Proof of Proposition J.9.(ai). Write $(\mathsf{t}^2, \mathsf{t}^0)$ for the (lax) strength of T. Thanks to Prop. J.2, the coherence diagrams of $(M, \mathsf{F}_T(u), \mathsf{F}_T(m))$ amount to the following in $\mathbf{Kl}(T)$.

- Coherence w.r.t. associativity amounts in $\mathbf{Kl}(T)$ to:

$$\begin{tikzcd}
(M\otimes M)\otimes M \arrow[r, "\mathsf{F}_T(\alpha)"] \arrow[d, "\mathsf{F}_T(m)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_M"'] & M\otimes(M\otimes M) \arrow[r, "\mathrm{id}^{\mathbf{Kl}}_M\otimes_{\mathbf{Kl}}\mathsf{F}_T(m)"] & M\otimes M \arrow[d, "\mathsf{F}_T(m)"] \\
M\otimes M \arrow[rr, "\mathsf{F}_T(m)"'] & & M
\end{tikzcd}$$

  Note that

$$\mathsf{F}_T(m)\circ_{\mathbf{Kl}}(\mathsf{F}_T(m)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_M) \quad=\quad \mu_M\circ T(\eta_{M\otimes M})\circ T(m)\circ \mathsf{t}^2_{M,M}\circ((\eta_M\circ m)\otimes(\eta_M))$$

  Reasonning similarly as in the proof of Prop. J.6, we have

$$\mathsf{F}_T(m)\circ_{\mathbf{Kl}}(\mathsf{F}_T(m)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_M) \quad=\quad T(m)\circ\eta_{M\otimes M}\circ(m\otimes\mathrm{id}_M) \quad=\quad \eta_M\circ m\circ(m\otimes\mathrm{id}_M) \quad=\quad \mathsf{F}_T(m\circ(m\otimes\mathrm{id}_M))$$

  We similarly obtain

$$\mathsf{F}_T(m)\circ_{\mathbf{Kl}}(\mathrm{id}^{K}l_M\otimes_{\mathbf{Kl}}\mathsf{F}_T(m)) \quad=\quad \mathsf{F}_T(m\circ(\mathrm{id}_M\otimes m))$$

  and we are done using the functoriality of F_T and the associativity coherence diagram (22) of monoids.

- Coherence w.r.t. units amounts in $\mathbf{Kl}(T)$ to:

$$\begin{tikzcd}
\mathbf{I}\otimes M \arrow[r, "\mathsf{F}_T(u)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_M"] \arrow[dr, "\mathsf{F}_T(\lambda)"'] & M\otimes M \arrow[d, "\mathsf{F}_T(m)"] & M\otimes\mathbf{I} \arrow[l, "\mathrm{id}^{\mathbf{Kl}}_M\otimes_{\mathbf{Kl}}\mathsf{F}_T(u)"'] \arrow[dl, "\mathsf{F}_T(\rho)"] \\
 & M &
\end{tikzcd}$$

  Reasonning as above, we obtain:

$$\mathsf{F}_T(m)\circ_{\mathbf{Kl}}(\mathsf{F}_T(u)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_M) \quad=\quad \mathsf{F}_T(m\circ(u\otimes\mathrm{id}_M)) \qquad\text{and}\qquad \mathsf{F}_T(m)\circ_{\mathbf{Kl}}(\mathrm{id}^{\mathbf{Kl}}_M\otimes_{\mathbf{Kl}}\mathsf{F}_T(u)) \quad=\quad \mathsf{F}_T(m\circ(\mathrm{id}_M\otimes u))$$

  and we are done using the units cohrence diagram (23)

- Coherence w.r.t. symmetry amounts in $\mathbf{Kl}(T)$ to:

$$\begin{tikzcd}
M\otimes M \arrow[rr, "\mathsf{F}_T(\gamma)"] \arrow[dr, "\mathsf{F}_T(m)"] & & M\otimes M \arrow[dl, "\mathsf{F}_T(m)"'] \\
 & M &
\end{tikzcd}$$

  and follows directly from diagram (24).

Proof of Proposition J.9.(aii). We proceed similarly as in the case (ai). We only detail the case of coherence w.r.t. associativity, which amounts in $\mathbf{Kl}(T)$ to:

$$\begin{tikzcd}
K \arrow[rr, "\mathsf{F}_T(d)"] \arrow[d, "\mathsf{F}_T(d)"'] & & K\otimes K \arrow[d, "\mathsf{F}_T(d)\otimes_{\mathbf{Kl}}\mathrm{id}_K"] \\
K\otimes K \arrow[r, "\mathrm{id}^{\mathbf{Kl}}_K\otimes_{\mathbf{Kl}}\mathsf{F}_T(d)"] & K\otimes(K\otimes K) & (K\otimes K)\otimes K \arrow[l, "\mathsf{F}_T(\alpha)"']
\end{tikzcd}$$

Note that

$$\begin{aligned}
(\mathrm{id}^{\mathbf{Kl}}_K\otimes_{\mathbf{Kl}}\mathsf{F}_T(d))\circ_{\mathbf{Kl}}\mathsf{F}_T(d) \quad&=\quad \mu_{K\otimes(K\otimes K)}\circ T(\mathrm{id}^{\mathbf{Kl}}_K\otimes_{\mathbf{Kl}}\mathsf{F}_T(d))\circ\eta_{K\otimes K}\circ d\\
&=\quad \mu_{K\otimes(K\otimes K)}\circ\eta_{T(K\otimes(K\otimes K))}\circ(\mathrm{id}^{\mathbf{Kl}}_K\otimes_{\mathbf{Kl}}\mathsf{F}_T(d))\circ\circ d\\
&=\quad (\mathrm{id}^{\mathbf{Kl}}_K\otimes_{\mathbf{Kl}}\mathsf{F}_T(d))\circ d\\
&=\quad \eta_{K\otimes(K\otimes K)}\circ(\mathrm{id}_K\otimes d)\circ d\\
&=\quad \mathsf{F}_T((\mathrm{id}_K\otimes d)\circ d)
\end{aligned}$$

We similarly obtain

$$(\mathsf{F}_T(d)\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_K)\circ_{\mathbf{Kl}}\mathsf{F}_T(d) \quad=\quad \mathsf{F}_T((d\otimes\mathrm{id}_K)\circ d)$$

and we conclude using the functoriality of F_T and the associativity coherence diagram (25) of comonoids.

J.3.6 The Monad of Monoid Indexing

Following [17, §2.5], a monoid (M,u,m) in a monoidal category $\mathbb{C}$ gives rise to a monad $T=(T,\eta,\mu)$ where $T(-):=(-)\otimes M$,

$$\eta_A \quad:=\quad (\mathrm{id}_A\otimes u)\circ\rho_A^{-1} \quad:\quad A\to A\otimes M \qquad\text{and}\qquad \mu_A \quad:=\quad (\mathrm{id}_A\otimes m)\circ\alpha_{A,M,M} \quad:\quad (A\otimes M)\otimes M\to A\otimes M$$

It is well-known (see e.g. [17, §2.5] or [27, §6.6]) that (T, η, μ) is a monad. We show here that T is actually a monoidal monad. The strength of T is

$$m^2_{A,B} \quad : \quad (A \otimes M) \otimes (B \otimes M) \to (A \otimes B) \otimes M \qquad \text{and} \qquad m^0 : \mathbf{I} \to \mathbf{I} \otimes M$$

where $m^2_{A,B}$ is the composite

$$(A \otimes M) \otimes (B \otimes M) \quad \stackrel{\theta_{A,B}}{\longrightarrow} \quad (A \otimes B) \otimes (M \otimes M) \quad \stackrel{\mathrm{id} \otimes m}{\longrightarrow} \quad (A \otimes B) \otimes M$$

where $\theta_{A,B}$ is a natural map made of identities and structure maps of $\mathbb{C}$, and where m^0 is the composite

$$\mathbf{I} \quad \stackrel{\lambda_{\mathbf{I}}^{-1}}{\longrightarrow} \quad \mathbf{I} \otimes \mathbf{I} \quad \stackrel{\mathrm{id}_{\mathbf{I}} \otimes u}{\longrightarrow} \quad \mathbf{I} \otimes M$$

The map $\theta_{A,B}$ is explicitely defined as the following composite:

$$(A \otimes M) \otimes (B \otimes M) \quad \stackrel{\alpha}{\longrightarrow} \quad A \otimes (M \otimes (B \otimes M)) \quad \stackrel{\mathrm{id}_A \otimes \gamma}{\longrightarrow} \quad A \otimes ((B \otimes M) \otimes M) \quad \stackrel{\mathrm{id}_A \otimes \alpha}{\longrightarrow} \quad A \otimes (B \otimes (M \otimes M)) \quad \stackrel{\alpha^{-1}}{\longrightarrow} \quad (A \otimes B) \otimes (M \otimes M)$$

Note that (T, η, μ) is only a *lax* monad, since the structure maps of monoid objects are in general not isos.

Proposition J.10. *(T, η, μ) is a (lax) symmetric monoidal monad.*

By applying Prop. J.2 to Prop. J.10 we thus get:

Corollary J.11. ***Kl**(T) is symmetric monoidal.*

J.3.7 Proof of Proposition J.10

$T(-) = (-) \otimes M$ is a (strong) symmetric monoidal functor. The diagrams to check amount to the following:

$$\begin{tikzcd}
((A \otimes M) \otimes (B \otimes M)) \otimes (C \otimes M) \arrow[r, "\alpha_{TA,TB,TC}"] \arrow[d, "((\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A,B}) \otimes \mathrm{id}_{C \otimes M}"'] & (A \otimes M) \otimes ((B \otimes M) \otimes (C \otimes M)) \arrow[d, "\mathrm{id}_{A \otimes M} \otimes ((\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{B,C})"] \\
((A \otimes B) \otimes M) \otimes (C \otimes M) \arrow[d, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A \otimes B, C}"'] & (A \otimes M) \otimes ((B \otimes C) \otimes M) \arrow[d, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A, B \otimes C}"] \\
((A \otimes B) \otimes C) \otimes M \arrow[r, "\alpha_{A,B,C} \otimes \mathrm{id}_M"'] & (A \otimes (B \otimes C)) \otimes M
\end{tikzcd}$$

which follows from the monoid coherence law (22) of (M, u, m) and the monoidal coherence $\mathbb{C}$, and to

$$\begin{tikzcd}
\mathbf{I} \otimes (A \otimes M) \arrow[r, "\lambda_{A \otimes M}"] \arrow[d, "((\mathrm{id}_{\mathbf{I}} \otimes u) \circ \lambda_{\mathbf{I}}^{-1}) \otimes \mathrm{id}_{A \otimes M}"] & A \otimes M \\
(\mathbf{I} \otimes M) \otimes (A \otimes M) \arrow[r, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{\mathbf{I},A}"'] & (\mathbf{I} \otimes A) \otimes M \arrow[u, "\lambda_A \otimes \mathrm{id}_M"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes M) \otimes \mathbf{I} \arrow[r, "\rho_{A \otimes M}"] \arrow[d, "\mathrm{id}_{A \otimes M} \otimes ((\mathrm{id}_{\mathbf{I}} \otimes u) \circ \lambda_{\mathbf{I}}^{-1})"] & A \otimes M \\
(A \otimes M) \otimes (\mathbf{I} \otimes M) \arrow[r, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A,\mathbf{I}}"'] & (A \otimes \mathbf{I}) \otimes M \arrow[u, "\rho_A \otimes \mathrm{id}_M"']
\end{tikzcd}$$

which follow from the monoid coherence laws (23) of (M, u, m) and the monoidal coherence of $\mathbb{C}$ and finally

$$\begin{tikzcd}
(A \otimes M) \otimes (B \otimes M) \arrow[r, "\gamma_{TA,TB}"] \arrow[d, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A,B}"'] & (B \otimes M) \otimes (A \otimes M) \arrow[d, "(\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{B,A}"] \\
(A \otimes B) \otimes M \arrow[r, "\gamma_{A,B} \otimes \mathrm{id}_M"'] & (B \otimes A) \otimes M
\end{tikzcd}$$

which follows from commutative monoid coherence law (24) of (M, u, m) together with the symmetric monoidal coherence of $\mathbb{C}$.

The map $\eta_A : A \to A \otimes M$ is monoidal. We have to check:

$$\begin{tikzcd}
A \otimes B \arrow[r, "((\mathrm{id}_A \otimes u) \circ \rho_A^{-1}) \otimes ((\mathrm{id}_B \otimes u) \circ \rho_B^{-1})"] \arrow[d, equal] & (A \otimes M) \otimes (B \otimes M) \arrow[d, "(\mathrm{id} \otimes m) \circ \theta_{A,B}"] \\
A \otimes B \arrow[r, "(\mathrm{id}_{A \otimes B} \otimes u) \circ \rho_{A \otimes B}^{-1}"'] & (A \otimes B) \otimes M
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& \mathbf{I} \arrow[dr, "(\mathrm{id}_{\mathbf{I}} \otimes u) \circ \lambda_{\mathbf{I}}^{-1}"] & \\
\mathbf{I} \arrow[ur, equal] \arrow[rr, "(\mathrm{id}_{\mathbf{I}} \otimes u) \circ \rho_{\mathbf{I}}^{-1}"'] & & \mathbf{I} \otimes M
\end{tikzcd}$$

The first diagram follows from Prop. J.8. The second one directly follows from the fact that $\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}$ (see e.g. [27, Prop. 2]).

The map $\mu_A : (A \otimes M) \otimes M \to A \otimes M$ ***is monoidal.*** We check:

$$\begin{tikzcd}
((A \otimes M) \otimes M) \otimes ((B \otimes M) \otimes M) \arrow[r, "\mu_A \otimes \mu_B"] \arrow[d, "(m^2_{A,B} \otimes \mathrm{id}_M) \circ m^2_{A \otimes M, B \otimes M}"'] & (A \otimes M) \otimes (B \otimes M) \arrow[d, "m^2_{A,B}"] \\
((A \otimes B) \otimes M) \otimes M \arrow[r, "\mu_{A \otimes B}"'] & (A \otimes B) \otimes M
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& \mathbf{I} \arrow[dl, "(m^0 \otimes \mathrm{id}_M) \circ m^0"'] \arrow[dr, "m^0"] & \\
(\mathbf{I} \otimes M) \otimes M \arrow[rr, "\mu_{\mathbf{I}}"'] & & \mathbf{I} \otimes M
\end{tikzcd}$$

for

$$m^2_{A,B} = (\mathrm{id}_{A \otimes B} \otimes m) \circ \theta_{A,B} \qquad \text{and} \qquad m^0 = (\mathrm{id}_{\mathbf{I}} \otimes u) \circ \lambda_{\mathbf{I}}^{-1} \qquad \text{and} \qquad \mu_A = (\mathrm{id}_A \otimes m) \circ \alpha_{A,M,M}$$

The first diagram follows from the monoid coherence laws (22) and (24) together with the symmetric monoidal coherence of $\mathbb{C}$. The second diagram follows from Prop. J.8.

J.3.8 The Comonad of Comonoid Indexing

Dually, a comonoid (K, e, d) in a monoidal category $\mathbb{C}$ gives rise to a comonad $G = (G, \epsilon, \delta)$ where $G(-) := K \otimes (-)$, and

$$\epsilon_A \quad := \quad \lambda_A \circ (e \otimes \mathrm{id}_A) \quad : \quad K \otimes A \to A \qquad \text{and} \qquad \delta_A \quad := \quad \alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A) \quad : \quad K \otimes A \to K \otimes (K \otimes A)$$

Since a comonoid on $\mathbb{C}$ is a monoid on $\mathbb{C}^{\mathrm{op}}$, it is also well-known (again from e.g. [17, §2.5] or [27, §6.8]) that G is a comonad. Dually to §J.3.6, G is actually *oplax* symmetric monoidal. Its strength is

$$g^2_{A,B} \quad : \quad K \otimes (A \otimes B) \quad \longrightarrow \quad (K \otimes A) \otimes (K \otimes B) \qquad \text{and} \qquad g^0 \quad : \quad K \otimes \mathbf{I} \quad \longrightarrow \quad \mathbf{I}$$

where $g^2_{A,B}$ is the composite

$$K \otimes (A \otimes B) \quad \xrightarrow{d \otimes \mathrm{id}} \quad (K \otimes K) \otimes (A \otimes B) \quad \xrightarrow{\vartheta_{A,B}} \quad (K \otimes A) \otimes (K \otimes B)$$

where $\vartheta_{A,B}$ is a natural map made of identities and structure maps of $\mathbb{C}$, and where g^0 is the composite

$$K \otimes \mathbf{I} \quad \xrightarrow{e \otimes \mathrm{id}_{\mathbf{I}}} \quad \mathbf{I} \otimes \mathbf{I} \quad \xrightarrow{\lambda_{\mathbf{I}}} \quad \mathbf{I}$$

The map $\vartheta_{A,B}$ is explicitely defined as the following composite:

$$(K \otimes K) \otimes (A \otimes B) \quad \xrightarrow{\alpha} \quad K \otimes (K \otimes (A \otimes B)) \quad \xrightarrow{\mathrm{id}_A \otimes \alpha^{-1}} \quad K \otimes ((K \otimes A) \otimes B) \quad \xrightarrow{\gamma} \quad ((K \otimes A) \otimes B) \otimes K \quad \xrightarrow{\alpha} \quad (K \otimes A) \otimes (K \otimes B)$$

By duality, from Prop. J.10 we get:

Proposition J.12. *(G, ϵ, δ) is an oplax symmetric monoidal comonad.*

Similarly to Cor. J.11, by applying Prop. J.4 to Prop. J.12 we get:

Corollary J.13. ***Kl**(G) is symmetric monoidal.*

J.3.9 The Distributive Law of Comonoid over Monoid Indexing

We now check that there is distributive law Φ of (the comonad of) comonoid indexing over (the monad of) monoid indexing. Moreover, Φ is monoidal in the sense of Prop. J.6.

Proposition J.14. *Consider, in an SMC $(\mathbb{C}, \otimes, \mathbf{I})$, a comonoid (K, e, d) and a monoid (M, u, m), inducing respectivelly the comonad (G, ϵ, δ) with*

$$GA := K \otimes A \qquad \epsilon_A := \lambda_A \circ (e \otimes \mathrm{id}_A) : K \otimes A \longrightarrow A \qquad \delta_A := \alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A) : K \otimes A \longrightarrow K \otimes (K \otimes A)$$

and the monad (T, η, μ) with

$$TA := A \otimes M \qquad \eta_A := (\mathrm{id}_A \otimes u) \circ \rho_A^{-1} : A \longrightarrow A \otimes M \qquad \mu_A := (\mathrm{id}_A \otimes m) \circ \alpha_{A,M,M} : (A \otimes M) \otimes M \longrightarrow A \otimes M$$

Then,

(i) the associativity structure map

$$\Phi_A := \alpha^{-1}_{K,A,M} \quad : \quad GTA = K \otimes (A \otimes M) \quad \Longrightarrow \quad (K \otimes A) \otimes M = TGA$$

is a distributive law of G over T, *(ii) and it is moreover monoidal (in the sense of Prop. J.6), that is:*

$$\begin{tikzcd}
G(TA \otimes TB) \arrow[rr, "G(m^2_{A,B})"] \arrow[d, "g^2_{TA,TB}"'] & & GT(A \otimes B) \arrow[d, "\Phi_{A\otimes B}"] \\
GTA \otimes GTB \arrow[d, "\Phi_A \otimes \Phi_B"'] & & TG(A \otimes B) \arrow[d, "T(g^2_{A,B})"] \\
TGA \otimes TGB \arrow[rr, "m^2_{GA,GB}"'] & & T(GA \otimes GB)
\end{tikzcd} \tag{28}$$

where (m^2, m^0) is the (lax) strength of T and (g^2, g^0) is the oplax strength of G.

Proof of Proposition J.14.(i). First, note that $\Phi_{(-)}$ is natural by assumption. The diagrams of §J.2 unfold to:

$$\begin{tikzcd}
 & & (K \otimes A) \otimes M \arrow[drr, "\delta_A \otimes \mathrm{id}_M"] & & \\
K \otimes (A \otimes M) \arrow[urr, "\Phi_A"] \arrow[dr, "\delta_{A\otimes M}"'] & & & & (K \otimes (K \otimes A)) \otimes M \\
 & K \otimes (K \otimes (A \otimes M)) \arrow[rr, "\mathrm{id}_K \otimes \Phi_A"'] & & K \otimes ((K \otimes A) \otimes M) \arrow[ur, "\Phi_{K\otimes A}"'] &
\end{tikzcd} \tag{29}$$

$$\begin{tikzcd}
 & & K \otimes (A \otimes M) \arrow[drr, "\Phi_A"] & & \\
K \otimes ((A \otimes M) \otimes M) \arrow[urr, "\mathrm{id}_K \otimes \mu_A"] \arrow[dr, "\Phi_{A\otimes M}"'] & & & & (K \otimes A) \otimes M \\
 & (K \otimes (A \otimes M)) \otimes M \arrow[rr, "\Phi_A \otimes \mathrm{id}_M"'] & & ((K \otimes A) \otimes M) \otimes M \arrow[ur, "\mu_{K\otimes A}"'] &
\end{tikzcd} \tag{30}$$

$$\begin{tikzcd}
 & (K \otimes A) \otimes M \arrow[dr, "\epsilon_A \otimes \mathrm{id}_M"] & \\
K \otimes (A \otimes M) \arrow[ur, "\Phi_A"] \arrow[rr, "\epsilon_{A\otimes M}"'] & & A \otimes M
\end{tikzcd} \tag{31}$$

$$\begin{tikzcd}
 & K \otimes (A \otimes M) \arrow[dr, "\Phi_A"] & \\
K \otimes A \arrow[ur, "\mathrm{id}_K \otimes \eta_A"] \arrow[rr, "\eta_{K\otimes A}"'] & & (K \otimes A) \otimes M
\end{tikzcd} \tag{32}$$

- Diagram (29) amounts to

$$\begin{tikzcd}
 & & (K \otimes A) \otimes M \arrow[dll, "\alpha_{K,A,M}"'] \arrow[drr, "(\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M"] & & \\
K \otimes (A \otimes M) \arrow[dr, "\alpha_{K,K,A\otimes M} \circ (d \otimes \mathrm{id}_{A\otimes M})"'] & & & & (K \otimes (K \otimes A)) \otimes M \arrow[dl, "\alpha_{K,K\otimes A,M}"'] \\
 & K \otimes (K \otimes (A \otimes M)) & & K \otimes ((K \otimes A) \otimes M) \arrow[ll, "\mathrm{id}_K \otimes \alpha_{K,A,M}"'] &
\end{tikzcd}$$

By functoriality of $\otimes$ we have

$$(\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M \quad = \quad (\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes (\mathrm{id}_M \circ \mathrm{id}_M) \quad = \quad (\alpha_{K,K,A} \otimes \mathrm{id}_M) \circ ((d \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

and therefore

$$(\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ ((\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M) \quad = \quad (\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ (\alpha_{K,K,A} \otimes \mathrm{id}_M) \circ ((d \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

From the pentagon law, it follows that

$$(\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ ((\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M) \quad = \quad \alpha_{K,K,A\otimes M} \circ \alpha_{K\otimes K,A,M} \circ ((d \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

and from by naturality of α we get

$$(\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ ((\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M) \quad = \quad \alpha_{K,K,A\otimes M} \circ (d \otimes (\mathrm{id}_A \otimes \mathrm{id}_M)) \circ \alpha_{K,A,M}$$

and we are done since $\mathrm{id}_A \otimes \mathrm{id}_M = \mathrm{id}_{A\otimes M}$ by bifunctoriality of $\otimes$.

- Diagram (30), which unfolds to

$$\begin{tikzcd}
 & K \otimes (A \otimes M) & \\
K \otimes ((A \otimes M) \otimes M) \arrow[ur, "\mathrm{id}_K \otimes \mu_A"] & & (K \otimes A) \otimes M \arrow[ul, "\alpha_{K,A,M}"'] \\
(K \otimes (A \otimes M)) \otimes M \arrow[u, "\alpha_{K,A\otimes M,M}"] & & ((K \otimes A) \otimes M) \otimes M \arrow[ll, "\alpha_{K,A,M} \otimes \mathrm{id}_M"] \arrow[u, "\mu_{K\otimes A}"']
\end{tikzcd}$$

  is dealt-with similarly.

- Diagram (31) amounts to

$$\begin{tikzcd}
 & (K \otimes A) \otimes M \arrow[dl, "\alpha_{K,A,M}"'] \arrow[dr, "(\lambda_A \circ (e \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M"] & \\
K \otimes (A \otimes M) \arrow[rr, "\lambda_{A\otimes M} \circ (e \otimes \mathrm{id}_{A\otimes M})"'] & & A \otimes M
\end{tikzcd}$$

  By bi-functoriality of $\otimes$, we have $\mathrm{id}_{A\otimes M} = \mathrm{id}_A \otimes \mathrm{id}_M$, and by naturality of α it follows that

$$\lambda_{A\otimes M} \circ (e \otimes \mathrm{id}_{A\otimes M}) \circ \alpha_{K,A,M} \quad = \quad \lambda_{A\otimes M} \circ \alpha_{\mathbf{I},A,M} \circ ((e \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

  On the other hand, by functoriality of $\otimes$, we have

$$(\lambda_A \circ (e \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M \quad = \quad (\lambda_A \circ (e \otimes \mathrm{id}_A)) \otimes (\mathrm{id}_M \circ \mathrm{id}_M) \quad = \quad (\lambda_A \otimes \mathrm{id}_M) \circ ((e \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

  and we are done since $\lambda_{A\otimes M} \circ \alpha_{\mathbf{I},A,M} = \lambda_A \otimes \mathrm{id}_M$ by [27, Prop. 1].

- Diagram (32) unfolds to

$$\begin{tikzcd}
 & K \otimes (A \otimes M) & \\
K \otimes A \arrow[ur, "\mathrm{id}_K \otimes ((\mathrm{id}_A \otimes u) \circ \rho_A^{-1})"] \arrow[rr, "(\mathrm{id}_{K\otimes A} \otimes u) \circ \rho_{K\otimes A}^{-1}"'] & & (K \otimes A) \otimes M \arrow[ul, "\alpha_{K,A,M}"']
\end{tikzcd}$$

  and is dealt-with similarly, but with [27, Prop. 1] used as follows: Reasoning as for Diagram (31), we are left to show that

$$\alpha_{K,A,\mathbf{I}} \circ \rho_{K\otimes A}^{-1} \quad = \quad \mathrm{id}_K \otimes \rho_A^{-1}$$

  which amounts to

$$\rho_{K\otimes A}^{-1} \quad = \quad \alpha_{K,A,\mathbf{I}}^{-1} \circ (\mathrm{id}_K \otimes \rho_A^{-1})$$

  and we are done by applying [27, Prop. 1].

Proof of Proposition J.14.(ii). Diagram (28) unfolds to

$$\begin{tikzcd}
K \otimes ((A \otimes M) \otimes (B \otimes M)) \arrow[r, "\mathrm{id}_K \otimes ((\mathrm{id} \otimes m) \circ \theta_{A,B})"] \arrow[d, "\vartheta_{TA,TB} \circ (d \otimes \mathrm{id})"'] & K \otimes ((A \otimes B) \otimes M) \arrow[d, "\alpha^{-1}"] \\
(K \otimes (A \otimes M)) \otimes (K \otimes (B \otimes M)) \arrow[d, "\alpha^{-1} \otimes \alpha^{-1}"'] & (K \otimes (A \otimes B)) \otimes M \arrow[d, "((\vartheta_{A,B} \circ (d \otimes \mathrm{id})) \otimes \mathrm{id}_M"] \\
((K \otimes A) \otimes M) \otimes ((K \otimes B) \otimes M) \arrow[r, "(\mathrm{id} \otimes m) \circ \theta_{K\otimes A,K\otimes B}"'] & ((K \otimes A) \otimes (K \otimes B)) \otimes M
\end{tikzcd}$$

But we are done, since modulo symmetric monoidal coherence, the above amounts to

$$\begin{tikzcd}
K \otimes M \otimes M \arrow[r, "\mathrm{id}_K \otimes m"] \arrow[d, "d \otimes \mathrm{id}_{M \otimes M}"'] & K \otimes M \arrow[d, "d \otimes \mathrm{id}_M"] \\
K \otimes K \otimes M \otimes M \arrow[r, "\mathrm{id}_{K \otimes K} \otimes m"'] & K \otimes K \otimes M
\end{tikzcd}$$

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