Monoidal Closed Categories of Tree Automata

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Abstract. We propose a fibred monoidal closed category of automata on infinite trees, with existential and universal quantifications. Our notion is inspired from Dialectica-like categories, suggested by the specific logical form of the transitions of alternating automata, and which gives the shape of linear implication automata and a notion of $\exists\forall$-normal form of automata. We thus obtain a realizability interpretation where proofs in a first-order multiplicative linear logic over automata are interpreted as winning strategies in a generalization of usual acceptance games.

1 Introduction

This paper proposes a fibred monoidal closed category of automata on infinite trees, following and extending the Curry-Howard like approach of [32]: "*automata as objects, executions as morphisms*".

We consider a variation of alternating automata on infinite trees. Alternating tree automata (see e.g. [28, 29, 12] and also [36]) are equivalent in expressive power to the Monadic Second-Order Logic (MSO) on infinite trees, which subsumes most of the logics used in verification [35]. Alternating tree automata are linearly closed under complement, and together with the translation of alternating automata to non-deterministic ones (the *Simulation Theorem* [28, 9, 29]) this provides a convenient decomposition of the translation of MSO formulas to automata (see e.g. [12] and [36]), implying the decidability of MSO [31].

Tree automata and MSO are traditionally viewed as positive objects: one is primarily interested in satisfaction or satisfiability, the primitive notion of quantification is existential, and the primitive Boolean connectives tend to be disjunction and negation. In contrast, Curry-Howard approaches tend to favor negative settings, in which the predominant logical connective is implication, and where the predominant form of quantification is universal. This is conveniently handled with fibrations (see e.g. [19]), which model universally quantified implications. The model of [32] already put tree automata in a negative setting.

The main difficulty in the translation of MSO formula to tree automata is the interplay between negation and existential quantification. Alternating automata have no correct primitive notion of existential quantification, while non-deterministic automata have existential quantification but no linear complement, and they simulate alternating automata at an exponential cost. It follows that

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quantifier alternations in MSO formulas reflect the non-elementary complexity of the translation to tree automata.

This paper shows that the decomposition *via* alternating automata of the translation of MSO formulas to tree automata corresponds to some extent to the decomposition of intuitionistic logic in linear logic [11]. The model presented here provides a realizability interpretation of a deduction system for a first order multiplicative linear logic over tree automata. The fibred symmetric monoidal closed structure allows to handle existential and universal quantifications, as well as a multiplicative conjunction and a linear implication (inducing a linear complement). Moreover, we show that the simulation of alternating automata by non-deterministic ones can be performed using a powerset operation satisfying the deduction rules of the '!' modality of linear logic. This gives constructions on automata reflecting the connectives of MSO.

Most modern approaches to MSO and tree automata use games (see e.g. [35, 12, 30]), because game determinacy provides a convenient approach to the complementation of alternating tree automata. Following [32], our models are based on game semantics (see e.g. [1, 15]). The notion of morphisms is given by a synchronous restriction of the linear arrow of *simple games*. This restriction allows to internalize homsets in tree automata (as required by the *closed* structure), so that a realizer in our computational interpretation can always be described as an accepting run of some tree automaton (with decidable emptiness checking in the case of *regular* automata, equivalent to MSO).

The monoidal closed structure on automata is inspired from the closed structure of Dialectica categories (see e.g. [8, 16]), which are based on Gödel's Dialectica interpretation (see e.g. [4, 23]). Dialectica can be seen as a constructive notion of prenex $\exists\forall$-formulas, which in particular allows to see an implication of $\exists\forall$-formulas as an $\exists\forall$-formula. This gives the transition function of the linear implication automata, and motivates our notion of automata (see §3.2).

Our main contributions (wrt [32]) are the *closed* structure on automata and a primitive notion of universal quantification (§5.1 & §5.2). We also explicit a deduction system (§5.3), as well as the fact that the simulation of alternating automata by non-deterministic ones satisfies the deduction rules (but unfortunately not cut-elimination) of the ! modality of intuitionistic linear logic (§5.4). As a by-product, the fibred structure of [32], based on codomain fibrations, is simplified to variants of simple fibrations (see e.g. [19]).

The paper is organized as follows. We begin in §2 with an overview of some basic material on game semantics and monoidal categories. We present our notion of automata in §3, as well as the notions of substituted acceptance games and of linear synchronous arrow games, which lead in §4 to the fibration DialAut. Then §5 presents some constructions and properties of automata and DialAut: monoidal closed structure in §5.1, quantifications in §5.2, the deduction system in §5.3, and the interpretation of simulation as exponential rules in §5.4.

Notations. Concatenation of sequences s, t is denoted either $s.t$ or $s \cdot t$, and ε is the empty sequence. Alphabets (denoted Σ, Γ, etc) are finite non-empty sets.

Let $\mathbf{A}$ be the category of alphabets and functions $\mathtt{f} : \Sigma \to \Gamma$. Fix a singleton set $\mathbf{1} := \{\bullet\}$, and two-elements sets $\mathbf{2} := \{0,1\}$ and $\mathbb{B} := \{\mathbb{t}, \mathbb{f}\}$. Fix also a non-empty finite set $\mathfrak{D}$ of *tree directions*. A Σ-labeled $\mathfrak{D}$-ary tree is a function $T : \mathfrak{D}^* \to \Sigma$.

2 Games

Following [32], the morphisms of our categories of automata are based on a restriction of the linear arrow of *simple games* (see e.g. [1, 15]) between (generalized) acceptance games. We review in §2.1 some basic material on simple games. More specific aspects of *zig-zag* games are discussed in §2.2 and §2.3.

2.1 Simple Games

Simple games are two-player games where the *Proponent* P ($\exists$loïse) and the *Opponent* O ($\forall$belard) play in turn moves, producing plays subject to specified rules. Formally, a simple game A has the form

$$A \quad = \quad (A^+, A^-, \xi_A, L_A)$$

where A^+ and A^- are resp. the sets of P-moves and O-moves, $\xi_A \in \{+,-\}$ is the *polarity*, and $L_A \subseteq \wp_A^{\xi_A}$ is a non-empty prefix-closed set of *legal plays*, where $\wp_A^{\xi} := (A^{\xi} \cdot A^{-\xi})^* + (A^{\xi} \cdot A^{-\xi})^* \cdot A^{\xi}$ for $\xi \in \{+,-\}$. So P starts in a positive game and O starts in a negative one. The game A is *full* if A^+ and A^- are non-empty and $L_A = \wp_A^{\xi_A}$. Write $A = (U, X)$ for the full positive game with $A^+ := U$ and $A^- := X$. Let $s, t, \dots$ range of over plays and $m, n, \dots$ range over moves.

A play s is a P-play (resp. an O-play) if either $s = \varepsilon$ or s ends with a P-move (resp. an O-move). A (P-)*strategy* on A is a non-empty set of legal P-plays $\sigma \subseteq L_A$ such that if $s.t \in \sigma$ and s is a P-play, then $s \in \sigma$ (P-prefix-closure), and if $s.n \in \sigma$ and $s.m \in \sigma$, then $n = m$ (P-determinism).

A *simple game with winning* is a simple game A equipped with a *set of winning plays* (or *winning condition*) $\mathcal{W}_A \subseteq (A^{\xi_A} \cdot A^{-\xi_A})^\omega$. Consider a P-strategy σ on A and an O-play $s \in L_A$. We say that s is an O-*interrogation of* σ if either $s = \varepsilon$ and A is positive, or if $s = t.m$ for some $t \in \sigma$. We say that σ is *total* if for every O-interrogation s of σ, we have $s.n \in \sigma$ for some n. A *winning* (P-)strategy on $(A, \mathcal{W}_A)$ is a total strategy σ on A s.t. for all $\varpi \in (A^{\xi_A} \cdot A^{-\xi_A})^\omega$, we have $\varpi \in \mathcal{W}_A$ whenever $\exists^\infty k \in \mathbb{N}.\ \varpi(0).\cdots.\varpi(k) \in \sigma$.

Simple games form a category $\mathbf{SG}$. The maps from A to B, with A, B *of the same polarity*, are P-strategies in the *linear arrow game*

$$A \multimap B \quad := \quad (A^- + B^+, A^+ + B^-, -, L_{A \multimap B})$$

where $L_{A \multimap B}$ consists of those negative plays s such that $s_{\restriction A} \in L_A$ and $s_{\restriction B} \in L_B$, where $s_{\restriction A}$ is the restriction of s to $A^+ + A^-$, and similarly for $s_{\restriction B}$.

Note that the polarity of moves in component B is preserved while the polarity of moves in A is reversed. The plays of $A \multimap B$ start in component A iff A and B are both positive. Moreover, plays satisfy the *switching condition*: given

$s.m.n \in L_{A \multimap B}$, with $n \in (A \multimap B)^-$, then m and n are in the same component (*i.e.* only P is allowed to switch between A and B).

In order to describe identities and composition in **SG**, we rely on the following fact from [17]. There is a faithful functor HS : $\mathbf{SG} \longrightarrow \mathbf{Rel}$ (the category of sets and relations) mapping a game A to L_A and a strategy $\sigma : A \multimap B$ to $\mathrm{HS}(\sigma) := \{(s_{\restriction A}, s_{\restriction B}) \mid s \in \sigma\} \subseteq L_A \times L_B$. Explicitly, $\mathrm{HS}(\mathrm{id}_A)$ is identity relation on L_A, and given $\sigma : A \multimap B$, $\tau : B \multimap C$, $\tau \circ \sigma$ is the unique strategy such that $\mathrm{HS}(\tau \circ \sigma)$ is the relation $\mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$.

2.2 Zig-Zag Games

In this paper, we only consider a subcategory of **SG** whose maps $\sigma : A \multimap B$ are required to be (total) *zig-zag strategies*. This constraint is easily expressed using Hyland & Schalk's representation of strategies as relations [17].

Given a set Tr, the category $\mathbf{Rel}(\mathbf{Set}_{/\mathrm{Tr}})$ of relations over Tr has the same objects as $\mathbf{Set}_{/\mathrm{Tr}}$ (pairs of a set A and a function $\mathrm{tr}_A : A \to \mathrm{Tr}$). Maps from (A, tr_A) to (B, tr_B) are relations $R \subseteq A \times B$ s.t. $\mathrm{tr}_A(a) = \mathrm{tr}_B(b)$ for all $(a, b) \in R$.

A *zig-zag strategy* $\sigma : A \multimap B$ is a strategy such that for every (even-length) play $s \in \sigma$, the projections $s_{\restriction A}$ and $s_{\restriction B}$ have the same length (see Fig. 1, top left). Hence, $\sigma : A \multimap B$ is a zig-zag strategy if and only if $\mathrm{HS}(\sigma)$ is a $\mathbf{Rel}(\mathbf{Set}_{/\mathbb{N}})$-map from $(L_A, \mathrm{length} : L_A \to \mathbb{N})$ to $(L_B, \mathrm{length} : L_B \to \mathbb{N})$. Simple games and zig-zag strategies therefore form a lluf subcategory of **SG**.

It is well-known (see e.g. [1, 15]), that total and winning strategies compose and form a category. The case of zig-zag strategies over full games is particularly simple. First, it is easy to see that if $\sigma : A \multimap B$ and $\tau : B \multimap C$ are both zig-zag and total, then $\tau \circ \sigma$ is total. Consider now full games with winning $(A, \mathcal{W}_A)$ and $(B, \mathcal{W}_B)$. Note that if $\sigma : A \multimap B$ is zig-zag and total, then for every $\varpi \in ((A^+ + B^-) \cdot (A^- + B^+))^\omega$, if ϖ has infinitely many finite prefixes in σ, then $\varpi_{\restriction A}$ and $\varpi_{\restriction B}$ are both infinite. Given $(A, \mathcal{W}_A)$ and $(B, \mathcal{W}_B)$, we can therefore let $\mathcal{W}_{A \multimap B} \subseteq ((A^+ + B^-) \cdot (A^- + B^+))^\omega$ be the set of infinite plays ϖ such that $\varpi_{\restriction B} \in \mathcal{W}_B$ whenever $\varpi_{\restriction A} \in \mathcal{W}_A$.

Definition 2.1 (The Categories DZ and DZ$^{\mathrm{W}}$). *The category* $\mathbf{DZ}^{(\mathrm{W})}$ *has full positive games (with winning) as objects. Maps from A to B are total (winning) zig-zag strategies $\sigma : A \multimap B$.*

The category **DZ** has a particularly simple monoidal structure, which differs from the usual ones in game semantics.

Proposition 2.2. *The category* **DZ** *is symmetric monoidal with unit* $\mathbf{I} := (\mathbf{1}, \mathbf{1})$ *and* $A \otimes B := (U \times V, X \times Y)$ *where* $A = (U, X)$ *and* $B = (V, Y)$.

2.3 Monoids and Comonoids

A commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is an object M equipped with structure maps $m : M \otimes M \to M$ and $u : \mathbf{I} \to M$ subject to coherence conditions (see e.g. [27]). A (commutative) comonoid in $\mathbb{C}$ is a (commutative) monoid in $\mathbb{C}^{\mathrm{op}}$. In this paper, by (co)monoid we always mean *commutative* (co)monoid. Write $\mathbf{Comon}(\mathbb{C})$ for the category of comonoids in $\mathbb{C}$. Its maps from (K, d, e) to (K', d', e') are $\mathbb{C}$-maps $K \to K'$ which commute with the comonoid structure. Note that if $(\mathbb{C}, \otimes, \mathbf{I})$ is Cartesian, then every $\mathbb{C}$-object has a canonical comonoid structure. Moreover, $\mathbf{1}$ is a monoid in $\mathbf{Set}$.

Proposition 2.3. *If M, K are non-empty, then $M := (\mathbf{1}, M)$ is a monoid and $K := (K, \mathbf{1})$ is a comonoid in $\mathbf{DZ}$. Structure maps can be depicted as follows:*

	$M \otimes M \xrightarrow{m_M} M$	
O	$(\bullet, \bullet)$	
	$\bullet$	P
	m	O
P	(m, m)	

	$\mathbf{I} \xrightarrow{u_M} M$	
O	$\bullet$	
	$\bullet$	P
	m	O
P	$\bullet$	

	$K \xrightarrow{d_K} K \otimes K$	
O	k	
	(k, k)	P
	$(\bullet, \bullet)$	O
P	$\bullet$	

	$K \xrightarrow{e_K} \mathbf{I}$	
O	k	
	$\bullet$	P
	$\bullet$	O
P	$\bullet$	

Following [17, 18], a monoid (M, m, u) in $\mathbb{C}$ induces a monad (M, μ^M, η^M) of *monoid indexing*. The functor M takes an object A to $A \otimes M$ and a map $f : A \to B$ to $f \otimes \mathrm{id}_M : A \otimes M \to B \otimes M$. The natural maps μ^M and η^M are

$$\begin{array}{lllrll} \mu^M_A & := & (\mathrm{id}_A \otimes m) \circ \alpha & : & (A \otimes M) \otimes M & \longrightarrow & A \otimes M \\ \eta^M_A & := & (\mathrm{id}_A \otimes u) \circ \rho^{-1} & : & A & \longrightarrow & A \otimes M \end{array}$$

(where $\rho : A \otimes \mathbf{I} \to A$ and $\alpha : (A \otimes M) \otimes M \to A \otimes (M \otimes M)$ are structural isos of $(\mathbb{C}, \otimes, \mathbf{I})$). The monad (M, μ^M, η^M) is actually lax symmetric monoidal, so that its Kleisli category $\mathbf{Kl}(M)$ is symmetric monoidal with structure induced by the (identity on objects) canonical functor $\mathsf{F}_M : \mathbb{C} \to \mathbf{Kl}(M)$ and the lax structure of (M, μ^M, η^M). In particular, $\otimes_{\mathbf{Kl}(M)}$ acts as $\otimes$ on objects.

Proposition 2.4. *Given a monoid M in $\mathbb{C}$, the Kleisli category $\mathbf{Kl}(M)$ is symmetric monoidal with $A \otimes_{\mathbf{Kl}(M)} B := A \otimes B$ on objects and unit $\mathbf{I}$. Moreover, each comonoid (K, d, e) in $\mathbb{C}$ induces a comonoid $(K,\, \eta^M_{K \otimes K} \circ d,\, \eta^M_{\mathbf{I}} \circ e)$ in $\mathbf{Kl}(M)$.*

3 Tree Automata and Generalized Acceptance Games

We present here our notion of tree automata (§3.2) as well as the morphisms of our categories of automata (§3.4). They are strategies in a generalization of the *linear synchronous arrow games* [32] obtained by a suitable restriction of the linear arrow of simple games between generalized substituted acceptance games. Usual acceptance games can be seen as automata instantiated with trees, and substituted acceptances games (§3.3) can be seen as automata instantiated with morphisms from a base category $\mathbf{T}$ of trees (§3.1).

3.1 The Base Category T of Trees

It follows from Prop. 2.3 that the $\mathbf{DZ}$-object $\mathfrak{D} := (\mathbf{1}, \mathfrak{D})$ is a monoid in $\mathbf{DZ}$. We write $\mathbf{DZ}_{\mathfrak{D}}$ for the Kleisli category of indexing with $\mathfrak{D}$. Thanks to Prop. 2.4,

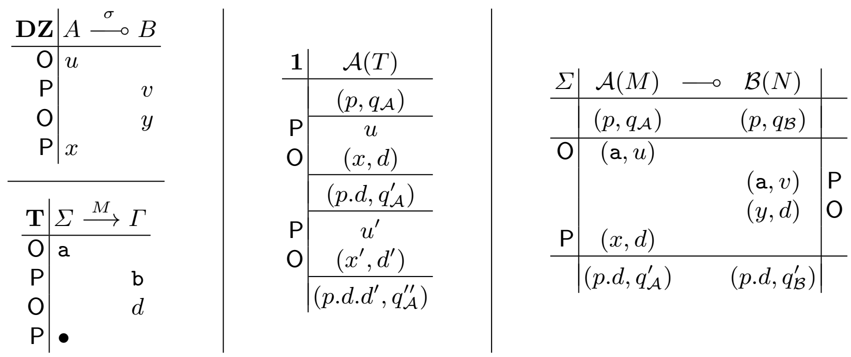


Fig. 1. Plays in Games

$\mathbf{DZ}_{\mathfrak{D}}$ is symmetric monoidal with $A \otimes B := (U \times V, X \times Y)$ and $\mathbf{I} := (\mathbf{1}, \mathbf{1})$. Moreover, each alphabet Σ induces a $\mathbf{DZ}_{\mathfrak{D}}$-comonoid $\Sigma := (\Sigma, \mathbf{1})$.

Definition 3.1 (The Base Category T). *The objects of* $\mathbf{T}$ *are alphabets. Maps from* Σ *to* Γ *are strategies* $M \in \mathbf{DZ}_{\mathfrak{D}}[\Sigma, \Gamma]$ *(see Fig. 1, down left).*

Remark 3.2. Note that $\mathbf{T}$-maps $\Sigma \to \Gamma$ are determined by functions of the form $\left(\bigcup_{n\in\mathbb{N}} \Sigma^{n+1} \times \mathfrak{D}^n\right) \to \Gamma$. In particular $\mathbf{T}[\mathbf{1}, \Sigma] \simeq (\mathfrak{D}^* \to \Sigma)$. We write $\dot{T}$ for the $\mathbf{T}$-map corresponding to the tree $T : \mathfrak{D}^* \to \Sigma$ and $M^\bullet$ for the tree corresponding to the map $M \in \mathbf{T}[\mathbf{1}, \Sigma]$.

Proposition 3.3. *The category* $\mathbf{T}$ *embeds to* $\mathbf{Comon}(\mathbf{DZ}_{\mathfrak{D}})$ *via the functor* $\mathsf{E}_{\mathbf{T}}$ *mapping* Σ *to the comonoid* $(\Sigma, d_\Sigma, e_\Sigma)$ *and* $M : \mathbf{T}[\Gamma, \Sigma]$ *to itself.*

Example 3.4. (i) A $\mathbf{2}$-labelled tree $T : \mathfrak{D}^* \to \mathbf{2}$ is the characteristic function of the set $S \subseteq \mathfrak{D}^*$ such that $p \in S$ iff $T(p) = 1$.
(ii) Each $\mathbf{A}$-map $\mathtt{f} : \Sigma \to \Gamma$ induces a $\mathbf{T}$-map $M_{\mathtt{f}} := \boldsymbol{\lambda}\bar{\mathsf{a}}.\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\mathsf{a}.\mathtt{f}(\mathsf{a})$. We write $\mathsf{p} \in \mathbf{T}[\overline{\Sigma}, \Sigma]$ for the $\mathbf{T}$-projection induced by the $\mathbf{A}$-projection $\mathsf{p} \in \mathbf{A}[\overline{\Sigma}, \Sigma]$.

3.2 Tree Automata

Definition 3.5 (Tree Automata). *A* tree automaton $\mathcal{A}$ *over alphabet* Σ *(notation* $\mathcal{A} : \Sigma$*) has the form*

$$\mathcal{A} \quad = \quad (Q_{\mathcal{A}}\,,\, q^{\imath}_{\mathcal{A}}\,,\, U\,,\, X\,,\, \delta_{\mathcal{A}}\,,\, \Omega_{\mathcal{A}}) \tag{1}$$

where $Q_{\mathcal{A}}$ *is the finite set of* states*,* $q^{\imath}_{\mathcal{A}} \in Q_{\mathcal{A}}$ *is the* initial state*,* U *and* X *are finite non-empty sets of resp.* P *and* O*-moves,* $\Omega_{\mathcal{A}} \subseteq Q^\omega$ *is the* acceptance condition*, and the* transition function $\delta_{\mathcal{A}}$ *has the form*

$$\delta_{\mathcal{A}} \quad : \quad Q_{\mathcal{A}} \times \Sigma \quad \longrightarrow \quad U \times X \quad \longrightarrow \quad (\mathfrak{D} \longrightarrow Q_{\mathcal{A}})$$

Following the usual terminology (see e.g. [35, 12, 30]), an automaton $\mathcal{A}$ as in (1) is *non-deterministic* if $X = \mathbf{1}$, and *deterministic* if moreover $U = \mathbf{1}$. We say that $\mathcal{A}$ is *regular* if $\Omega_{\mathcal{A}}$ is an ω-regular set (see e.g. [35, 12, 30]). *Parity* automata are regular automata $\mathcal{A}$ such that $\Omega_{\mathcal{A}}$ is generated from a map $c_{\mathcal{A}} : Q_{\mathcal{A}} \to \mathbb{N}$ as the set of sequences $(q_k)_k$ such that the maximal number occurring infinitely often in $(c_{\mathcal{A}}(q_k))_k$ is even. Regular automata are equivalent in expressive power with MSO (see e.g. [35, 12]).

Example 3.6. (i) The *unit automaton* $\mathbf{I}_{\Sigma} : \Sigma$ is the unique deterministic automaton over Σ with state set $\mathbf{1}$ (so that $\bullet$ is initial) and $\Omega_{\mathbf{I}} := \mathbf{1}^{\omega}$.

(ii) Usually (see e.g. [28, 29], and also [36]), an alternating tree automaton $\mathcal{A}$ over Σ with state set $Q_{\mathcal{A}}$ has transitions given by a map $\delta_{\mathcal{A}}$ taking a state q and an input letter $\mathtt{a}$ to an irredundant disjunctive normal form over $Q_{\mathcal{A}} \times \mathfrak{D}$, so that we can assume $\delta_{\mathcal{A}}(q, \mathtt{a}) \in \mathcal{P}(\mathcal{P}(Q_{\mathcal{A}} \times \mathfrak{D}))$.
This leads to an automaton $\widehat{\mathcal{A}}$ in the sense of Def. 3.5 with states $Q_{\mathcal{A}} + \mathbb{B}$, P-moves $U := \mathcal{P}(Q_{\mathcal{A}} \times \mathfrak{D})$ and O-moves $X := Q_{\mathcal{A}}$, with $\Omega_{\widehat{\mathcal{A}}} := \Omega_{\mathcal{A}} + Q_{\mathcal{A}}^{*}.\mathbb{t}^{\omega}$, and with $\delta_{\widehat{\mathcal{A}}}(\mathbb{b}, \mathtt{a}, \_, \_, \_) := \mathbb{b}$ if $\mathbb{b} \in \mathbb{B}$ and for $q \in Q_{\mathcal{A}}$, $\delta_{\widehat{\mathcal{A}}}(q, \mathtt{a}, \gamma, q', d) = q''$ where $q'' := q'$ if $(q', d) \in \gamma \in \delta_{\mathcal{A}}(q, \mathtt{a})$, $q'' := \mathbb{t}$ if $(q', d) \notin \gamma \in \delta_{\mathcal{A}}(q, \mathtt{a})$, and $q'' := \mathbb{f}$ if $\gamma \notin \delta_{\mathcal{A}}(q, \mathtt{a})$.

3.3 Substituted Acceptance Games

A *substituted acceptance game* [32] is a full positive game $\Gamma \vdash \mathcal{A}(M)$ obtained by instantiating an automaton $\mathcal{A} : \Sigma$ as in (1) with a tree map $M \in \mathbf{T}[\Gamma, \Sigma]$. The P-moves of $\Gamma \vdash \mathcal{A}(M)$ are $\Gamma \times U$ and its O-moves are $X \times \mathfrak{D}$. Note that this game has no maximal finite play. We equip it with a winning condition $\mathcal{W}$. Each infinite play $\varpi \in ((\Sigma \times U) \cdot (X \times \mathfrak{D}))^{\omega}$ generates an infinite sequence of states $(q_k)_k \in Q_{\mathcal{A}}^{\omega}$ as follows. We let $q_0 := q_{\mathcal{A}}^{\imath}$ and (using Rem. 3.2)

$$q_{k+1} \quad := \quad \delta_{\mathcal{A}}(q_k, M(\mathtt{b}_0. \cdots .\mathtt{b}_k\,,\, d_0. \cdots .d_{k-1}), u_k, x_k, d_k)$$

where $\varpi = ((\mathtt{b}_k, u_k) \cdot (x_k, d_k))_k$. Then ϖ is winning (*i.e.* $\varpi \in \mathcal{W}$) iff $(q_k)_k$ is accepting (*i.e.* $(q_k)_k \in \Omega_{\mathcal{A}}$). Note that P plays input characters $\mathtt{b}_0, \dots, \mathtt{b}_k \in \Gamma$, which are transmitted by M to the transition function of $\mathcal{A}$. On the other hand, the tree directions $d_0, \dots, d_k \in \mathfrak{D}$ are played by O. Write $\Sigma \vdash \mathcal{A}$ for $\Sigma \vdash \mathcal{A}(\mathrm{Id}_{\Sigma})$.

If $M = M_{\mathtt{f}}$ for an $\mathbf{A}$-map $\mathtt{f} : \Gamma \to \Sigma$, then the game $\Gamma \vdash \mathcal{A}(M_{\mathtt{f}})$ is the same as the game $\Gamma \vdash \mathcal{A}[\mathtt{f}]$, where the automaton $\mathcal{A}[\mathtt{f}] : \Gamma$ is defined as $\mathcal{A}$, but with $\delta_{\mathcal{A}[\mathtt{f}]}(q, \mathtt{b}, u, x, d) := \delta_{\mathcal{A}}(q, \mathtt{f}(\mathtt{b}), u, x, d)$.

Given a tree $T : \mathfrak{D}^* \to \Sigma$, the game $\mathbf{1} \vdash \mathcal{A}(\dot{T})$ (also written $\mathcal{A}(T)$) is similar to usual acceptance games (see e.g. [35, 12, 30]). A typical play of $\mathcal{A}(T)$ is depicted on Fig. 1 (middle). Note that the input alphabet of $\mathcal{A}(T)$ is $\mathbf{1}$, so that P only plays moves in U.

Definition 3.7. *Given $\mathcal{A} : \Sigma$ and $T : \mathfrak{D}^* \to \Sigma$, we say that $\mathcal{A}$* accepts *T if P has a winning strategy in $\mathcal{A}(T)$. We let $\mathcal{L}(\mathcal{A})$ be the set of $T : \mathfrak{D}^* \to \Sigma$ such that $\mathcal{A}$ accepts T. A set of trees $\mathcal{L} \subseteq (\mathfrak{D}^* \to \Sigma)$ is* regular *if there is a regular automaton $\mathcal{A} : \Sigma$ such that $\mathcal{L} = \mathcal{L}(\mathcal{A})$.*

3.4 Linear Synchronous Arrow Games

Morphisms of our categories of automata are strategies in *linear synchronous arrow games* [32], obtained by a suitable restriction of the linear arrow of simple games on generalized substituted acceptance games.

Fix an alphabet Σ and consider full positive games $A = (\Sigma \times U, X \times \mathfrak{D}, \mathcal{W}_A)$ and $B = (\Sigma \times V, Y \times \mathfrak{D}, \mathcal{W}_B)$. We define a synchronous restriction of $A \multimap B$. Let the *trace* $\mathrm{tr}(t) \in (\Sigma + \mathfrak{D})^*$ of a play $t \in \wp^+_A$ (resp. $\wp^+_B$) be the restriction of t to $\Sigma + \mathfrak{D}$. We say that an even-length play $s \in L_{A \multimap B}$ is *synchronous* if $\mathrm{tr}(s_{\restriction A}) = \mathrm{tr}(s_{\restriction B})$. Note that P has to play the same $\mathsf{a} \in \Sigma$ and tree directions $d \in \mathfrak{D}$ as chosen by O (see also Fig. 1 (right)).

When $A = (\Sigma \vdash \mathcal{A}(M))$ and $B = (\Sigma \vdash \mathcal{B}(N))$, we write $\Sigma \vdash \mathcal{A}(M) \multimap \mathcal{B}(N)$ for the game $A \multimap B$ restricted to (prefixes of) synchronous plays $s \in L_{A \multimap B}$ (see Fig. 1 (right)). Note that $s_{\restriction A}$ and $s_{\restriction B}$ explore the same path of the input tree, with the same input characters $\mathsf{a} \in \Sigma$. We write $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ for the game $\Sigma \vdash \mathcal{A}(\mathrm{Id}_\Sigma) \multimap \mathcal{B}(\mathrm{Id}_\Sigma)$ where $\mathcal{A}, \mathcal{B} : \Sigma$.

A strategy $\sigma : A \multimap B$ is *synchronous* if all its plays are synchronous, or equivalently if $\mathrm{HS}(\sigma)$ is a $\mathbf{Rel}(\mathbf{Set}_{/(\Gamma + \mathfrak{D})^*})$-map from $(\wp^+_A, \mathrm{tr})$ to $(\wp^+_B, \mathrm{tr})$. Note that synchronous strategies are zig-zag. We thus get categories of winning synchronous strategies by equipping games of the form $A \multimap B$ with the winning condition $\mathcal{W}_{A \multimap B}$ of zig-zag games.

Definition 3.8 (The Categories $\mathbf{AG}^{(\mathrm{W})}_\Sigma$). *Objects of $\mathbf{AG}_\Sigma$ are pairs (U, X), and maps from (U, X) to (V, Y) are* synchronous *$\mathbf{DZ}$-maps from $(\Sigma \times U, X \times \mathfrak{D})$ to $(\Sigma \times V, Y \times \mathfrak{D})$.*

Objects of $\mathbf{AG}^{\mathrm{W}}_\Sigma$ are tuples $(U, X, \mathcal{W}_A)$, where $\mathcal{W}_A \subseteq ((\Sigma \times U) \cdot (X \times \mathfrak{D}))^\omega$. $\mathbf{AG}^{\mathrm{W}}_\Sigma$-maps from $(U, X, \mathcal{W}_A)$ to $(V, Y, \mathcal{W}_B)$ are synchronous *$\mathbf{DZ}^{\mathrm{W}}$-maps from $(\Sigma \times U, X \times \mathfrak{D}, \mathcal{W}_A)$ to $(\Sigma \times V, Y \times \mathfrak{D}, \mathcal{W}_B)$.*

Given $A = (\Sigma \vdash \mathcal{A}(M))$ write A for the $\mathbf{AG}^{(\mathrm{W})}_\Sigma$-object (U, X) (resp. $(U, X, \mathcal{W}_A)$).

Example 3.9. Given $\mathcal{A} : \Sigma$ and $T : \mathfrak{D}^* \to \Sigma$, P has a winning strategy in $\mathcal{A}(T)$ (*i.e.* $T \in \mathcal{L}(\mathcal{A})$) iff P has a winning strategy in $\mathbf{1} \vdash \mathbf{I_1} \multimap \mathcal{A}(\dot{T})$.

4 Fibrations of Tree Automata

We now present our category $\mathsf{DialAut}$ of tree automata and generalized substituted acceptance games. The category $\mathsf{DialAut}$ is fibred over $\mathbf{T}$ (Def. 3.1), and its fibre over Σ is equivalent to the category $\mathbf{AG}^{\mathrm{W}}_\Sigma$ (Def. 3.8).

The fibred structure of $\mathsf{DialAut}$ is based on an indexed category DialZ, induced from $\mathbf{DZ}_{\mathfrak{D}}$ by *comonoid indexing* (inspired from [17, 18]). This allows a smooth treatment of monoidal closure and universal quantifications.

4.1 The Indexed Category DialZ

The indexed category DialZ is similar to the *simple fibration* $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ (see e.g. [19, 16, 14]). Given $\mathbb{B}$ with finite products, the objects of $\mathsf{s}(\mathbb{B})$ are pairs (I, X) of $\mathbb{B}$-objects. Maps $(I, X) \to (J, Y)$ are pairs (f_0, f) with $f_0 : I \to J$ and $f : I \times X \to Y$. The functor $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ is the first projection, and the fibre over I is the Kleiseli category of indexing with the comonoid I. Using *comonoid indexing* [17, 18] (dual to monoid indexing, see §2.3), since comonoids have finite products, the same pattern can be applied to a symmetric monoidal category. This gives a fibration whose fibre over the comonoid K is the Kleiseli category of indexing with K.

Dually to monoid indexing, a comonoid (K, d, e) in a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ induces a comonad $(K, \delta^K, \epsilon^K)$, where we let $K(A) := K \otimes A$ and $K(f) := \mathrm{id}_K \otimes f$ for $f : A \to B$, and where

$$\begin{array}{lllllll} \delta^K_A & := & \alpha \circ (d \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & K \otimes (K \otimes A) \\ \epsilon^K_A & := & \lambda \circ (e \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & A \end{array}$$

Moreover, the dual of Prop. 2.4 holds, so that $\mathbf{Kl}(K)$ is symmetric monoidal with on objects $A \otimes_{\mathbf{Kl}(K)} B := A \otimes B$ and unit $\mathbf{I}$.

Now, a comonoid morphism $u : K \to L$ induces a strict monoidal functor $u^{\mathsf{Cl}} : \mathbf{Kl}(L) \to \mathbf{Kl}(K)$ which is the identity on objects and such that $u^{\mathsf{Cl}}(f) := f \circ (u \otimes \mathrm{id}_A) : K \otimes A \to B$ for $f : L \otimes A \to B$. Since $(-)^{\mathsf{Cl}}$ is itself functorial, we thus have an indexed category $\mathsf{Cl}(\mathbb{C}) : \mathbf{Comon}(\mathbb{C})^{\mathrm{op}} \to \mathbf{Cat}$.

Given a comonoid K and a monoid M in $\mathbb{C}$, the comonad K is related to the monad M by a distributive law. A distributive law Λ of a comonad G over a monad T on $\mathbb{C}$ is a natural map $\Lambda : G \circ T \Rightarrow T \circ G$ subject to some coherence conditions (see e.g. [13]), which ensure that we have a category $\mathbf{Kl}(\Lambda)$ with the same objects as $\mathbb{C}$ and with $\mathbf{Kl}(\Lambda)[A, B] := \mathbb{C}[GA, TB]$, and that there is a lifting functor $(-)^{\uparrow} : \mathbf{Kl}(\Lambda) \to \mathbb{C}$ taking $f : GA \to TB$ to $f^{\uparrow} : GTA \to GTB$. A distributive law of K over M is given by the natural associativity maps:

$$\Phi_{(-)} \quad := \quad \alpha^{-1}_{K,(-),M} \quad : \quad K \otimes ((-) \otimes M) \quad \Longrightarrow \quad (K \otimes (-)) \otimes M$$

Moreover, by Prop. 2.4 we have $\mathbf{Kl}(\Phi)[A, B] = \mathsf{Cl}(\mathbf{Kl}(M))(K)[A, B]$.

We now turn to the specific cases of $\mathbf{DZ}$ and $\mathbf{DZ}_{\mathfrak{D}}$. Recall that $\mathbf{DZ}$-objects of the form $\Sigma = (\Sigma, \mathbf{1})$ are comonoids in $\mathbf{DZ}$ and $\mathbf{DZ}_{\mathfrak{D}}$. Moreover, $\mathbf{T}$-maps induce comonoid morphisms in $\mathbf{DZ}_{\mathfrak{D}}$ by Prop. 3.3. We thus get an indexed category $\mathsf{DialZ} \; := \; \mathsf{Cl}(\mathbf{DZ}_{\mathfrak{D}}) \circ \mathsf{E}_{\mathbf{T}} \; : \; \mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$.

4.2 The Fibred Category $\mathsf{DialAut}$

The fibration $\mathsf{da} : \mathsf{DialAut} \to \mathbf{T}$ is obtained by Grothendieck completion of an indexed category $(-)^* : \mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$, which takes an alphabet Σ to a category equivalent to $\mathbf{AG}^{\mathrm{W}}_{\Sigma}$ (see Def. 3.8). The action of $(-)^*$ on $\mathbf{T}$-maps is based on the indexed category DialZ defined above.

We first note that $\mathsf{DialZ}(\Sigma)$ is equivalent to $\mathbf{AG}_\Sigma$. On the one hand, a strategy $\sigma \in \mathsf{DialZ}(\Sigma)[A,B] \simeq \mathbf{DZ}[\Sigma \otimes A, B \otimes \mathfrak{D}]$ is actually lifted to a *synchronous* $\sigma^\uparrow \in \mathbf{AG}_\Sigma[A,B]$. On the other hand, given $\sigma \in \mathbf{AG}_\Sigma[A,B]$, we have $\sigma = (\epsilon^\Sigma \circ \sigma \circ (\mathrm{id}_\Sigma \otimes \eta^{\mathfrak{D}}))^\uparrow$ where $\epsilon^\Sigma \circ \sigma \circ (\mathrm{id}_\Sigma \otimes \eta^{\mathfrak{D}}) \in \mathsf{DialZ}(\Sigma)[A,B]$. Since $(-)^\uparrow$ is faithful on $\mathsf{DialZ}(\Sigma)$, it follows that $\sigma \mapsto \epsilon^\Sigma \circ \sigma \circ (\mathrm{id}_\Sigma \otimes \eta^{\mathfrak{D}})$ is functorial.

For each alphabet Σ, we define a category $\mathsf{DialAut}_\Sigma$. It has the same objects as $\mathbf{AG}^{\mathrm{W}}_\Sigma$, namely tuples $(U, X, \mathcal{W}_A)$ where $\mathcal{W}_A \subseteq ((\Sigma \times U) \cdot (X \times \mathfrak{D}))^\omega$. Maps from $(U, X, \mathcal{W}_A)$ to $(V, Y, \mathcal{W}_B)$ are $\mathbf{DZ}$-strategies $\sigma : (\Sigma \times U, X) \multimap (V, Y \times \mathfrak{D})$ (a.k.a. $\mathsf{DialZ}(\Sigma)$-maps) whose lift $\sigma^\uparrow$ are $\mathbf{AG}^{\mathrm{W}}_\Sigma$-maps from $(U, X, \mathcal{W}_A)$ to $(V, Y, \mathcal{W}_B)$. Composition and identities of $\mathsf{DialAut}_\Sigma$ are induced by composition and identities of $\mathsf{DialZ}(\Sigma)$ thanks to the functoriality of $(-)^\uparrow$. $\mathsf{DialAut}_\Sigma$ is equivalent to $\mathbf{AG}^{\mathrm{W}}_\Sigma$.

Maps $L \in \mathbf{T}[\Gamma, \Sigma]$ induce functors $L^* : \mathsf{DialAut}_\Sigma \to \mathsf{DialAut}_\Gamma$. Given a $\mathsf{DialAut}_\Sigma$-object $A = (U, X, \mathcal{W}_A)$, let $L^*(A) := (U, X, L^*(\mathcal{W}_A))$, where $((\mathsf{b}_k, u_k) \cdot (x_k, d_k))_k \in L^*(\mathcal{W}_A)$ iff $((L(\mathsf{b}_0 . \cdots . \mathsf{b}_k,\ d_0 . \cdots . d_{k-1}), u_k) \cdot (x_k, d_k))_k \in \mathcal{W}_A$.

The action of L^* on maps is induced by $\mathsf{Cl}(\mathbf{DZ}_{\mathfrak{D}})(L) : \mathsf{DialZ}(\Sigma) \to \mathsf{DialZ}(\Gamma)$, so that for $\sigma \in \mathsf{DialAut}_\Sigma[A,B]$, we let $L^*(\sigma) := \sigma \circ (L \otimes \mathrm{id}_A)$ (where $\circ$, $\otimes$ and id_A are taken in $\mathbf{DZ}_{\mathfrak{D}}$). It is easy to check that $L^*(\sigma)^\uparrow$ is winning.

We obtain an indexed category $(-)^* : \mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$ since $(-)^*$ is itself functorial. We let $\mathsf{da} : \mathsf{DialAut} \to \mathbf{T}$ be its Grothendieck completion. The objects of $\mathsf{DialAut}$ are pairs $(\Sigma, (U, X, \mathcal{W}_A))$ where $(U, X, \mathcal{W}_A)$ is an object of $\mathsf{DialAut}_\Sigma$. Maps from $(\Sigma, (U, X, \mathcal{W}_A))$ to $(\Gamma, (V, Y, \mathcal{W}_B))$ are pairs (L, σ) of a $\mathbf{T}$-map $L : \Sigma \to \Gamma$ and a $\mathsf{DialAut}_\Sigma$-map σ from $(U, X, \mathcal{W}_A)$ to $L^*(V, Y, \mathcal{W}_B)$.

Substitution in Games and Language Inclusion. We now discuss substitution for the acceptance games of §3.3 and check that $\mathsf{DialAut}_\Sigma$ (and $\mathbf{AG}^{\mathrm{W}}_\Sigma$) is correct w.r.t. language inclusion.

First, note that for a $\mathsf{DialAut}_\Sigma$-object $A := (\Sigma \vdash \mathcal{A}(M))$ and $L \in \mathbf{T}[\Gamma, \Sigma]$, we have $L^*(A) = (\Gamma \vdash \mathcal{A}(M \circ L))$. Given also $B := (\Sigma \vdash \mathcal{B}(N))$ and some $\sigma \in \mathsf{DialAut}_\Sigma[A,B]$, we have $L^*(\sigma) \in \mathsf{DialAut}_\Gamma[(\Gamma \vdash \mathcal{A}(M \circ L)), (\Gamma \vdash \mathcal{B}(N \circ L))]$.

Assume now that $A = (\Sigma \vdash \mathcal{A})$ and $B = (\Sigma \vdash \mathcal{B})$. Given $T : \mathfrak{D}^* \to \Sigma$, we thus have $\dot{T}^*(\sigma) \in \mathsf{DialAut}_{\mathbf{1}}[\mathcal{A}(T), \mathcal{B}(T)]$. If $T \in \mathcal{L}(\mathcal{A})$ then there is some $\tau \in \mathsf{DialAut}_{\mathbf{1}}[\mathbf{I}_{\mathbf{1}}, \mathcal{A}(T)]$, so that $\dot{T}^*(\sigma) \circ \tau \in \mathsf{DialAut}_{\mathbf{1}}[\mathbf{I}_{\mathbf{1}}, \mathcal{B}(T)]$ and $T \in \mathcal{L}(\mathcal{B})$. We therefore have shown:

Proposition 4.1. *If* P *has a winning strat. in* $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$*, then* $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$*.*

5 Categorical Structure and Operations on Automata

We present here some constructions and properties of automata and $\mathsf{DialAut}$: monoidal closed structure in §5.1, quantifications in §5.2, the deduction system in §5.3, and the interpretation of simulation as exponential rules in §5.4.

5.1 Monoidal Closed Structure

The main contribution of this paper is that automata and generalized substituted acceptance games are equipped with a monoidal *closed* structure. We introduce a *linear implication* connective on automata, satisfying

$$\mathcal{A} \otimes \mathcal{B} \multimap \mathcal{C} \qquad \simeq \qquad \mathcal{B} \multimap (\mathcal{A} \multimap \mathcal{C}) \tag{2}$$

and which is compatible with cut-elimination (see Rem. 5.14). As a consequence, we get universal quantifications as right adjoints to weakening.

The category $\mathbf{DZ}$ has a monoidal closed structure which follows the pattern of the Gödel's Dialectica interpretation: Given $A = (U, X)$ and $B = (V, Y)$, a total zig-zag strategy $\sigma : A \multimap B$ as in Fig. 1 (top left) provides infinite sequences of maps $f : U \to V$ (for the P-move v in B after the O-move u in A) and $F : U \times Y \to X$ (for the P-move x in A after the O-moves u and y).

Proposition 5.1. *The category* $\mathbf{DZ}$ *is symmetric monoidal closed. The linear exponent of* $A = (U, X)$ *and* $B = (V, Y)$ *is* $A \multimap_{\mathbf{DZ}} B \;:=\; (V^U \times X^{U \times Y}, U \times Y)$.

The monoidal closed structure of $\mathbf{DZ}$ thus departs from traditional game semantics since the natural isomorphism $A \otimes B \multimap C \simeq B \multimap (A \multimap_{\mathbf{DZ}} C)$ relates only strategies, but not *plays*.

The monoidal closed structure of $\mathbf{DZ}$ lifts to $\mathbf{DZ}_{\mathfrak{D}}$ and to the fibres $\mathsf{DialZ}(\Sigma)$. Since $\mathbf{DZ}_{\mathfrak{D}}[A \otimes B, C] = \mathbf{DZ}[A \otimes B, C \otimes \mathfrak{D}] \simeq \mathbf{DZ}[A, B \multimap_{\mathbf{DZ}} C \otimes \mathfrak{D}]$ we should have $(A \multimap_{\mathbf{DZ}_{\mathfrak{D}}} B) \otimes \mathfrak{D} \simeq (A \multimap_{\mathbf{DZ}} B \otimes \mathfrak{D})$. This leads to $((U, X) \multimap_{\mathbf{DZ}_{\mathfrak{D}}} (V, Y)) = (W, Z)$ with $(W, Z \times \mathfrak{D}) \simeq (V^U \times X^{U \times Y \times \mathfrak{D}}, U \times Y \times \mathfrak{D})$. We therefore let $(U, X) \multimap_{\mathbf{DZ}_{\mathfrak{D}}} (V, Y) := (V^U \times X^{U \times Y \times \mathfrak{D}}, U \times Y)$. This lifts directly to $\mathsf{DialZ}(\Sigma)$ since $\mathsf{DialZ}(\Sigma)[A \otimes B, C] = \mathbf{DZ}_{\mathfrak{D}}[\Sigma \otimes A \otimes B, C] \simeq \mathbf{DZ}_{\mathfrak{D}}[\Sigma \otimes A, B \multimap_{\mathbf{DZ}_{\mathfrak{D}}} C]$.

Proposition 5.2. $\mathbf{DZ}_{\mathfrak{D}}$ *and* $\mathsf{DialZ}(\Sigma)$ *are symmetric monoidal closed.*

This gives the fibrewise symmetric monoidal closed structure of $\mathsf{DialAut}$ (in the sense of [19, §1.8]). The unit over Σ is $\mathbf{I}_\Sigma := (\mathbf{1}, \mathbf{1}, \mathbf{1}^\omega)$. Given $\mathsf{DialAut}_\Sigma$-objects $A = (U, X, \mathcal{W}_A)$ and $B = (V, Y, \mathcal{W}_B)$, let

$$\begin{array}{rcl} A \otimes_{\mathsf{DA}} B & := & (U \times V,\; X \times Y,\; \mathcal{W}_A \sqcap \mathcal{W}_B) \\ A \multimap_{\mathsf{DA}} B & := & (V^U \times X^{U \times Y \times \mathfrak{D}},\; U \times Y,\; \mathcal{W}_A \sqsupset \mathcal{W}_B) \end{array}$$

where $\varpi \in \mathcal{W}_A \sqcap \mathcal{W}_B$ iff $(\varpi_{\restriction(\Sigma \times U)+(X \times \mathfrak{D})} \in \mathcal{W}_A \wedge \varpi_{\restriction(\Sigma \times V)+(Y \times \mathfrak{D})} \in \mathcal{W}_B)$, and $((\mathsf{a}_k, f_k, F_k) \cdot (u_k, y_k, d_k))_k \in \mathcal{W}_A \sqsupset \mathcal{W}_B$ iff $(\alpha \in \mathcal{W}_A \Rightarrow \beta \in \mathcal{W}_B)$ with $\alpha := ((\mathsf{a}_k, u_k) \cdot (F_k(u_k, y_k, d_k), d_k))_k$ and $\beta := ((\mathsf{a}_k, f_k(u_k)) \cdot (y_k, d_k))_k$.

Proposition 5.3. *The fibration* $\mathsf{DialAut}$ *is fibrewise monoidal closed.*

The symmetric monoidal closed structure of $\mathsf{DialAut}_\Sigma$ gives a monoidal product and a linear implication on automata.

Definition 5.4. *Given* $\mathcal{A} : \Sigma$ *as in (1) and* $(\mathcal{B} : \Sigma) = (Q_\mathcal{B}, q^\imath_\mathcal{B}, V, Y, \delta_\mathcal{B}, \Omega_\mathcal{B})$*, let*

$$\begin{array}{rcl} \mathcal{A} \otimes \mathcal{B} & := & (Q_\mathcal{A} \times Q_\mathcal{B},\; (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}),\; U \times V,\; X \times Y,\; \delta_{\mathcal{A} \otimes \mathcal{B}},\; \Omega_{\mathcal{A} \otimes \mathcal{B}}) \\ \mathcal{A} \multimap \mathcal{B} & := & (Q_\mathcal{A} \times Q_\mathcal{B},\; (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}),\; V^U \times X^{U \times Y \times \mathfrak{D}},\; U \times Y,\; \delta_{\mathcal{A} \multimap \mathcal{B}},\; \Omega_{\mathcal{A} \multimap \mathcal{B}}) \end{array}$$

with $\delta_{\mathcal{A} \otimes \mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), \mathsf{a}, (u, v), (x, y), d) := (\delta_\mathcal{A}(q_\mathcal{A}, \mathsf{a}, u, x, d),\; \delta_\mathcal{B}(q_\mathcal{B}, \mathsf{a}, v, y, d))$ *and with* $\delta_{\mathcal{A} \multimap \mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), \mathsf{a}, (f, F), (u, y), d) := (q'_\mathcal{A}, q'_\mathcal{B})$ *where* $q'_\mathcal{B} = \delta_\mathcal{B}(q_\mathcal{B}, \mathsf{a}, fv, y, d)$ *and* $q'_\mathcal{A} = \delta_\mathcal{A}(q_\mathcal{A}, \mathsf{a}, u, F(u, y, d), d)$.

We moreover let $(q_k, q'_k)_k \in \Omega_{\mathcal{A} \otimes \mathcal{B}}$ *iff* $((q_k)_k \in \Omega_\mathcal{A} \wedge (q'_k)_k \in \Omega_\mathcal{B})$*, and* $(q_k, q'_k)_k \in \Omega_{\mathcal{A} \multimap \mathcal{B}}$ *iff* $((q_k)_k \in \Omega_\mathcal{A} \Rightarrow (q'_k)_k \in \Omega_\mathcal{B})$.

Note that $\mathcal{A} \otimes \mathcal{B}$ is non-deterministic if both $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic. Moreover, given $\mathcal{A}, \mathcal{B} : \Gamma$ and $M \in \mathbf{T}[\Sigma, \Gamma]$, we have, as $\mathsf{DialAut}_\Sigma$-objects, $\Sigma \vdash (\mathcal{A}(M) \multimap_{\mathsf{DA}} \mathcal{B}(M)) = \Sigma \vdash (\mathcal{A} \multimap \mathcal{B})(M)$ and $\Sigma \vdash (\mathcal{A}(M) \otimes_{\mathsf{DA}} \mathcal{B}(M)) = \Sigma \vdash (\mathcal{A} \otimes \mathcal{B})(M)$.

Example 5.5. (i) Given $\mathcal{A}, \mathcal{B}$ as in Def. 5.4, there is a winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \mathcal{A}$. It maps $(u, v) \in U \times V$ to $u \in U$ and $x \in X$ to $(x, y) \in X \times Y$, where $y \in Y$ is arbitrary (recall that Y is non-empty). It follows that P has a winning strategy in $\Sigma \vdash \mathcal{A} \multimap (\mathcal{B} \multimap \mathcal{A})$.

(ii) If $\mathcal{A} : \Sigma$ is non-deterministic, then there is a winning P-strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{A} \otimes \mathcal{A}$. Its maps $u \in U$ to $(u, u) \in U \times U$. Note that such strategy may not exist when $X \neq \mathbf{1}$, since O can play two different $(x, x') \in X \times X$ in the component $\mathcal{A} \otimes \mathcal{A}$.

Proposition 5.6. *Given $\mathcal{A}, \mathcal{B} : \Sigma$, we have $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.*

Falsity and Complementation. Alternating automata enjoy a complementation construction linear in the number of states (see e.g. [28]). Using the monoidal closed structure, a similar construction can be done with our automata.

The *falsity automaton* $\perp\!\!\!\!\perp : \Sigma$ is $(\mathbb{B}, \mathbb{f}, \mathfrak{D}, \mathbf{1}, \delta_{\perp\!\!\!\perp}, \Omega_{\perp\!\!\!\perp})$ where $\Omega_{\perp\!\!\!\perp} := \mathbb{B}^*.\mathbb{t}^\omega$ and where $\delta_{\perp\!\!\!\perp}(\mathbb{t}, \_, d', \bullet, d) := \mathbb{t}$ and $\delta_{\perp\!\!\!\perp}(\mathbb{f}, \_, d, \bullet, d) := \mathbb{f}$ and $\delta_{\perp\!\!\!\perp}(\mathbb{f}, \_, d', \bullet, d) := \mathbb{t}$ if $d' \neq d$. Note that $\perp\!\!\!\!\perp$ accepts no tree since in an acceptance game, O can always play the same d as P. Given $\mathcal{A} : \Sigma$, let $\mathcal{A}^{\perp\!\!\!\perp} := \mathcal{A} \multimap \perp\!\!\!\!\perp$. Note that $\mathcal{A}^{\perp\!\!\!\perp}$ is equivalent to an automaton with states $Q_\mathcal{A} + \mathbf{1}$. We say that $\mathcal{A}$ is *Borel* if its acceptance condition is a Borel set (regular sets are Borel). Thanks to the determinacy of Borel games [26], we get:

Proposition 5.7. *If $\mathcal{A} : \Sigma$ is Borel then $\mathcal{L}(\mathcal{A}^{\perp\!\!\!\perp}) = \Sigma^{\mathfrak{D}^*} \setminus \mathcal{L}(\mathcal{A})$.*

5.2 Quantifications

A fibration $\mathsf{p} : \mathbb{E} \to \mathbb{B}$ has existential quantifications (also called simple coproducts [19]) when the weakening functors $\pi^* : \mathbb{E}_I \to \mathbb{E}_{I \times J}$ (induced by the $\mathbb{B}$-projections $\pi : I \times J \to I$) have left adjoints $\coprod_{I,J} : \mathbb{E}_{I \times J} \to \mathbb{E}_I$ satisfying a *Beck-Chevalley* coherence condition, insuring that the adjunction $\coprod_{I,J} \dashv \pi^*$ is preserved by substitution. Universal quantifications (simple products [19]) are given by a right adjoint $\prod_{I,J} : \mathbb{E}_{I \times J} \to \mathbb{E}_I$ to π^* (also with a Beck-Chevalley condition). The simple fibration $\mathsf{s} : \mathsf{s}(\mathbb{B}) \to \mathbb{B}$ always has simple coproducts, and has simple products iff $\mathbb{B}$ is Cartesian closed. They are given by

$$\coprod_{I,J}(I \times J, X) \quad := \quad (I, J \times X) \qquad \text{and} \qquad \prod_{I,J}(I \times J, X) \quad := \quad (I, X^J)$$

This extends to DialZ, where given a $\mathbf{T}$-projection $\mathsf{p} : \Sigma \times \Gamma \to \Sigma$, the weakening functor $\mathsf{p}^* : \mathsf{DialZ}(\Sigma) \to \mathsf{DialZ}(\Sigma \times \Gamma)$ has left and right adjoints

$$\coprod_{\Sigma,\Gamma}(U, X) := (\Gamma \times U, X) \quad \text{and} \quad \prod_{\Sigma,\Gamma}(U, X) := (U^\Gamma, \Gamma \times X) \simeq (\Gamma \multimap_{\mathbf{DZ}_\mathfrak{D}} (U, X))$$

The Beck-Chevalley condition amounts, for $L \in \mathbf{T}[\Delta, \Sigma]$, to $L^*(\square_{\Sigma,\Gamma}(U, X)) = \square_{\Delta,\Gamma}(L \times \mathrm{Id}_\Gamma)^*(U, X)$ where $\square \in \{\coprod, \prod\}$. It holds since substitution functors are identities on objects. Quantifications in $\mathsf{DialAut}$ are given by

$$\coprod_{\Sigma,\Gamma}(U, X, \mathcal{W}_A) := (\Gamma \times U, X, \mathcal{W}_A) \quad \text{and} \quad \prod_{\Sigma,\Gamma}(U, X, \mathcal{W}_A) := (U^\Gamma, \Gamma \times X, \prod_{\Sigma,\Gamma} \mathcal{W}_A)$$

where $((\mathsf{a}_k, f_k) \cdot (\mathsf{b}_k, x_k, d_k))_k \in \prod_{\Sigma,\Gamma} \mathcal{W}_A$ iff $((\mathsf{a}_k, \mathsf{b}_k, f_k(\mathsf{b}_k)) \cdot (x_k, d_k))_k \in \mathcal{W}_A$.

Proposition 5.8. $\mathsf{DialAut}$ *has existential and universal quantifications.*

The same constructions give quantifiers on automata.

Definition 5.9. *Given* $\mathcal{A} : \Sigma \times \Gamma$ *as in (1), let*

$$\begin{array}{lll} (\exists_\Gamma \mathcal{A} : \Sigma) & := & (Q_\mathcal{A}\,,\, q^\imath_\mathcal{A}\,,\, \Gamma \times U\,,\, X\,,\, \delta_{\exists_\Gamma \mathcal{A}}\,,\, \Omega_\mathcal{A}) \\ (\forall_\Gamma \mathcal{A} : \Sigma) & := & (Q_\mathcal{A}\,,\, q^\imath_\mathcal{A}\,,\, U^\Gamma\,,\, \Gamma \times X\,,\, \delta_{\forall_\Gamma \mathcal{A}}\,,\, \Omega_\mathcal{A}) \end{array}$$

where $\delta_{\exists_\Gamma \mathcal{A}}(q, \mathsf{a}, (\mathsf{b}, u), x, d) := \delta_\mathcal{A}(q, (\mathsf{a}, \mathsf{b}), u, x, d)$ *and* $\delta_{\forall_\Gamma \mathcal{A}}(q, \mathsf{a}, f, (\mathsf{b}, x), d) := \delta_\mathcal{A}(q, (\mathsf{a}, \mathsf{b}), f(\mathsf{b}), x, d)$.

Note that given $\mathcal{A} : \Sigma \times \Gamma$ we have $(\Sigma \vdash \coprod_{\Sigma,\Gamma} \mathcal{A}) = (\Sigma \vdash \exists_\Sigma \mathcal{A})$ and $(\Sigma \vdash \prod_{\Sigma,\Gamma} \mathcal{A}) = (\Sigma \vdash \forall_\Sigma \mathcal{A})$ as $\mathsf{DialAut}_\Sigma$-objects.

Let $\mathsf{p}_\Sigma \in \mathbf{T}[\Sigma \times \Gamma, \Sigma]$ be the first projection. It is easy to see that if $\mathcal{A}$ accepts $T : \mathfrak{D}^* \to (\Sigma \times \Gamma)$, then $\exists_\Gamma \mathcal{A}$ accepts $(\mathsf{p}_\Sigma \circ \dot{T})^\bullet$, so that $\mathsf{p}_\Sigma(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\exists_\Gamma \mathcal{A})$. The converse only holds for *non-deterministic* automata.

Proposition 5.10. *If* $\mathcal{A} : \Sigma \times \Gamma$ *is* non-deterministic *then* $\mathcal{L}(\exists_\Gamma \mathcal{A}) = \mathsf{p}_\Sigma(\mathcal{L}(\mathcal{A}))$.

Example 5.11.

(i) Given $\mathcal{A} : \Sigma$ as in (1), let $\mathcal{D} : \Sigma \times U \times X$ be the deterministic automaton $(Q_\mathcal{A}, q^\imath_\mathcal{A}, \mathbf{1}, \mathbf{1}, \delta_\mathcal{D}, \Omega_\mathcal{A})$ with $\delta_\mathcal{D} : Q_\mathcal{A} \times (\Sigma \times U \times X) \to \mathfrak{D} \to Q_\mathcal{A}$ obtained from $\delta_\mathcal{A}$ in the obvious way. In $\mathsf{DialAut}_\Sigma$ we have $\mathcal{A} \simeq \exists_U \forall_X \mathcal{D}$.

(ii) The Beck-Chevalley conditions imply $\coprod_{\Sigma,\Gamma} \mathcal{A}(M \times \mathrm{Id}_\Gamma) = (\exists_\Gamma \mathcal{A})(M)$ and $\prod_{\Sigma,\Gamma} \mathcal{A}(M \times \mathrm{Id}_\Gamma) = (\forall_\Gamma \mathcal{A})(M)$. Thanks to the adjunctions $\coprod \dashv \mathsf{p}^* \dashv \prod$, in $\mathsf{DialAut}$ we have

$$\begin{array}{lcl} \Sigma \vdash (\exists_\Gamma \mathcal{A})(M) \multimap \mathcal{B}(N) & \simeq & \Sigma \times \Gamma \vdash \mathcal{A}(M \times \mathrm{Id}_\Gamma) \multimap \mathcal{B}(N \circ \mathsf{p}_\Sigma) \\ \Sigma \vdash \mathcal{B}(N) \multimap (\forall_\Gamma \mathcal{A})(M) & \simeq & \Sigma \times \Gamma \vdash \mathcal{B}(N \circ \mathsf{p}_\Sigma) \multimap \mathcal{A}(M \times \mathrm{Id}_\Gamma) \end{array}$$

Hence P has a winning strategy in $\Sigma \vdash (\forall_\Sigma \mathcal{A})[\mathbf{1}_\Sigma] \multimap \mathcal{A}$ (mod $\Sigma \simeq \mathbf{1} \times \Sigma$).

(iii) If $\mathcal{A}, \mathcal{B} : \Sigma$ are regular, then the game $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ is equivalent to a finite regular game. Indeed, by (ii), P has a winning strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ iff he has a winning strategy in $\mathbf{1} \vdash \mathbf{I_1} \multimap \forall_\Sigma(\mathcal{A} \multimap \mathcal{B})$. But since in that game O can only play $\bullet$ in the component $\mathbf{I_1}$, similarly as in Ex. 3.9 it is equivalent to the acceptance game of the automaton $\forall_\Sigma(\mathcal{A} \multimap \mathcal{B}) : \mathbf{1}$ on the unique tree $\mathbf{1} : \mathfrak{D}^* \to \mathbf{1}$. Since $\forall_\Sigma(\mathcal{A} \multimap \mathcal{B})$ is regular, this game is equivalent to a finite regular game and the winner always effectively has a finite state winning strategy (see e.g. [35, Ex. 6.12 & Thm. 6.18]).

(iv) If $\mathcal{A}, \mathcal{B} : \Sigma$ are non-deterministic Borel and $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, then P has a winning strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$ and in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}^\perp$.

5.3 A Deduction System for Automata

We now present a deduction system for tree automata. It allows to derive sequents $M \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}$ where $\overline{\mathcal{A}} = \mathcal{A}_1, \dots, \mathcal{A}_n : \Sigma$, $\mathcal{A} : \Sigma$ and $M \in \mathbf{T}[\Gamma, \Sigma]$. The rules are given in Fig. 2. They are correct in the following sense:

Proposition 5.12. *If $M \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}$ is derivable, then there is a winning P-strategy in $\Gamma \vdash \mathcal{A}_1(M) \otimes \ldots \otimes \mathcal{A}_n(M) \multimap \mathcal{A}(M)$.*

A game of the form $\Gamma \vdash \mathcal{A}(M) \multimap \mathcal{B}(N)$ is represented as $\langle M, N\rangle \;;\; \mathcal{A}[\mathsf{p}] \vdash \mathcal{B}[\mathsf{q}]$ where p and q are suitable projections. Write $\mathcal{A} \vdash \mathcal{B}$ for the sequent $\mathrm{Id} \;;\; \mathcal{A} \vdash \mathcal{B}$.

Example 5.13.

(i) If $\mathcal{A}, \mathcal{B} : \Sigma$ are both non-deterministic, then one can derive $\mathcal{A} \vdash \mathcal{A} \otimes \mathcal{A}$ and $\mathcal{B} \otimes (\mathcal{B} \multimap \mathcal{A}) \vdash \mathcal{A} \otimes \mathcal{B}$, so that by Prop. 5.12, P has a winning strategy in $\Sigma \vdash \mathcal{B} \otimes (\mathcal{B} \multimap \mathcal{A}) \longrightarrow\!\!\circ\; \mathcal{A} \otimes \mathcal{B}$.

(ii) Continuing (i), if $\mathcal{A}, \mathcal{B}$ are moreover Borel and $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, by Ex. 5.11.(iv) there is a winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$. By composing it with the strategy of (i), we obtain a winning P-strategy in $\Sigma \vdash \mathcal{B} \otimes (\mathcal{B} \multimap \mathcal{A}) \longrightarrow\!\!\circ\; \perp$ and thus in $\Sigma \vdash (\mathcal{B} \multimap \mathcal{A}) \longrightarrow\!\!\circ\; \mathcal{B}^{\perp}$. It then follows from Ex. 5.5.(i) and Prop. 4.1 that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B} \multimap \mathcal{A}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$.

Remark 5.14. The rules of Fig. 2 are compatible with cut-elimination (see e.g. [27]). For instance, the following two derivations are interpreted by the same strategy:

$$\dfrac{\dfrac{\dfrac{\Delta_1}{\mathcal{A} \vdash \mathcal{B}}}{\mathbf{I} \vdash \mathcal{A} \multimap \mathcal{B}} \qquad \dfrac{\dfrac{\Delta_2}{\mathbf{I} \vdash \mathcal{A}} \qquad \dfrac{}{\mathcal{B} \vdash \mathcal{B}}}{\mathcal{A} \multimap \mathcal{B} \vdash \mathcal{B}}}{\mathbf{I} \vdash \mathcal{B}} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \Delta_1[\Delta_2/\mathcal{A}]\end{array}}{\mathbf{I} \vdash \mathcal{B}}$$

5.4 Non-Deterministic Automata

We have seen in §5.1 that similarly to usual alternating automata, our automata have linear complements. Moreover, by Prop. 5.10 the projection operation is correct on non-deterministic automata. However, complementation is not linear on non-deterministic automata and projection is not correct in general on alternating automata. On the other hand it is well-known that regular alternating and non-deterministic automata are equivalent in expressive power:

Theorem 5.15 (Simulation [28, 9] (see also [29])). *Given a regular $\mathcal{A} : \Sigma$, one can effectively build a regular non-deterministic $!\mathcal{A} : \Sigma$ s.t. $\mathcal{L}(\mathcal{A}) = \mathcal{L}(!\mathcal{A})$.*

The automaton $!\mathcal{A}$ is in general exponentially larger than $\mathcal{A}$.

In our context, it is possible to adapt the construction for $!\mathcal{A}$ of [36] so that the operation $!(-)$ satisfies the *deduction* rules of the usual exponential modality ! of intuitionistic linear logic [11] (see also [27]). For regular $\overline{\mathcal{A}}, \mathcal{B}, \mathcal{A}$:

$$\frac{M \;;\; !\overline{\mathcal{A}} \vdash \mathcal{A}}{M \;;\; !\overline{\mathcal{A}} \vdash !\mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}}, \mathcal{B} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}}, !\mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}}, \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}}, !\mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}}, !\mathcal{B}, !\mathcal{B} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}}, !\mathcal{B} \vdash \mathcal{A}}$$

Propositional rules (for $\mathcal{N}$ non-deterministic):

$$\frac{}{M \;;\; \overline{\mathcal{A}},\, \mathcal{A} \vdash \mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}} \vdash \mathcal{B} \qquad M \;;\; \overline{\mathcal{C}},\, \mathcal{B} \vdash \mathcal{C}}{M \;;\; \overline{\mathcal{A}},\, \overline{\mathcal{C}} \vdash \mathcal{C}} \qquad \frac{M \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{A}}$$

$$\frac{M \;;\; \overline{\mathcal{B}} \vdash \mathcal{B} \qquad M \;;\; \overline{\mathcal{C}},\, \mathcal{C} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{B}},\, \overline{\mathcal{C}},\, \mathcal{B} \multimap \mathcal{C} \vdash \mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{C}}{M \;;\; \overline{\mathcal{A}} \vdash \mathcal{B} \multimap \mathcal{C}} \qquad \frac{M \;;\; \overline{\mathcal{A}},\, \mathcal{N},\, \mathcal{N} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}},\, \mathcal{N} \vdash \mathcal{A}}$$

$$\frac{M \;;\; \overline{\mathcal{A}},\, \mathcal{B},\, \mathcal{C} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}},\, \mathcal{B} \otimes \mathcal{C} \vdash \mathcal{A}} \qquad \frac{M \;;\; \overline{\mathcal{A}} \vdash \mathcal{A} \qquad M \;;\; \overline{\mathcal{B}} \vdash \mathcal{B}}{M \;;\; \overline{\mathcal{A}},\, \overline{\mathcal{B}} \vdash \mathcal{A} \otimes \mathcal{B}} \qquad \frac{}{M \;;\; \overline{\mathcal{A}} \vdash \mathbf{I}}$$

Quantification and Substitution Rules:
(where M, N are composable, p is a suitable projection, and $\mathsf{a} \in \mathbf{A}[\Sigma, \Gamma]$)

$$\frac{M \times \mathrm{Id}_\Gamma \;;\; \overline{\mathcal{A}[\mathsf{p}]},\, \mathcal{B} \vdash \mathcal{A}[\mathsf{p}]}{M \;;\; \overline{\mathcal{A}},\, \exists_\Gamma \mathcal{B} \vdash \mathcal{A}} \qquad \frac{M \times N \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}}{M \times N \;;\; \overline{\mathcal{A}} \vdash (\exists_\Gamma \mathcal{A})[\mathsf{p}]} \qquad \frac{M \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}}{M \circ N \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}}$$

$$\frac{M \times N \;;\; \overline{\mathcal{A}},\, \mathcal{B} \vdash \mathcal{A}}{M \times N \;;\; \overline{\mathcal{A}},\, (\forall_\Gamma \mathcal{B})[\mathsf{p}] \vdash \mathcal{A}} \qquad \frac{M \times \mathrm{Id}_\Gamma \;;\; \overline{\mathcal{A}[\mathsf{p}]} \vdash \mathcal{A}}{M \;;\; \overline{\mathcal{A}} \vdash \forall_\Gamma A} \qquad \frac{M \times \mathrm{Id}_\Gamma \;;\; \overline{\mathcal{A}} \vdash \mathcal{A}}{M \times \mathrm{Id}_\Sigma \;;\; \overline{\mathcal{A}[\mathrm{id} \times \mathsf{a}]} \vdash \mathcal{A}[\mathrm{id} \times \mathsf{a}]}$$

Fig. 2. Deduction Rules for Automata

It follows that the exponential ! allows to define, using Girard's decomposition, an intuitionistic implication $\Rightarrow$ as $\mathcal{A} \Rightarrow \mathcal{B} := !\mathcal{A} \multimap \mathcal{B}$.

The last two rules (called *Weakening* and *Contraction*) are already part of the basic system (Fig. 2). The second rule (*Dereliction*) easily follows from the construction of $!\mathcal{A}$. The most difficult rule is the first one (*Promotion*). It relies on the existence of positional P-strategies in regular games equipped with a disjunction of parity conditions (also called a *Rabin* condition) [21, 22, 20, 37]. Unfortunately, positionality is not preserved by composition, and this rule is not compatible with cut-elimination (in the sense of Rem. 5.14).

Proposition 5.16. *Let $\mathcal{N}, \mathcal{A} : \Sigma$ be regular, with $\mathcal{N}$ non-deterministic.*

(i) There is a winning P*-strategy in $\Sigma \vdash !\mathcal{A}(M) \multimap \mathcal{A}(M)$.*
(ii) If there is a winning P*-strategy in $\Sigma \vdash \mathcal{N}(L) \multimap \mathcal{A}(M)$ then there is a winning* P*-strategy in $\Sigma \vdash \mathcal{N}(L) \multimap !\mathcal{A}(M)$.*

Example 5.17. Let $\mathcal{A}, \mathcal{B} : \Sigma$ be regular, and write $?\mathcal{A}$ for $(!(\mathcal{A}^\perp))^\perp$.

(i) There is a winning P-strategy in $\Sigma \vdash ((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \Longrightarrow ?\mathcal{A}$.
(ii) If $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, then there is a winning P-strategy in $\Sigma \vdash !\mathcal{A} \multimap ?\mathcal{B}$.
(iii) Extending [7, Thm. 1], for each regular language $\mathcal{L} \subseteq \Sigma^{\mathfrak{D}^*}$, there is a non-deterministic automaton $\mathcal{B}$ with $\mathcal{L}(\mathcal{B}) = \mathcal{L}$, and such that for every non-deterministic parity automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}$, there is a winning P-strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ induced by a function $Q_\mathcal{A} \times Q_\mathcal{B} \times \Sigma \times U \to V$.

Further works include e.g. intuitionistic and linear versions of MSO, accounts of separation properties (as in e.g. [33]), and exponential modalities in **DZ**.

References

[1] S. Abramsky. Semantics of Interaction. In A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, volume 14 of *Publications of the Newton Institute*, page 1. Cambridge University Press, 1997. 2, 3, 4, 20

[2] A. Arnold. The μ-calculus alternation-depth hierarchy is strict on binary trees. *ITA*, 33(4/5):329–340, 1999.

[3] A. Arnold and D. Niwinski. Continuous Separation of Game Languages. *Fundam. Inform.*, 81(1-3):19–28, 2007.

[4] J. Avigad and S. Feferman. Gödel's functional ("Dialectica") interpretation. In S. Buss, editor, *Handbook Proof Theory*, volume 137 of *Studies in Logic and the Foundations of Mathematics*, pages 337–405. Elsevier, 1998. 2

[5] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8(4), 2012. 44, 45, 49

[6] M. Bojańczyk and T. Colcombet. Bounds in w-Regularity. In *Proceedings of LICS'06*, pages 285–296. IEEE Computer Society, 2006.

[7] T. Colcombet and C. Löding. The Non-deterministic Mostowski Hierarchy and Distance-Parity Automata. In *ICALP 2008*, volume 5126 of *Lecture Notes in Computer Science*, pages 398–409. Springer, 2008. 15, 36, 38

[8] V. de Paiva. The Dialectica categories. Technical Report 213, University of Cambridge Computer Laboratory, January 1991. 2, 41

[9] E. A. Emerson and C. S. Jutla. Tree Automata, Mu-Calculus and Determinacy (Extended Abstract). In *FOCS*, pages 368–377. IEEE Computer Society, 1991. 1, 14, 33, 95

[10] A. Facchini, F. Murlak, and M. Skrzypczak. Rabin-Mostowski Index Problem: A Step beyond Deterministic Automata. In *LICS*, pages 499–508. IEEE Computer Society, 2013.

[11] J.-Y. Girard. Linear Logic. *Theoretical Computer Science*, 50:1–102, 1987. 2, 14

[12] E. Grädel, W. Thomas, and T. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research*, volume 2500 of *LNCS*. Springer, 2002. 1, 2, 7, 23

[13] R. Harmer, M Hyland, and P.-A. Melliès. Categorical combinatorics for innocent strategies. In *LICS 2007*, pages 379–388, 2007. 9, 23, 59, 76

[14] P. J. W. Hofstra. The dialectica monad and its cousins. In M. Makkai and B.T. Hart, editors, *Models, Logics, and Higher-dimensional Categories: A Tribute to the Work of Mihály Makkai*, CRM proceedings & lecture notes. American Mathematical Society, 2011. 9

[15] J. M. E. Hyland. Game Semantics. In A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, volume 14 of *Publications of the Newton Institute*, page 131. Cambridge University Press, 1997. 2, 3, 4, 20

[16] J. M. E. Hyland. Proof theory in the abstract. *Ann. Pure Appl. Logic*, 114(1-3):43–78, 2002. 2, 9, 41

[17] J. M. E. Hyland and A. Schalk. Abstract Games for Linear Logic. *Electr. Notes Theor. Comput. Sci.*, 29:127–150, 1999. 4, 5, 8, 9, 18, 94

[18] J. M. E. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294(1/2):183–231, 2003. 5, 8, 9, 40, 41, 42, 44, 87, 89

[19] B. Jacobs. *Categorical Logic and Type Theory*. Studies in logic and the foundations of mathematics. Elsevier, 2001. 1, 2, 9, 11, 12

[20] C. S. Jutla. Determinization and Memoryless Winning Strategies. *Inf. Comput.*, 133(2):117–134, 1997. 15, 33, 34

[21] N. Klarlund. Progress measures, immediate determinacy, and a subset construction for tree automata. *Annals of Pure and Applied Logic*, 69(2-3):243–268, 1994. 15, 33, 34

[22] N. Klarlund and D. Kozen. Rabin Measures. *Chicago J. Theor. Comput. Sci.*, 1995, 1995. 15, 33, 34

[23] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Springer, 2008. 2

[24] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 2nd edition, 1998. 26, 68

[25] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer, 1992. 45

[26] D. A. Martin. Borel Determinacy. *The Annals of Mathematics, Second Series*, 102(2):363–371, 1975. 12, 30

[27] P.-A. Melliès. Categorical semantics of linear logic. In *Interactive models of computation and program behaviour*, volume 27 of *Panoramas et Synthèses*. SMF, 2009. 5, 14, 20, 26, 43, 52, 68, 70, 71, 72, 74, 75, 82, 87, 89, 92, 93

[28] D. E. Muller and P. E. Schupp. Alternating Automata on Infinite Trees. *Theor. Comput. Sci.*, 54:267–276, 1987. 1, 7, 12, 14, 33, 95

[29] D. E. Muller and P. E Schupp. Simulating Alternating Tree Automata by Nondeterministic Automata: New Results and New Proofs of the Theorems of Rabin, McNaughton and Safra. *Theor. Comput. Sci.*, 141(1&2):69–107, 1995. 1, 7, 14, 33, 95

[30] D. Perrin and J.-É. Pin. *Infinite Words: Automata, Semigroups, Logic and Games*. Pure and Applied Mathematics. Elsevier, 2004. 2, 7, 23

[31] M. O. Rabin. Decidability of Second-Order Theories and Automata on Infinite Trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969. 1

[32] C. Riba. Fibrations of tree automata. In *TLCA*, volume 38 of *LIPIcs*, pages 302–316. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. 1, 2, 3, 5, 7, 8, 18, 20, 26

[33] L. Santocanale and A. Arnold. Ambiguous classes in mu-calculi hierarchies. *Theor. Comput. Sci.*, 333(1-2):265–296, 2005. 15, 36, 39, 95

[34] M. Skrzypczak. Separation Property for wB- and wS-regular Languages. *Logical Methods in Computer Science*, 10(1), 2014.

[35] W. Thomas. Languages, Automata, and Logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, volume III, pages 389–455. Springer, 1997. 1, 2, 7, 13, 23, 32, 38

[36] I. Walukiewicz. Monadic second-order logic on tree-like structures. *Theor. Comput. Sci.*, 275(1-2):311–346, 2002. 1, 7, 14, 23, 30, 33

[37] Z. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200(1-2):135–183, 1998. 15, 33, 34

A Proofs and Additional Material for §2 (Games)

This appendix gathers proofs and some supplementary basic material on §2. We discuss the Hyland & Schalk functor [17], the definition of the categories $\mathbf{DZ}$ and $\mathbf{DZ}^{\mathrm{W}}$, and some related easy basic facts about relations in slices which will lead in §B to a simple proof that $\mathbf{AG}_{\Sigma}$ and $\mathbf{AG}^{\mathrm{W}}_{\Sigma}$ are categories (see Def. 3.8). We also discuss general facts on monoids and comonoids in symmetric monoidal categories. We finally discuss the monoidal structure of $\mathbf{DZ}$, using a representation of $\mathbf{DZ}$ as a subcategory of the simple self-dualization $\mathbf{G}(\mathscr{S})$ of the topos of trees $\mathscr{S}$. The construction of simple self dualization is presented in §I. The representation of zig-zag strategies in $\mathbf{G}(\mathscr{S})$ is presented in §J. Some material on monoidal categories is recalled in §M.

Notations. Given a full positive game $A = (, U, X)$, we let $\wp^{\mathsf{even}}_A \subseteq \wp^+_A = L_A$ be the set of its even-length plays. Note that there is a bijection

$$\partial \quad = \quad \langle \partial_U, \partial_X \rangle \quad : \quad \wp^{\mathsf{even}}_A \quad \longrightarrow \quad \cup_{n \in \mathbb{N}} (U^n \times X^n)$$

with $\partial(\varepsilon) = (\bullet, \bullet)$ and $\partial(s.u.x) = (\partial_U(s).u, \partial_X(s).x)$.

A.1 The Hyland-Schalk Functor [17]

The faithfull functor $\mathrm{HS} : \mathbf{SG} \longrightarrow \mathbf{Rel}$ is defined in [17] as a functor from the full subcategory of $\mathbf{SG}$ consisting of *negative* games only. The extension to $\mathbf{SG}$ is discussed in §4 of the Appendix of the long version of [32][1].

Proposition A.1. *The map taking $\sigma : A \multimap B$ to*

$$\mathrm{HS}(\sigma) \quad := \quad \{(s_{\restriction A}, s_{\restriction B}) \mid s \in \sigma\} \quad \subseteq \quad L_A \times L_B$$

is a faithfull functor $\mathrm{HS} : \mathbf{SG} \to \mathbf{Rel}$.

We can therefore faithfully represent strategies $\sigma : A \multimap B$ as spans

$$L_A \quad \xleftarrow{\pi_1} \quad \mathrm{HS}(\sigma) \quad \xrightarrow{\pi_2} \quad L_B$$

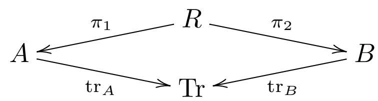
Explicitly, $\mathrm{HS}(\mathrm{id}_A)$ is identity relation on L_A, and given $\sigma : A \multimap B$, $\tau : B \multimap C$, $\tau \circ \sigma$ is the unique strategy such that $\mathrm{HS}(\tau \circ \sigma)$ is the relation $\mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$.

A.2 The Category $\mathbf{SG}_{/\mathbf{Tr}}$ of Sliced Games over Tr

Given a set Tr, the category $\mathbf{Rel}(\mathbf{Set}_{/\mathrm{Tr}})$ of relations over Tr has the same objects as $\mathbf{Set}_{/\mathrm{Tr}}$ (pairs of a set A and a function $\mathrm{tr}_A : A \to \mathrm{Tr}$). Maps from

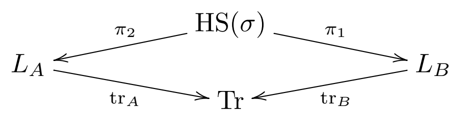
[1] Available at `https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf`.

(A, tr_A) to (B, tr_B) are relations $R \subseteq A \times B$ s.t. $\mathrm{tr}_A(a) = \mathrm{tr}_B(b)$ for all $(a, b) \in R$, that is



A *sliced game over* Tr is a pair of a simple game (with winning) A and a function $\mathrm{tr}_A : L_A \to \mathrm{Tr}$.

The category $\mathbf{SG}_{/\mathrm{Tr}}$ has sliced games as objects. The maps from (A, tr_A) to (B, tr_B) are strategies $\sigma : A \multimap B$ such that $\mathrm{HS}(\sigma)$ is a $\mathbf{Rel}(\mathbf{Set}_{/\mathrm{Tr}})$-map from (L_A, tr_A) to (L_B, tr_B). We therefore require:

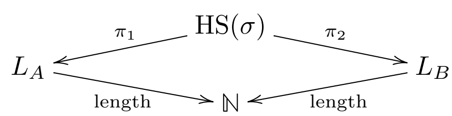


It directly follows from Prop. A.1 that $\mathbf{SG}_{/\mathrm{Tr}}$ is a category.

Proposition A.2. $\mathbf{SG}_{/\mathrm{Tr}}$ *is a category.*

A.3 The Category $\mathbf{DZ}^0$ of Zig-Zag Games

A *zig-zag strategy* $\sigma : A \multimap B$ is a strategy such that for every (even-length) play $s \in \sigma$, the projections $s_{\restriction A}$ and $s_{\restriction B}$ have the same length (see Fig. 1, top left). Hence, $\sigma : A \multimap B$ is a zig-zag strategy if and only if $\mathrm{HS}(\sigma)$ is a $\mathbf{Rel}(\mathbf{Set}_{/\mathbb{N}})$-map from $(L_A, \mathrm{length} : L_A \to \mathbb{N})$ to $(L_B, \mathrm{length} : L_B \to \mathbb{N})$:



Let $\mathbf{DZ}^0$ have full positive games as objects and zig-zag strategies as morphisms. Proposition A.2 directly gives:

Proposition A.3. $\mathbf{DZ}^0$ *is a category.*

A.4 Simple Games with Winning

A *simple game with winning* is a simple game A equipped with a *set of winning plays* (or *winning condition*) $\mathcal{W}_A \subseteq (A^{\xi_A} \cdot A^{-\xi_A})^\omega$.

Consider a P-strategy σ on A and an O-play $s \in L_A$. We say that s is an O*-interrogation of* σ if either $s = \varepsilon$ and A is positive, or if $s = t.m$ for some $t \in \sigma$. We say that σ is *total* if for every O-interrogation s of σ, we have $s.n \in \sigma$ for some n. A *winning* (P-)strategy on $(A, \mathcal{W}_A)$ is a total strategy σ on A s.t. for all $\varpi \in (A^{\xi_A} \cdot A^{-\xi_A})^\omega$, we have $\varpi \in \mathcal{W}_A$ whenever $\exists^\infty k \in \mathbb{N}.\ \varpi(0). \cdots .\varpi(k) \in \sigma$.

It is well-known (see e.g. [1, 15]), that on negaive simple games, total and winning strategies compose and form a category. The case of **SG** is discussed in §5 of the Appendix of the long version of [32][2].

Proposition A.4. *Simple games with winning define a category,* $\mathbf{SG}^{\mathrm{W}}$*, where maps from* $(A, \mathcal{W}_A)$ *to* $(B, \mathcal{W}_B)$ *are total strategies* $\sigma : A \multimap B$ *which are winning for the winning condition* $\mathcal{W}^{\mathbf{SG}}_{A\multimap B}$ *consisting of infinite plays* ϖ *such that if* ($\varpi_{\restriction A}$ *is either a finite legal* P*-play of* A *or* $\varpi_{\restriction A} \in \mathcal{W}_A$) *then* ($\varpi_{\restriction B}$ *is either a finite legal* P*-play of* A *or* $\varpi_{\restriction B} \in \mathcal{W}_B$)*.*

A.5 The Categories DZ and $\mathbf{DZ}^{\mathbf{W}}$

Its easy to see that total zig-zag strategies compose: $\sigma : A \multimap B$ and $\tau : B \multimap C$ are both zig-zag and total, then $\tau \circ \sigma$ is zig-zag and total.

– *Proof.* Indeed, consider $(s, t) \in \mathrm{HS}(\tau \circ \sigma) = \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$, and u such that $(s, u) \in \mathrm{HS}(\sigma)$ and $(u, t) \in \mathrm{HS}(\tau)$. Given a legal $(A \multimap C)_{\mathsf{O}}$-move m in (say) component A, since σ is zig-zag and total, there is some n such that $(s.m, u.n) \in \mathrm{HS}(\sigma)$. Since $n \in B_{\mathsf{P}} \subseteq (B \multimap C)_{\mathsf{O}}$, and since τ is zig-zag and total, there is some $r \in C_{\mathsf{P}}$ such that $(u.n, t.r) \in \mathrm{HS}(\tau)$, from which it follows that $(s.m, t.r) \in \mathrm{HS}(\tau \circ \sigma)$. The case of $m \in C_{\mathsf{O}}$ is similar. ◻

Consider full games with winning $(A, \mathcal{W}_A)$ and $(B, \mathcal{W}_B)$. Note that if $\sigma : A \multimap B$ is zig-zag and total, then for every $\varpi \in ((A^+ + B^-) \cdot (A^- + B^+))^\omega$, if ϖ has infinitely many finite prefixes in σ, then $\varpi_{\restriction A}$ and $\varpi_{\restriction B}$ are both infinite. We let $\mathcal{W}_{A\multimap B} \subseteq ((A^+ + B^-) \cdot (A^- + B^+))^\omega$ be the set of infinite plays ϖ such that $\varpi_{\restriction B} \in \mathcal{W}_B$ whenever $\varpi_{\restriction A} \in \mathcal{W}_A$. Note that for an infinite play ϖ of a total zig-zag $\sigma : A \multimap B$ (*i.e.* s.t. $\exists^\infty k \in \mathbb{N}. \varpi(0). \cdots .\varpi(k) \in \sigma$), we have $\varpi \in \mathcal{W}^{\mathbf{SG}}_{A\multimap B}$ iff $\varpi \in \mathcal{W}_{A\multimap B}$. We therefore get:

Proposition A.5. **DZ** *and* $\mathbf{DZ}^{\mathrm{W}}$ *are categories.*

A.6 Monoids and Comonoids

A commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is an object M equipped with structure maps $m : M \otimes M \to M$ and $u : \mathbf{I} \to M$ subject to coherence conditions (see e.g. [27]). A (commutative) comonoid in $\mathbb{C}$ is a (commutative) monoid in $\mathbb{C}^{\mathrm{op}}$. In this paper, by (co)monoid we always mean *commutative* (co)monoid. Note that if $(\mathbb{C}, \otimes, \mathbf{I})$ is Cartesian, then every object has a canonical comonoid structure.

The following is Prop. M.12 together with Prop. M.14.

Proposition A.6. *(a) A monoid* (M, m, u) *in* $\mathbb{C}$ *induces a (lax) symmetric monoidal monad* (M, μ^M, η^M) *of* monoid indexing*. The functor* M *takes*

[2] Available at `https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf`.

an object A to $A \otimes M$ and a map $f : A \to B$ to $f \otimes \mathrm{id}_M : A \otimes M \to B \otimes M$. The natural maps μ^M and η^M are

$$\begin{array}{rcllrcl} \mu^M_A & := & (\mathrm{id}_A \otimes m) \circ \alpha & : & (A \otimes M) \otimes M & \longrightarrow & A \otimes M \\ \eta^M_A & := & (\mathrm{id}_A \otimes u) \circ \rho^{-1} & : & A & \longrightarrow & A \otimes M \end{array}$$

(where $\rho : A \otimes \mathbf{I} \to A$ and $\alpha : (A \otimes M) \otimes M \to A \otimes (M \otimes M)$ are structural isos of $(\mathbb{C}, \otimes, \mathbf{I})$).

(b) Dually, a comonoid $K = (K, d, e)$ in $\mathbb{C}$ induces an oplax symmetric monoidal comonad $(K, \delta^K, \epsilon^K)$, where we let $K(A) := K \otimes A$ and $K(f) := \mathrm{id}_K \otimes f$ for $f : A \to B$, and where

$$\begin{array}{rcllrcl} \delta^K_A & := & \alpha \circ (d \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & K \otimes (K \otimes A) \\ \epsilon^K_A & := & \lambda \circ (e \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & A \end{array}$$

It then follows from Prop. M.2 and Prop. M.4 that for a monoid M and a comonoid K, the Kleisely categories $\mathbf{Kl}(M)$ and $\mathbf{Kl}(K)$ are symmetric monoidal, for a monoidal product $\otimes_{\mathbf{Kl}}$ which has the same actions as $\otimes$ on objects and the same unit $\mathbf{I}$. Moreover, thanks to Prop. M.11, comonoids in $\mathbb{C}$ lift to comonoids in $\mathbf{Kl}(M)$ in a canonical way (and dually for monoids and $\mathbf{Kl}(K)$), so that we have:

Proposition A.7 (Prop 2.4).

(a) Given a monoid M in $\mathbb{C}$, the Kleisli category $\mathbf{Kl}(M)$ is symmetric monoidal with $A \otimes_{\mathbf{Kl}(M)} B := A \otimes B$ on objects and unit $\mathbf{I}$. Moreover, each comonoid (K, d, e) in $\mathbb{C}$ induces a comonoid $(K,\ \eta^M_{K \otimes K} \circ d,\ \eta^M_{\mathbf{I}} \circ e)$ in $\mathbf{Kl}(M)$.

(b) Dually, given a comonoid K in $\mathbb{C}$, the Kleisli category $\mathbf{Kl}(K)$ is symmetric monoidal with $A \otimes_{\mathbf{Kl}(K)} B := A \otimes B$ on objects and unit $\mathbf{I}$. Moreover, each comonoid (M, m, u) in $\mathbb{C}$ induces a comonoid $(K,\ m \circ \epsilon^K_{M \otimes M},\ u \circ \epsilon^K_{\mathbf{I}})$.

A.7 The Monoidal Structure of DZ (§2.2 and §2.3)

We discuss here Prop. 2.2 and Prop. 2.3 concerning the symmetric monoidal structure of $\mathbf{DZ}$. The proofs are based on an interpretation of $\mathbf{DZ}$ as a subcategory of the simple self-dualization $\mathbf{G}(\mathscr{S})$ of the topos of trees $\mathscr{S}$. The construction of simple self dualization is presented in §I. The representation of zig-zag strategies in $\mathbf{G}(\mathscr{S})$ is presented in §J. Some material on monoidal categories is recalled in §M.

The monoidal structure of $\mathbf{DZ}$ is given by the following data, (where $A = (U, X)$, $B = (V, Y)$ and $C = (W, Z)$):

$$A \otimes B \quad := \quad (U \times V, X \times Y) \qquad \text{with unit} \quad \mathbf{I} := (\mathbf{1}, \mathbf{1})$$

and the natural structure maps:

	$(A \otimes B) \otimes C$	$\xrightarrow{\alpha_{A,B,C}}$	$A \otimes (B \otimes C)$	
O	$((u, v), w)$			
			$(u, (v, w))$	P
			$(x, (y, z))$	O
P	$((x, y), z)$			

	$\mathbf{I} \otimes A$ $\overset{\lambda_A}{\multimap}$	A	
O	$(\bullet, u)$		
		u	P
		x	O
P	$(\bullet, x)$		

	$A \otimes \mathbf{I}$ $\overset{\rho_A}{\multimap}$	A	
O	$(u, \bullet)$		
		u	P
		x	O
P	$(x, \bullet)$		

	$A \otimes B$ $\overset{\gamma_{A,B}}{\multimap}$	$B \otimes A$	
O	(u, v)		
		(v, u)	P
		(y, x)	O
P	(x, y)		

The following is shown in Prop. J.5 using a represention of total zig-zag strategies in the topos of trees.

Proposition A.8 (Prop. 2.2). *The category* $\mathbf{DZ}$ *equipped with the above data is symmetric monoidal.*

Proposition 2.3 is Prop. J.6, proved using a represention of total zig-zag strategies in the topos of trees.

Proposition A.9 (Prop. 2.3). *If* M, K *are non-empty, then* $M := (\mathbf{1}, M)$ *is a monoid and* $K := (K, \mathbf{1})$ *is a comonoid in* $\mathbf{DZ}$*. Structure maps can be depicted as follows:*

	$M \otimes M$ $\overset{m_M}{\multimap}$	M	
O	$(\bullet, \bullet)$		
		$\bullet$	P
		m	O
P	(m, m)		

	$\mathbf{I}$ $\overset{u_M}{\multimap}$	M	
O	$\bullet$		
		$\bullet$	P
		m	O
P	$\bullet$		

	K $\overset{d_K}{\multimap}$	$K \otimes K$	
O	k		
		(k, k)	P
		$(\bullet, \bullet)$	O
P	$\bullet$		

	K $\overset{e_K}{\multimap}$	$\mathbf{I}$	
O	k		
		$\bullet$	P
		$\bullet$	O
P	$\bullet$		

B Proofs of §3 (Tree Automata and Generalized Acceptance Games)

B.1 Proofs of §3.1 (The Base Category T of Trees)

The following is shown in Prop. J.7 using a represention of total zig-zag strategies in the topos of trees (see also §A.7).

Proposition B.1 (Prop. 3.3). *The category* $\mathbf{T}$ *embeds to* $\mathbf{Comon}(\mathbf{DZ}_{\mathfrak{D}})$ *via the functor* $\mathsf{E}_{\mathbf{T}}$ *mapping an object* Σ *of* $\mathbf{T}$ *to the comonoid* $(\Sigma, e_\Sigma, d_\Sigma)$ *and a morphism* $M : \mathbf{T}[\Gamma, \Sigma]$ *to itself.*

Remark B.2 (Comonoids in $\mathbf{DZ}_{\mathfrak{D}}$*).* Each $\mathbf{DZ}$-comonoid $\Sigma = (\Sigma, \mathbf{1})$ is lifted to a comonoid $\Sigma = (\Sigma, \mathbf{1})$ whose structure maps $\tilde{d}_\Sigma = \mathsf{F}_{\mathfrak{D}}(d_\Sigma)$ and $\tilde{e}_\Sigma = \mathsf{F}_{\mathfrak{D}}(e_\Sigma)$ can be depicted as follows:

	Σ $\overset{\tilde{d}_\Sigma}{\multimap}$	$\Sigma \otimes \Sigma$	
O	a		
		(a, a)	P
		d	O
P	$\bullet$		

	Σ $\overset{\tilde{e}_\Sigma}{\multimap}$	$\mathbf{I}$	
O	a		
		$\bullet$	P
		d	O
P	$\bullet$		

Proof. From Prop. J.6.(ii) together with Prop. M.11 applied to Prop. M.12 and Prop. J.6.(i). □

B.2 Proofs of §3.4 (Linear Synchronous Arrow Games)

Given $\mathbf{AG}_\Sigma$-objects $A = (U, X)$ and $B = (V, Y)$, a strategy $\sigma : (\Sigma \times U, X \times \mathfrak{D}) \multimap (\Sigma \times V, Y \times \mathfrak{D})$ is a $\mathbf{AG}_\Sigma$-map iff $\mathrm{HS}(\sigma)$ is a $\mathbf{Rel}(\mathbf{Set}_{/(\Gamma+\mathfrak{D})^*})$-map from $(\wp^+_A, \mathrm{tr})$ to $(\wp^+_B, \mathrm{tr})$. It therefore follows from Prop. A.2 that

Proposition B.3. *For each alphabet Σ, $\mathbf{AG}_\Sigma$ is a category.*

Since $\mathbf{AG}_\Sigma$-maps are zig-zag, we moreover obtain from Prop. A.5 that

Proposition B.4. *For each alphabet Σ, $\mathbf{AG}^{\mathrm{W}}_\Sigma$ is a category.*

We record here for future use the following property:

Proposition B.5. *For every regular automaton $\mathcal{A} : \Sigma$, there is a parity automaton $\mathcal{A}^\dagger : \Sigma$ such that $\mathcal{A}^\dagger \simeq \mathcal{A}$ in $\mathbf{AG}^{\mathrm{W}}_\Sigma$.*

Proof. Recall (from e.g. [35, 12, 30]) that every regular language L of ω-words can be recognized by a deterministic ω-word parity automaton $(Q_L, q^\imath_L, \delta_L, c_L)$. Following [36], given a regular tree automaton $\mathcal{A} : \Sigma$ as in (1), let

$$\mathcal{A}^\dagger \quad := \quad (Q_\mathcal{A} \times Q_L \,,\, (q^\imath_\mathcal{A}, q^\imath_L) \,,\, U \,,\, X \,,\, \delta_{\mathcal{A}^\dagger} \,,\, \Omega_{\mathcal{A}^\dagger})$$

where $L = \Omega_\mathcal{A}$, $\Omega_{\mathcal{A}^\dagger}$ is generated from c_L *via* second projection, and

$$\delta_{\mathcal{A}^\dagger}((q_\mathcal{A}, q_L), \mathsf{a}, u, x, d) \quad := \quad (q'_\mathcal{A} \,,\, \delta_L(q_L, q'_\mathcal{A}))$$

with $q'_\mathcal{A} := \delta_\mathcal{A}(q_\mathcal{A}, \mathsf{a}, u, x, d)$. Note that $\mathcal{A}$ and $\mathcal{A}^\dagger$ have the same P and O-moves, so that identity strategies provide an isomorphism $\mathcal{A} \simeq \mathcal{A}^\dagger$. □

C Proofs of §4 (Fibrations of Tree Automata)

In this appendix, we gather facts related to §4 and show that we indeed have an indexed category

$$(-)^* \quad : \quad \mathbf{T} \quad \longrightarrow \quad \mathbf{Cat}$$

taking Σ to $\mathsf{DialAut}_\Sigma$ and $L \in \mathbf{T}[\Gamma, \Sigma]$ to $L^* : \mathsf{DialAut}_\Sigma \to \mathsf{DialAut}_\Gamma$.

First, the functoriality of $u^{\mathsf{Cl}} : \mathbf{Kl}(K') \to \mathbf{Kl}(K)$ for K, K' comonoids in $(\mathbb{C}, \otimes, \mathbf{I})$ trivially follows from the bifunctoriality of $\otimes$.

Note that given a distributive law $\Lambda : GT \Rightarrow TG$ (in the sense of [13]), the lifting functor $(-)^\uparrow$ takes $f : GA \to TB$ to

$$f^\uparrow \quad := \quad G(\mu_B \circ Tf \circ \Lambda_A) \circ \delta_{TA} \quad : \quad GTA \quad \longrightarrow \quad GTB$$

The dual of Prop. 2.4, (on the comonad of comonoid indexing) is Prop. A.7. Moreover, given a comonoid K and a monoid M, it follows from Prop. M.16 that the associativity map

$$\Phi_{(-)} \quad := \quad \alpha^{-1}_{K,(-),M} \quad : \quad K \otimes ((-) \otimes M) \quad \Longrightarrow \quad (K \otimes (-)) \otimes M$$

is a distributive law. We therefore have:

Proposition C.1. *The family of associativity maps* $\Phi^{\Sigma}_{A} : \Sigma \otimes (A \otimes \mathfrak{D}) \multimap (\Sigma \otimes A) \otimes \mathfrak{D}$ *forms a distributive law.*

Remark C.2. It is well-known (and stated in Prop. M.5) that the Kleisli category $\mathbf{Kl}(\Phi^{\Sigma})$ of Φ^{Σ} is equivalent to the Kleisli category $\mathsf{DialZ}(\Sigma)$ of the lift to $\mathbf{DZ}_{\mathfrak{D}}$ of the $\mathbf{DZ}$-comonad Σ.

Proposition C.3. *For each alphabet* Σ*,* $\mathsf{DialAut}_{\Sigma}$ *is a category.*

Proof. Since $\mathsf{DialZ}(\Sigma)$ is a category, it remains to check that $\mathsf{DialAut}_{\Sigma}$ has identities and composition. This follows from the fuctoriality of $(-)^{\uparrow} : \mathsf{DialZ}(\Sigma) \to \mathbf{AG}_{\Sigma}$. Indeed,

- ${\mathrm{id}_A}^{\uparrow} = \mathrm{id}^{\mathbf{AG}}_{A}$ is winning since $\mathbf{AG}^{\mathrm{W}}_{\Sigma}$ is a category ;
- given composable $\mathsf{DialAut}_{\Sigma}$-maps τ and σ, since $(\tau \circ \sigma)^{\uparrow} = \tau^{\uparrow} \circ \sigma^{\uparrow}$, we have that $(\tau \circ \sigma)^{\uparrow}$ is winning since $\tau^{\uparrow}$ and $\sigma^{\uparrow}$ are winning. ☐

C.1 Substitution in DialAut

We now check that given $L \in \mathbf{T}[\Gamma, \Sigma] = \mathbf{DZ}_{\mathfrak{D}}[\Gamma, \Sigma]$, the operation taking $\sigma \in \mathsf{DialAut}_{\Sigma}[A, B]$ to

$$L^{*}(\sigma) \quad := \quad \sigma \circ_{\mathbf{DZ}_{\mathfrak{D}}} (L \otimes_{\mathbf{DZ}_{\mathfrak{D}}} \mathrm{id}^{\mathbf{DZ}_{\mathfrak{D}}}_{A})$$

is indeed a functor $\mathsf{DialAut}_{\Sigma} \to \mathsf{DialAut}_{\Gamma}$. It trivially follows from the bifunctoriality of $\otimes_{\mathbf{DZ}_{\mathfrak{D}}}$ that L^{*} is a functor from $\mathsf{DialZ}(\Gamma) \to \mathsf{DialZ}(\Sigma)$. Hence we only have to show that $L^{*}(\sigma)^{\uparrow}$ is winning from $L^{*}(A)$ to $L^{*}(B)$ as soon as $\sigma^{\uparrow}$ is winning from A to B.

We first discuss the action of $(-)^{\uparrow} : \mathsf{DialZ}(\Sigma) \to \mathbf{AG}_{\Sigma}$. Consider $A = (U, X)$, $B = (V, Y)$ and $\sigma \in \mathsf{DialZ}(\Sigma)[A, B]$, as in:

Σ	A	$\overset{\sigma}{\multimap}$	B	
O	(a, u)			
			v	P
			(y, d)	O
P	x			

Modulo associatiy, the map $\sigma^{\uparrow} \in \mathbf{AG}_{\Sigma}[A, B]$ is given by

$$(\mathrm{id}_{\Sigma} \otimes ((\mathrm{id}_B \otimes m_{\mathfrak{D}}) \circ (\sigma \otimes \mathrm{id}_{\mathfrak{D}}))) \circ (d_{\Sigma} \otimes \mathrm{id}_{A \otimes \mathfrak{D}}) \quad : \quad \Sigma \otimes A \otimes \mathfrak{D} \quad \multimap \quad \Sigma \otimes B \otimes \mathfrak{D}$$

Note that

$$\tilde{\sigma} \quad := \quad (\mathrm{id}_B \otimes m_{\mathfrak{D}}) \circ (\sigma \otimes \mathrm{id}_{\mathfrak{D}}) \quad : \quad \Sigma \otimes A \otimes \mathfrak{D} \quad \longrightarrow \quad B \otimes \mathfrak{D}$$

plays as follows:

	$\Sigma\otimes A\otimes\mathfrak{D}$	$\xrightarrow{\sigma\otimes\mathrm{id}_{\mathfrak{D}}}$	$B\otimes\mathfrak{D}\otimes\mathfrak{D}$	$\xrightarrow{\mathrm{id}_B\otimes m_{\mathfrak{D}}}$	$B\otimes\mathfrak{D}$	
O	(a,u)					
			v			
					v	P
					(y,d)	O
			(y,d,d)			
P	(x,d)					

It follows that $\sigma^{\uparrow}=(\mathrm{id}_\Sigma\otimes\tilde{\sigma})\circ(d_\Sigma\otimes\mathrm{id}_{A\otimes\mathfrak{D}})$ plays as follows:

	$\Sigma\otimes A\otimes\mathfrak{D}$	$\xrightarrow{d_\Sigma\otimes\mathrm{id}_{A\otimes\mathfrak{D}}}$	$\Sigma\otimes\Sigma\otimes A\otimes\mathfrak{D}$	$\xrightarrow{\mathrm{id}_\Sigma\otimes\tilde{\sigma}}$	$\Sigma\otimes B\otimes\mathfrak{D}$	
O	(a,u)					
			$(\mathsf{a},\mathsf{a},u)$			
					(a,v)	P
					(y,d)	O
			(x,d)			
P	(x,d)					

That is

Σ	A	$\xrightarrow{\sigma^{\uparrow}}$	B	
O	(a,u)			
			(a,v)	P
			(y,d)	O
P	(x,d)			

On the other hand, note that $L^*(\sigma)$ plays as follows:

	$\Gamma\otimes A$	$\xrightarrow{L\otimes_{\mathbf{DZ}_{\mathfrak{D}}}\mathrm{id}_A^{\mathbf{DZ}_{\mathfrak{D}}}}$	$\Sigma\otimes A\otimes\mathfrak{D}$	$\xrightarrow{(\mathrm{id}_B\otimes m)\circ(\sigma\otimes\mathrm{id}_{\mathfrak{D}})}$	$B\otimes\mathfrak{D}$	
O	(b,u)					
			(a,u)			
					v	P
					(y,d)	O
			(x,d)			
P	x					

so that $L^*(\sigma)^{\uparrow}$ plays as:

Σ	A	$\xrightarrow{L^*(\sigma)^{\uparrow}}$	B	
O	(b,u)			
			(b,v)	P
			(y,d)	O
P	(x,d)			

Proposition C.4. *If $\sigma^{\uparrow}$ is winning then $L^*(\sigma)^{\uparrow}$ is winning.*

Proof. Note first for an arbitrary total zig-zag strategy $\tau : A \multimap B$, every infinite play ϖ such that $\exists^\infty k.\varpi(0).\cdots.\varpi(k) \in \tau$ is uniquely determined by $\varpi_{\restriction A}$ and $\varpi_{\restriction B}$. In the following, we write $\varpi = (\varpi_{\restriction A}, \varpi_{\restriction B})$.

Consider now a strategy $\sigma \in \mathsf{DialAut}_{\Sigma}[A, B]$, and an infinite play ϖ such that $\exists^\infty k.\varpi(0).\cdots.\varpi(k) \in L^*(\sigma)^{\uparrow}$. Let

$$\varpi \quad = \quad (((\mathtt{b}_k, u_k) \cdot (x_k, d_k))_k \ , \ ((\mathtt{b}_k, v_k) \cdot (y_k, d_k))_k)$$

By definition of the action of L^* on $\mathsf{DialAut}_{\Sigma}$-objects, we have

$$\begin{array}{rl} & ((L(\mathtt{b}_0.\cdots.\mathtt{b}_k,\, d_0.\cdots.d_{k-1}), u_k) \cdot (x_k, d_k))_k \in \mathcal{W}_A \\ \text{and} & ((L(\mathtt{b}_0.\cdots.\mathtt{b}_k,\, d_0.\cdots.d_{k-1}), v_k) \cdot (y_k, d_k))_k \in \mathcal{W}_B \end{array}$$

Let

$$\varpi' \quad := \quad (((L(\mathtt{b}_0.\cdots.\mathtt{b}_k,\, d_0.\cdots.d_{k-1}), u_k)\cdot(x_k, d_k))_k \ , \ ((L(\mathtt{b}_0.\cdots.\mathtt{b}_k,\, d_0.\cdots.d_{k-1}), v_k)\cdot(y_k, d_k))_k)$$

We thus have $\varpi \in \mathcal{W}_{L^*(A) \multimap L^*(B)}$ if and only if $\varpi' \in \mathcal{W}_{A \multimap B}$

Now we are done since on the other hand, reasonning as in App. 7.2 & 8.2 of the long version of [32][3], it is easy to see that for all $k \in \mathbb{N}$ we have

$$\varpi'(0).\cdots.\varpi'(k) \in \sigma^{\uparrow} \qquad \text{iff} \qquad \varpi(0).\cdots.\varpi(k) \in L^*(\sigma)^{\uparrow}$$

so that $\exists^\infty k.\varpi'(0).\cdots.\varpi'(k) \in \sigma^{\uparrow}$. ☐

Proposition C.5 (Prop. 4.1). *Given $\mathcal{A} : \Sigma$ and $\mathcal{B} : \Sigma$, if there is a winning P-strategy σ in $\mathcal{A} \multimap \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.*

Proof. Let $T : \mathfrak{D}^* \to \Sigma$. Since σ is a $\mathsf{DialAut}_{\Sigma}$-map, it follows from Prop. C.4 that $T^*(\sigma)$ is a $\mathsf{DialAut}_{\mathbf{1}}$-map from $\mathcal{A}(T)$ to $\mathcal{B}(T)$.

Now, if $T \in \mathcal{L}(\mathcal{A})$, then there is a $\mathsf{DialAut}_{\mathbf{1}}$-strategy τ in $\mathbf{I_1} \multimap \mathcal{A}(T)$. It follows from Prop. C.3 that $T^*(\sigma) \circ \tau$ is winning on $\mathbf{I_1} \multimap \mathcal{B}(T)$, hence that $T \in \mathcal{L}(\mathcal{B})$. ☐

D Proofs of §5.1 (Monoidal Closed Structure)

D.1 The Monoidal Closed Structure of DZ

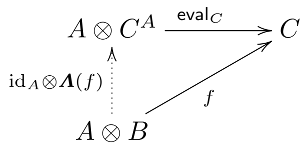
Proposition D.1 (Prop. 5.1). *The category* **DZ** *is symmetric monoidal closed.*

Recall from e.g. [27] that a symmetric monoidal category $\mathbb{C}$ is *closed* if for every object A, the functor $A \otimes (-)$ has a right adjoint $(-)^A$. Since $A \otimes (-)$ is already a functor, according to [24, Thm. IV.1.2] it is sufficient to show that for every object C there is an object C^A and map

$$\mathsf{eval}_C \quad : \quad A \otimes C^A \quad \longrightarrow \quad C$$

[3] Available at https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf.

such that for every $f : A \otimes B \to C$ there is a unique $\boldsymbol{\Lambda}(f) : B \to C^A$ such that



Proof (of Prop. D.1). We rely on Prop. A.1. Let $A = (U, X)$, $B = (V, Y)$ and $C = (W, Z)$. Recall that $A \multimap_{\mathbf{DZ}} C = (W^U \times X^{U \times Z}, U \times Z)$. We define the total zig-zag strategy $\mathsf{eval}_C : A \otimes (A \multimap_{\mathbf{DZ}} C) \multimap C$ as follows:

	$A \otimes (A \multimap_{\mathbf{DZ}} C)$	$\overset{\mathsf{eval}_C}{\multimap}$	C	
O	$(u, (f, F))$			
			$f(u)$	P
			z	O
P	$(F(u, z), (u, z))$			

Given any $\tau' : B \multimap (A \multimap_{\mathbf{DZ}} C)$, the composition $\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau')$ is given by:

	$A \otimes B$	$\overset{\mathrm{id}_A \otimes \tau'}{\multimap}$	$A \otimes (A \multimap_{\mathbf{DZ}} C)$	$\overset{\mathsf{eval}_C}{\multimap}$	C	
O	(u, v)					
			$(u, (f', F'))$			
					$f'(u)$	P
					z	O
			$(F'(u, z), (u, z))$			
P	$(F'(u, z), y')$					

It follows that $\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau') = \mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau'')$ implies $\tau' = \tau''$.

We show this by induction on pairs of even-length plays $(s, t) \in \wp_A^{\mathsf{even}} \times \wp_{A \multimap_{\mathbf{DZ}} C}^{\mathsf{even}}$. Assume toward a contradiction that for some such $(s, t) \in \mathrm{HS}(\tau') \cap \mathrm{HS}(\tau'')$, for some $v \in V$ we have $(s.v, t.(f', F')) \in \mathrm{HS}(\tau')$ and $(s.v, t.(f'', F'')) \in \mathrm{HS}(\tau'')$ with $f' \neq f''$. Then for some $u \in U$, we have say $f'(u) \neq f''(u)$. Then, for some r we have

$$\mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau') \quad \ni \quad r.(u, v).f'(u) \quad \neq \quad r.(u, v).f''(u) \quad \in \quad \mathsf{eval}_C \circ (\mathrm{id}_A \otimes \tau'')$$

Hence a contradiction. The case of $F' \neq F''$ is dealt-with similarly.

Fix now some total zig-zag $\sigma : A \otimes B \multimap C$.

We define $\tau = \boldsymbol{\Lambda}(\sigma) : B \multimap (A \multimap_{\mathbf{DZ}} C)$ by induction on plays. To each $(s, t) \in \mathrm{HS}(\tau)$, with s and t even-length, we associate $(s', t') \in \mathrm{HS}(\sigma)$, with s' and t' of the same length, and such that, for $(\overline{v}, \overline{y}) = \partial(s)$ and $((\overline{f}, \overline{F}), (\overline{u}, \overline{z})) = \partial(t)$, we have $\partial(s') = ((\overline{u}, \overline{v}), (\overline{F}(\overline{u}, \overline{z}), \overline{y}))$ and $\partial(t') = (\overline{f}(\overline{u}), \overline{z})$, where we take the pointwise application of sequences of functions and the map ∂ is defined in App. 2.

For the base case, we put $(\varepsilon, \varepsilon) \in \mathrm{HS}(\tau)$, and associate it to $(\varepsilon, \varepsilon) \in \mathrm{HS}(\sigma)$.

Assume now $(s,t) \in \mathrm{HS}(\tau)$, associated to $(s',t') \in \mathrm{HS}(\sigma)$. For each $v \in V$, we define the functions $f_v : U \to W$ and $F_v : U \times Z \to X$ as follows: given $u \in U$, let w such that $(s'.(u,v), t'.w) \in \mathrm{HS}(\sigma)$, and for each $z \in Z$, let x and $y_{u,z}$ such that $(s'.(u,v).(x, y_{u,z}), t'.w.z) \in \mathrm{HS}(\sigma)$. We then let $f_v(u) := w$ and $F_v(u,z) := x$. We now let $(s.v.y_{u,z}, t.(f_v, F_v).(u,z)) \in \mathrm{HS}(\tau)$, and associate it to $(s'.(u,v).(x, y_{u,z}), t'.w.z) = (s'.(u,v).(F_v(u,z), y_{u,z}), t'.f_v(u).z)$ so that the invariant is satisfied.

This conclude the definition of τ.

It then follows from the invariant that we indeed have $\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau = \sigma$.

First note that the map $(s,t) \in \mathrm{HS}(\tau) \mapsto (s',t') \in \mathrm{HS}(\sigma)$ is surjective. The property then follows from the fact that $(s,t) \in \mathrm{HS}(\tau)$ iff $(s',t') \in \mathrm{HS}(\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau)$. This is shown by induction on pairs of plays $(s,t) \in \wp^{\mathsf{even}}_B \times \wp^{\mathsf{even}}_{A \multimap_{\mathbf{DZ}} C}$. The base case is trivial. For the induction step, given such $(s.v.y_{u,z}, t.(f_v, F_v).(u,z))$, we have $(s.v.y_{u,z}, t.(f_v, F_v).(u,z)) \in \mathrm{HS}(\tau)$ if and only if

$$(s'.(u,v).(F_v(u,z), y_{u,z}), t'.f)v(u).z) \in \mathrm{HS}(\mathsf{eval}_C \circ \mathrm{id}_A \otimes \tau)$$

This concludes the proof of Prop. D.1. ☐

D.2 Symmetric Monoidal Closed Structure in $\mathbf{DZ}_{\mathfrak{D}}$, $\mathsf{DialZ}(\Sigma)$ and $\mathsf{DialAut}$

Proposition D.2 (Prop. 5.2). $\mathbf{DZ}_{\mathfrak{D}}$ *and* $\mathsf{DialZ}(\Sigma)$ *are symmetric monoidal closed.*

Proof. The symmetrict monoidal structure follows from Prop. A.7. The *closed* structure is presented in §5.1. ☐

Proposition D.3 (Prop. 5.3). *The fibration* $\mathsf{DialAut}$ *is fibrewise monoidal closed.*

Proof. We first show that substitution functors of DialZ are strong symmetric monoidal. Using Prop. B.1, this follows from a general fact on indexed categories $\mathsf{Cl}(\mathbb{C}) : \mathbf{Comon}(\mathbb{C})^{\mathrm{op}} \to \mathbf{Cat}$, for a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$. Given a commonoid morphism $u : K \to L$ in $\mathbf{Comon}(\mathbb{C})$, the substitution functor u^* is the identity on objects, so the strength is made of identities. It remains to show that the required diagrams commute (see §M.1), which amounts to

$$u^*(\alpha^{\mathbf{Kl}(L)}) = \alpha^{\mathbf{Kl}(K)} \qquad u^*(\rho^{\mathbf{Kl}(L)}) = \rho^{\mathbf{Kl}(K)}$$
$$u^*(\lambda^{\mathbf{Kl}(L)}) = \lambda^{\mathbf{Kl}(K)} \qquad u^*(\gamma^{\mathbf{Kl}(L)}) = \gamma^{\mathbf{Kl}(K)}$$

where $\alpha^{\mathbf{Kl}(-)}$, $\rho^{\mathbf{Kl}(-)}$, $\lambda^{\mathbf{Kl}(-)}$ and $\gamma^{\mathbf{Kl}(-)}$ are the symmetric monoidal structure maps of $\mathbf{Kl}(-)$. But by §M.3 each of these maps $f^{\mathbf{Kl}(-)}$ is $f \circ \lambda \circ (e \otimes \mathrm{id})$ (where f is the corresponding map of $\mathbb{C}$), so that in $\mathbb{C}$:

$$u^*(f^{\mathbf{Kl}(L)}) \quad = \quad f \circ \lambda \circ (e \otimes \mathrm{id}) \circ (u \otimes \mathrm{id})$$

and we are done since $e \circ u = e$ as u is a comonoid morphism (see §M.3).

The argument is the same for the fibrewise symmetric monoidal *closed* structure of DialZ, since the closed structure of $\mathsf{DialZ}(\Sigma)$ is directly lifted by the comonad Σ from the closed structure of $\mathbf{DZ}_{\mathfrak{D}}$.

The result then follows from the fact that the operations $\mathcal{W}_A \sqcap \mathcal{W}_B$ and $\mathcal{W}_A \sqsupset \mathcal{W}_B$ are preserved by substitution, and from the fact that all the symmetric monoidal structure maps as well as the evaluation map are (total) winning, and that the Currying map $\boldsymbol{\Lambda}(-)$ preserves (total) winning strategies. □

Proposition D.4 (Prop. 5.6). $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.

Proof. The inclusion ($\subseteq$) follows using the projections $\mathcal{A}\otimes\mathcal{B} \to \mathcal{A}$ and $\mathcal{A}\otimes\mathcal{B} \to \mathcal{B}$.

For the other direction, using Prop. D.3, tensor σ winning on $\mathbf{I_1} \multimap \mathcal{A}(T)$ with τ winning on $\mathbf{I_1} \multimap \mathcal{B}(T)$ and then precompose with a monoidal unit map. □

D.3 Falsity and Complementation

Recall that $\perp : \Sigma$ is $(\mathbb{B}, \mathbb{f}, \mathfrak{D}, \mathbf{1}, \delta_\perp, \Omega_\perp)$ where $\Omega_\perp := \mathbb{B}^*.\mathbb{t}^\omega$ and $\delta_\perp(\mathbb{t}, \_, d', \bullet, d) := \mathbb{t}$ and $\delta_\perp(\mathbb{f}, \_, d, \bullet, d) := \mathbb{f}$ and $\delta_\perp(\mathbb{f}, \_, d', \bullet, d) := \mathbb{t}$ if $d' \neq d$.

Let $\mathcal{A} : \Sigma$ as in (1). Note that $\mathcal{A}^\perp : \Sigma$ can be described as

$$(Q_\mathcal{A} \times \mathbb{B}, (q^\imath_\mathcal{A}, \mathbb{f}), \mathfrak{D}^U \times X^{U\times\mathfrak{D}}, U, \delta_{\mathcal{A}^\perp}, \Omega_{\mathcal{A}^\perp})$$

with $\delta_{\mathcal{A}^\perp}(a, (q_\mathcal{A}, \mathbb{f}), (f, F), u, d) = (q'_\mathcal{A}, \mathbb{b})$ where $\mathbb{b} = \mathbb{f}$ iff $f(u) = d$, and

$$\delta_{\mathcal{A}^\perp}(a, (q_\mathcal{A}, \mathbb{t}), (f, F), u, d) = (q'_\mathcal{A}, \mathbb{t})$$

where $q'_\mathcal{A} := \delta_\mathcal{A}(a, q_\mathcal{A}, u, F(u, d), d)$. Hence O looses as soon as he does not follow the direction proposed by P *via* f. Moreover, since $\perp$ has no transition from $\mathbb{t}$ to $\mathbb{f}$, and since $\Omega_\perp = \mathbb{B}^*.\mathbb{t}^\omega$, it follows that $\mathcal{A}^\perp$ is equivalent to an automaton with state set $Q_\mathcal{A} + \{\mathbb{t}\}$. (which is linear in $Q_\mathcal{A}$).

Proposition D.5 (Prop. 5.7). *If $\Omega_\mathcal{A}$ is Borel, then $T \in \mathcal{L}(\mathcal{A}^\perp)$ iff $T \notin \mathcal{L}(\mathcal{A})$.*

The proof uses the notion of O-strategy. In order to properly define winning O-strategies in our context, it is convenient to define an O-strategy on A as a P-stratey in the *dual* $\overline{A}$ of A. The *dual* of A is the game $\overline{A} := (A^-, A^+, -\xi_A, L_A)$. Note that $\overline{A}^\xi = A^{-\xi}$, so that $\overline{A}^{\xi_{\overline{A}}} = \overline{A}^{-\xi_A} = A^{\xi_A}$ and

$$\begin{array}{rcl}
\wp_A^{\xi_A} & = & \wp_A^{-\xi_{\overline{A}}} \\
& = & (A^{-\xi_{\overline{A}}} \cdot A^{\xi_{\overline{A}}})^* + (A^{-\xi_{\overline{A}}} \cdot A^{\xi_{\overline{A}}})^* \cdot A^{-\xi_{\overline{A}}} \\
& = & (\overline{A}^{\xi_{\overline{A}}} \cdot \overline{A}^{-\xi_{\overline{A}}})^* + (\overline{A}^{\xi_{\overline{A}}} \cdot \overline{A}^{-\xi_{\overline{A}}})^* \cdot \overline{A}^{\xi_{\overline{A}}} \\
& = & \wp_{\overline{A}}^{\xi_{\overline{A}}}
\end{array}$$

and we indeed have $L_A \subseteq \wp_{\overline{A}}^{\xi_{\overline{A}}}$. If A is a game with winning, then we let $\mathcal{W}_{\overline{A}} := (A^{\xi_A} \cdot A^{-\xi_A}) \setminus \mathcal{W}_A$.

Proof. The argument is an adaptation of the one given in [36]. By Martin's Theorem [26], it is equivalent to show that P wins the game $\mathcal{A}^{\perp}(T)$ iff O wins $\mathcal{A}(T)$, where, using the notions of §2, an O-strategy is just a P-strategy on the dual game.

For ($\Rightarrow$), assuming given a winning P-strat σ on $\mathcal{A}(T) \multimap \perp$, we build a winning O-strat τ in $\mathcal{A}(T)$. The strategy τ is build by induction on plays. To each play t of τ, we associate a play s of σ such that if t leads to state $q_{\mathcal{A}}$, then s leads to state $(q_{\mathcal{A}}, \mathbb{f})$. In the base case, both t and s are the empty plays, and the invariant is respected. For the induction step, assume that P plays u from t in $\mathcal{A}(T)$. Let (f, F) be the move of σ from s. We then let τ answer the pair $(F(u, f(u)), f(u))$ from $s.u$, and $\mathcal{A}$ goes to state $q'_{\mathcal{A}}$. In $\mathcal{A}(T) \multimap \perp$, we let O play the pair $(f(u), u)$. Then $\mathcal{A} \multimap \perp$ goes to state $(q'_{\mathcal{A}}, \mathbb{f})$ and the invariant is respected. Since σ is winning and $\mathcal{A} \multimap \perp$ stays in states of the form $(\_, \mathbb{f})$ the infinite sequence of states produced in $\mathcal{A}(T)$ is rejecting, as required.

For the conversion direction, assuming given a winning O-strat τ on $\mathcal{A}(T)$, we build a winning P-strat σ in $\mathcal{A}(T) \multimap \perp$. The strategy σ is build by induction on plays as long as $\mathcal{A} \multimap \perp$ stays in states of the form $(\_, \mathbb{f})$ (if it switches to $(\_, \mathbb{t})$ then P trivially wins). So to each play s of σ which leads to state $(q_{\mathcal{A}}, \mathbb{f})$, we associate a play t of τ which leads to state $q_{\mathcal{A}}$. The base case is trivial. For the induction step, we build (f, F) from σ as follows: to each u, σ associates (from t) a pair (x, d). We let $F(u, \_) := d$ and $f(u) := x$. Assume then that from $s.(f, F)$, O plays some (u, d). If $d \neq f(u)$ then we are done. Otherwise, $\mathcal{A} \multimap \perp$ switches to $(q'_{\mathcal{A}}, \mathbb{f})$. We then let P play u from t, so that by construction τ answers $(F(u, \_), d)$, and $\mathcal{A}$ goes to state $q'_{\mathcal{A}}$. But then, since τ is winning for O, the sequence of $\mathcal{A}$-states is rejecting, so that P wins in $\mathcal{A}(T) \multimap \perp$, as required. ◻

E Proofs of §5.2 (Quantifications)

Recall that quantifications in DialAut are given by

$$\coprod_{\Sigma,\Gamma}(U, X, \mathcal{W}_A) = (\Gamma \times U, X, \mathcal{W}_A) \quad \text{and} \quad \prod_{\Sigma,\Gamma}(U, X, \mathcal{W}_A) = (U^{\Gamma}, \Gamma \times X, \prod_{\Sigma,\Gamma} \mathcal{W}_A)$$

where $((\mathsf{a}_k, f_k) \cdot (\mathsf{b}_k, x_k, d_k))_k \in \prod_{\Sigma,\Gamma} \mathcal{W}_A$ iff $((\mathsf{a}_k, \mathsf{b}_k, f_k(\mathsf{b}_k)) \cdot (x_k, d_k))_k \in \mathcal{W}_A$.

Proposition E.1 (Prop. 5.8). DialAut *has existential and universal quantifications.*

Proof. We check the Beck-Chevalley conditions. Given a $\mathsf{DialAut}_{\Sigma\times\Gamma}$-object A and $L \in \mathbf{T}[\Delta, \Sigma]$, we have to show

$$L^*(\coprod_{\Sigma,\Gamma} A) = \coprod_{\Delta,\Gamma} (L \times \mathrm{Id}_{\Gamma})^*(A) \qquad \text{and} \qquad L^*(\prod_{\Sigma,\Gamma} A) = \prod_{\Delta,\Gamma} (L \times \mathrm{Id}_{\Gamma})^*(A)$$

– For existential quantifications, this follows from the fact that

$$((\mathsf{c}_k, \mathsf{b}_k, u_k) \cdot (x_k, d_k))_k \in \mathcal{W}_{L^*(\coprod A)}$$

iff

$$((L(\mathsf{c}_0.\cdots.\mathsf{c}_k, d_0.\cdots.d_{k-1}), \mathsf{b}_k, u_k)\cdot(x_k,d_k))_k \in \mathcal{W}_A$$

iff

$$((\mathsf{c}_k, \mathsf{b}_k, u_k)\cdot(x_k,d_k))_k \in \mathcal{W}_{\coprod(L\times\mathrm{Id})^*(A)}$$

– For universal quantification, we have

$$((\mathsf{c}_k, f_k)\cdot(\mathsf{b}_k, x_k, d_k))_k \in \mathcal{W}_{L^*(\prod A)}$$

iff

$$((L(\mathsf{c}_0.\cdots.\mathsf{c}_k, d_0.\cdots.d_{k-1}), \mathsf{b}_k, f_k(\mathsf{b}_k))\cdot(x_k,d_k))_k \in \mathcal{W}_A$$

iff

$$((\mathsf{c}_k, \mathsf{b}_k, f_k(\mathsf{b}_k))\cdot(x_k,d_k))_k \in \mathcal{W}_{(L\times\mathrm{Id})^*(A)}$$

iff

$$((\mathsf{c}_k, f_k)\cdot(\mathsf{b}_k, x_k, d_k))_k \in \mathcal{W}_{\prod(L\times\mathrm{Id})^*(A)}$$

□

Proposition E.2 (Prop. 5.10). *If* $\mathcal{A} : \Sigma\times\Gamma$ *is* non-deterministic *then* $\mathcal{L}(\exists_\Gamma\mathcal{A}) = \mathsf{p}_\Sigma(\mathcal{L}(\mathcal{A}))$.

Proof. Let $T : \mathfrak{D}^* \to \Sigma$. A winning P-strategy in $\mathbf{1} \vdash \mathbf{I} \multimap (\exists_\Gamma\mathcal{A})(T)$ is given by a map

$$\prod_{n\in\mathbb{N}}(\mathfrak{D}^n \longrightarrow \Gamma\times U)$$

hence by a pair of maps

$$\left[\prod_{n\in\mathbb{N}}(\mathfrak{D}^n \to \Gamma)\right] \times \left[\prod_{n\in\mathbb{N}}(\mathfrak{D}^n \to U)\right]$$

that is by a tree $T' : \mathfrak{D}^* \to \Gamma$ and a winning P-strategy in $\mathbf{1} \vdash \mathbf{I} \multimap \mathcal{A}\langle T, T'\rangle$. □

The following proposition contains an effective strengthening of Ex. 5.11.(iv). It asserts the existence of a winning P-strategy on $\Sigma \vdash \mathcal{A}\otimes\mathcal{B} \multimap \bot$ as soon as $\mathcal{A},\mathcal{B} : \Sigma$ are non-deterministic regular such that $\mathcal{L}(\mathcal{A})\cap\mathcal{L}(\mathcal{B}) = \emptyset$.

Proposition E.3 (Ex. 5.11.(iv)). *Given non-deterministic Borel* $\mathcal{A},\mathcal{B} : \Sigma$, *if* $\mathcal{L}(\mathcal{A})\cap\mathcal{L}(\mathcal{B}) = \emptyset$, *then there are winning* P*-strategies in* $\mathcal{A}\otimes\mathcal{B} \multimap \bot$ *and* $\mathcal{A} \multimap \mathcal{B}^\bot$. *If moreover* $\mathcal{A}$ *and* $\mathcal{B}$ *are regular then the* P*-strategies can be assumed to be regular.*

The effectiveness part of the statment can be seen to follow from Ex. 5.11.(iii). It is nevertheless interesting to note how the strategy can be effectivelly computed in this particular case.

Proof. Since $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$, we have $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \emptyset$ by Prop. D.4. Since $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic, so is $\mathcal{A} \otimes \mathcal{B}$. It then follows from Prop. E.2 that $\mathcal{L}(\exists_\Sigma(\mathcal{A} \otimes \mathcal{B})) = \emptyset$, hence, by Prop. D.5 that the automaton $(\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}))^{\perp} : \mathbf{1}$ accepts the unique tree $\mathbf{1} : \mathfrak{D}^* \to \mathbf{1}$. But winning P-strategies in $(\exists_\Sigma(\mathcal{A}\otimes\mathcal{B}))^{\perp}(\mathbf{1})$ can be lifted to winning P-strategies in

$$\mathbf{I_1} \quad \multimap \quad (\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}))^{\perp}(\mathbf{1})$$

But note that since $(\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}))^{\perp} : \mathbf{1}$, that game is actually the same as

$$\mathbf{I_1} \quad \multimap \quad (\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}))^{\perp}$$

It then follows from Prop. D.3 that there is a winning P-strategy in the game

$$\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}) \quad \multimap \quad \perp$$

and therefore by Prop. E.1 (in the form of Ex. 5.11.(ii)) that there is a winning P-strategy on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ and therefore also in $\mathcal{A} \multimap \mathcal{B}^{\perp}$.

If the automata $\mathcal{A}$ and $\mathcal{B}$ are regular, then the automaton $(\exists_\Sigma(\mathcal{A} \otimes \mathcal{B}))^{\perp}$ is regular. It is therefore effectively equivalent to a parity automaton by Prop. B.5. It is then well-known (see e.g. [35, Thm. 6.18]) that there is effectivelly a regular winning P-strategy in the acceptance game $(\exists_\Sigma(\mathcal{A}\otimes\mathcal{B}))^{\perp}(\mathbf{1})$. It is easy to see that this strategy is lifted (as above) to regular winning P-strategies in $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ and $\mathcal{A} \multimap \mathcal{B}^{\perp}$. □

F Proofs of §5.3 (A Deduction System for Automata)

Proposition F.1 (Prop. 5.12). *Let* $\overline{\mathcal{A}} = \mathcal{A}_1, \ldots, \mathcal{A}_n : \Sigma$ *and* $\mathcal{A} : \Sigma$ *and* $M \in \mathbf{T}[\Gamma, \Sigma]$. *If* $M \,;\, \overline{\mathcal{A}} \vdash \mathcal{A}$ *is derivable using the rules of Fig. 2, then there is a winning* P*-strategy in* $\Gamma \vdash \mathcal{A}_1(M) \otimes \ldots \otimes \mathcal{A}_n(M) \multimap \mathcal{A}(M)$.

Proof. The proof is as usual by induction on the derivations and by cases on the last applied rules.

- The *Propositional rules* of Fig. 2 follow from the facts that $\mathsf{DialAut}_\Sigma$ are categories (Prop. C.3), moreover equipped with a symmetric monoidal closed structure (Prop. D.3) and from Ex.5.5.(ii).
- The *Substitution rules* of Fig. 2 follow from the facts that $\mathsf{DialAut}$ is fibred over $\mathbf{T}$ (§C.1 and in part. Prop. C.4), and from the internalization of $\mathbf{A}$-maps in automata (§3.3).
- The *Quantifier rules* of Fig. 2 follow from the adjunctions $\coprod \dashv \mathsf{p}^* \dashv \prod$ (§5.2) and from the Beck-Chevalley conditions (Prop. E.1), and Ex. 5.11.(ii). □

G Proofs of §5.4 (Non-Deterministic Automata)

G.1 Game Graphs and Positionality

Fix $\mathcal{A}(M) : \Sigma$ and $\mathcal{B}(N) : \Sigma$. The *game graph* of $\Sigma \vdash \mathcal{A}(M) \multimap \mathcal{B}(N)$ is the graph G with vertices:

$$(A_\mathsf{P} \times B_\mathsf{P}) + (A_\mathsf{O} \times B_\mathsf{P}) + (A_\mathsf{O} \times B_\mathsf{O})$$

$$
\begin{array}{c|ccc|c}
 & ((p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{A}}) & , & (p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{B}})) & A_{\mathsf{P}} \times B_{\mathsf{P}} \\
\mathsf{O} & \downarrow & & & \\
 & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p\,,\,\bar{\mathsf{a}}\,,\,q_{\mathcal{B}})) & A_{\mathsf{O}} \times B_{\mathsf{P}} \\
\mathsf{P} & & & \downarrow & \\
 & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{B}}\,,\,v)) & A_{\mathsf{O}} \times B_{\mathsf{O}} \\
\mathsf{O} & & & \downarrow & \\
 & ((p\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q_{\mathcal{A}}\,,\,u) & , & (p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{B}})) & A_{\mathsf{O}} \times B_{\mathsf{P}} \\
\mathsf{P} & \downarrow & & & \\
 & ((p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{A}}) & , & (p.d\,,\,\bar{\mathsf{a}}.\mathsf{a}\,,\,q'_{\mathcal{B}})) & A_{\mathsf{P}} \times B_{\mathsf{P}}
\end{array}
$$

Fig. 3. Edges of the graph G

where

$$
\begin{array}{lll}
A_{\mathsf{P}} := \mathfrak{D}^* \times \Sigma^* \times Q_{\mathcal{A}} & & A_{\mathsf{O}} := \mathfrak{D}^* \times \Sigma^* \times Q_{\mathcal{A}} \times U \\
B_{\mathsf{P}} := \mathfrak{D}^* \times \Sigma^* \times Q_{\mathcal{B}} & & B_{\mathsf{O}} := \mathfrak{D}^* \times \Sigma^* \times Q_{\mathcal{B}} \times V
\end{array}
$$

and edges as in Fig. 3, with $q'_{\mathcal{A}} := \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(\bar{\mathsf{a}}.\mathsf{a}, p), u, x, d)$ (for some $x \in X$) and $q'_{\mathcal{B}} := \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(\bar{\mathsf{a}}.\mathsf{a}, p), v, y, d)$ (for some $y \in Y$). Write pos for the graph morphism from the set of plays of $\Sigma \vdash \mathcal{A}(M) \multimap \mathcal{B}(N)$ (seen as a tree) to G. We say that a strategy σ is *positional* if it agrees on plays with the same position, *i.e.* if $s.m \in \sigma$, $t.m' \in \sigma$ with $\mathsf{pos}(s) = \mathsf{pos}(t)$ implies $m = m'$.

Consider now parity automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{B}$. Then the winning condition of a game of the form $\mathcal{A}_1(M_1) \otimes \ldots \otimes \mathcal{A}_n(M_n) \multimap \mathcal{B}(N)$ is a disjunction of parity conditions, also called a *Rabin* condition, which is induced by colorings depending only on the vertices of its game graph G. It has been shown in [21, 22, 20, 37] that if P has a winning strategy σ in such a game, then he has a winning *positional* strategy (w.r.t. G), which according to [37] is recursive in σ.

G.2 Non-Determinization (or Simulation [28, 9, 29])

Our exponential construction $!(-)$ is an adapatation of the one used in the proof of Thm. 5.15 by [36]. Given a parity automaton $\mathcal{A} : \Sigma$, we let

$$
!\mathcal{A} \quad := \quad (Q_{!\mathcal{A}}\,,\, q^{\imath}_{!\mathcal{A}}\,,\, U^{Q_{\mathcal{A}}}\,,\, \mathbf{1}\,,\, \delta_{!\mathcal{A}}\,,\, \Omega_{!\mathcal{A}})
$$

where $Q_{!\mathcal{A}} := \mathcal{P}(Q_{\mathcal{A}} \times Q_{\mathcal{A}})$, $q^{\imath}_{!\mathcal{A}} := \{(q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{A}})\}$ and $\delta_{!\mathcal{A}}$ is defined as follows: Given $\mathsf{a} \in \Sigma$, $f \in U^{Q_{\mathcal{A}}}$, $d \in \mathfrak{D}$ and $S = \{(\_, q_1), \ldots, (\_, q_n)\} \in Q_{!\mathcal{A}}$, let

$$
\delta_{!\mathcal{A}}(S, \mathsf{a}, f, \bullet, d) \quad := \quad T_1 \cup \cdots \cup T_n
$$

where, for each $k \in \{1, \ldots, n\}$,

$$
T_k \quad := \quad \{(q_k, q) \mid \exists x \in X.\ q = \delta_{\mathcal{A}}(q_k, \mathsf{a}, f(q_k), x, d)\}
$$

Let a *trace* in an infinite sequence $(S_n)_n \in Q^{\omega}_{!\mathcal{A}}$ be a sequence $(q_n)_n$ such that for all n, $(q_n, q_{n+1}) \in S_{n+1}$. We let $\Omega_{!\mathcal{A}}$ be the set of sequences $(S_n)_n$ whose traces all belong to $\Omega_{\mathcal{A}}$. Note that $\Omega_{!\mathcal{A}}$ is ω-regular since $\Omega_{\mathcal{A}}$ is ω-regular.

Remark G.1 (Partiality). Note that $Q_{!\mathcal{A}} = \mathcal{P}(Q \times Q)$ contains a "true" state $\emptyset \in Q_{!\mathcal{A}}$, so the map

$$\tilde{\delta}_{!\mathcal{A}} \quad : \quad Q_{!\mathcal{A}} \times \Sigma \quad \longrightarrow \quad U^Q \quad \longrightarrow \quad (\mathfrak{D} \longrightarrow Q_{!\mathcal{A}})$$

is always total.

If $\mathcal{A}$ is a regular automaton, let $!\mathcal{A} := !(\mathcal{A}^\dagger)$, where $\mathcal{A}^\dagger$ is obtained from Prop. B.5.

G.3 Proofs of §5.4

Proposition G.2 (Prop. 5.16.(i)). *If $\mathcal{A} : \Sigma$ is regular, there is a winning* P*-strategy ϵ in $\Sigma \vdash !\mathcal{A}(N) \multimap \mathcal{A}(N)$.*

Proof. By Prop. B.5, we can assume $\mathcal{A}$ to be a parity automaton. Using Prop. A.1, we define $\mathrm{HS}(\epsilon)$ by induction on plays as follows, with the following invariant: for each $(s,t) \in \mathrm{HS}(\epsilon)$, with s,t of even length, writing q for the state of t and S for the state of s, we have $q \in S{\restriction}2$.

The base case is trivial. Let $(s,t) \in \mathrm{HS}(\epsilon)$ with s and t even-length, and with t in state q and s in state S. Given an O-move (a,h), we let $(s.(a,h), t.h(q)) \in \mathrm{HS}(\epsilon)$, and for all (x,d) we further let $(s.(a,h).(\bullet,d), t.h(q).(x,d)) \in \mathrm{HS}(\epsilon)$. Then the invariant is insured by definition of $!\mathcal{A}$.

The strategy τ is winning since the sequence of states produced in $\mathcal{A}$ is a trace in the sequence of states produced in $!\mathcal{A}$. ☐

Proposition G.3 (Prop. 5.16.(ii)). *Given $\mathcal{N}, \mathcal{A} : \Sigma$ regular with $\mathcal{N}$ non-deterministic, if there is a winning* P*-strategy in $\Sigma \vdash \mathcal{N}(L) \multimap \mathcal{A}(M)$ then there is a winning* P*-strategy in $\Sigma \vdash \mathcal{N}(L) \multimap !\mathcal{A}(M)$.*

Proof. By Prop. B.5, we can assume $\mathcal{N}$ and $\mathcal{A}$ to be parity automata. Write G for the game graph of $\Sigma \vdash \mathcal{N}(L) \multimap \mathcal{A}(M)$. Thanks to [21, 22, 20, 37], there is a positional (w.r.t. G) winning P-strategy σ in $\Sigma \vdash \mathcal{N}(L) \multimap \mathcal{A}(M)$.

We build a winning P-strategy τ on $\mathcal{N}(L) \multimap !\mathcal{A}(M)$ such that the following invariant is satisfied:

- To each play t of τ with $\mathsf{pos}(t) = ((p, \bar{\mathsf{a}}, q_\mathcal{N}), (p, \bar{\mathsf{a}}, S))$ where $S = \{(\_, q_1), \ldots, (\_, q_n)\}$, we associate a set $E(t) = \{s_1, \ldots, s_n\}$ of plays of σ, with $\mathsf{pos}(s_i) = ((p, \bar{\mathsf{a}}, q_\mathcal{N}), (p, \bar{\mathsf{a}}, q_i))$,
- and if moreover t' extends t and is such that $\mathsf{pos}(t') = ((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_\mathcal{N}), (p.d, \bar{\mathsf{a}}.\mathsf{a}, S'))$ then for all $s' \in E(t')$ there is some $s \in E(t)$ such that s' extends s.

The strategy τ is build by induction on plays as follows:

- For the base case (initial position ε), we have by definition $S = \{(q^\imath_\mathcal{A}, q^\imath_\mathcal{A})\}$ and $E(\varepsilon) = \{q^\imath_\mathcal{A}\}$.
- For the inductive step, let t with $\mathsf{pos}(t) = ((p, \bar{\mathsf{a}}, q_\mathcal{N}), (p, \bar{\mathsf{a}}, S))$ and let O play from t some (a, v) in component $\mathcal{N}(L)$ of $\mathcal{N}(L) \multimap !\mathcal{A}$.
  For $s_i \in E(t)$, let u_i be the move of σ from position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_\mathcal{N}, v), (p, \bar{\mathsf{a}}, q_i))$ (thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_\mathcal{N}, v), (p, \bar{\mathsf{a}}.\mathsf{a}, q_i, u_i)))$. This defines a map

$h_{t.(\mathsf{a},v)} : Q_{\mathcal{A}} \to U$ taking q_i to u_i (the definition of $h_{t.(\mathsf{a},v)}$ on irrelevant q's is arbitrary), and we let τ play $h_{t.(\mathsf{a},v)}$ in the component $!\mathcal{A}(M)$ of $\mathcal{N}(L) \multimap\ !\mathcal{A}(M)$, thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\ ,\ (p, \bar{\mathsf{a}}.\mathsf{a}, S, h_{t.(\mathsf{a},v)}))$. Then if O answers some $d \in \mathfrak{D}$ in the component $!\mathcal{A}(M)$, and we let P play $\bullet$ in the component $\mathcal{N}(L)$ (recall that both $!\mathcal{A}$ and $\mathcal{N}$ are non-deterministic), the current position in $\mathcal{N}(L) \multimap\ !\mathcal{A}(M)$ becomes $((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_{\mathcal{N}})\ ,\ (p.d, \bar{\mathsf{a}}.\mathsf{a}, S'))$ where

$$q'_{\mathcal{N}} \quad := \quad \delta_{\mathcal{N}}(q_{\mathcal{N}}, L(\bar{\mathsf{a}}.\mathsf{a}, p), v, \bullet, d)$$

and

$$S' \quad := \quad \delta_{!\mathcal{A}}(S, M(\bar{\mathsf{a}}.\mathsf{a}, p), h_{t.(\mathsf{a},v)}, \bullet, d)$$

Let

$$t' \quad := \quad t.(\mathsf{a}, v).h_{t.(\mathsf{a},v)}.d.\bullet$$

and write $S' = \{(\_, q'_1), \dots, (\_, q'_m)\}$. By definition of $!\mathcal{A}$, each q'_j is equal to $\delta_{\mathcal{A}}(q_{i_j}, M(\bar{\mathsf{a}}.\mathsf{a}, p), u_{i_j}, x_j, d)$ for fome i_j and some x_j (note that there might be several such i_j and x_j, but we select one). For each j, we let O play (x_j, d) in the component $\mathcal{A}(M)$ of $\mathcal{N}(L) \multimap \mathcal{A}(M)$ from position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\ ,\ (p, \bar{\mathsf{a}}.\mathsf{a}, q_{i_j}, u_{i_j}))$ thus going to position $((p, \bar{\mathsf{a}}.\mathsf{a}, q_{\mathcal{N}}, v)\ ,\ (p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_j))$. We then let P answer $\bullet$ in the component $\mathcal{N}(L)$, thus leading to position $((p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_{\mathcal{N}})\ ,\ (p.d, \bar{\mathsf{a}}.\mathsf{a}, q'_j))$.
We finally put

$$E(t') \quad := \quad \{s_{i_0}.(\mathsf{a}, v).u_{i_0}.(x_0, d).\bullet\ ,\ \cdots\ ,\ s_{i_m}.(\mathsf{a}, v).u_{i_m}.(x_m, d).\bullet\}$$

This completes the definition of τ.

We now show that τ is winning. Consider an infinite play $(t_i)_{i \in \mathbb{N}}$ of τ, and let $(q_n, S_n)_{n \in \mathbb{N}}$ be the associated sequence of states in $(Q_{\mathcal{N}} \times Q_{!\mathcal{A}})^\omega$. Assume that $(q_n)_n \in \Omega_{\mathcal{N}}$. We show that $(S_n)_n \in \Omega_{!\mathcal{A}}$. Let $(q'_n)_n$ be a trace in $(S_n)_n$, so that $(q'_n, q'_{n+1}) \in S_{n+1}$. We have to show that $(q'_n)_n \in \Omega_{\mathcal{A}}$. Note that for all $n \in \mathbb{N}$,

$$\mathsf{pos}(t_{4n}) \quad = \quad ((p_n, \bar{\mathsf{a}}_n, q_n)\ ,\ (p_n, \bar{\mathsf{a}}_n, S_n))$$

By construction, for each $n \in \mathbb{N}$ there are $s_n \in E(t_{4n})$ and $s'_n \in E(t_{4(n+1)})$ such that s'_n extends s_n:

$$s'_n \quad = \quad s_n.(\mathsf{a}_n, v_n).u_n.d_n.\bullet \qquad \text{where} \qquad \bar{\mathsf{a}}_{n+1} = \bar{\mathsf{a}}_n.\mathsf{a}_n \qquad \text{and} \qquad p_{n+1} = p_n.d_n$$

and such that moreover

$$\mathsf{pos}(s_n) \quad = \quad ((p_n, \bar{\mathsf{a}}_n, q_n)\ ,\ (p_n, \bar{\mathsf{a}}_n, q'_n))$$

and

$$\mathsf{pos}(s'_n) \quad = \quad ((p_{n+1}, \bar{\mathsf{a}}_{n+1}, q_{n+1})\ ,\ (p_{n+1}, \bar{\mathsf{a}}_{n+1}, q'_{n+1}))$$

so that

$$\mathsf{pos}(s'_n) \quad = \quad \mathsf{pos}(s_{n+1})$$

Since σ is positional, it follows that the infinite sequence

$$\varpi \quad := \quad \varepsilon.(\mathsf{a}_0, v_0).u_0.d_0.\cdots.p_n.(\mathsf{a}_n, v_n).u_n.d_n.\cdots$$

is an infinite play of σ. Since ϖ produces the sequence of states $(q_n, q'_n)_n \in (Q_{\mathcal{N}} \times Q_{\mathcal{A}})^\omega$, we get $(q'_n)_n \in \Omega_{\mathcal{A}}$ since $(q_n)_n \in \Omega_{\mathcal{N}}$ by assumption. □

By combinining Props. G.3 and G.2 we obtain:

Corollary G.4 (Thm. 5.15). *$\mathcal{L}(\mathcal{A}) = \mathcal{L}(!\mathcal{A})$ for a regular $\mathcal{A}$.*

Proposition G.5 (Ex. 5.17.(i)). *The law of Peirce $!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \vdash ?\mathcal{A}$, (where $?\mathcal{A} = (!\mathcal{A}^{\perp})^{\perp}$) can be derived using the exponential rules.*

Proof. We can derive

$$!\mathcal{A}^{\perp}\,,\,?\mathcal{A} \vdash \perp$$

so that (since $?\mathcal{B} = (!\mathcal{B}^{\perp})^{\perp}$)

$$!\mathcal{A}^{\perp}\,,\,?\mathcal{A} \vdash ?\mathcal{B}$$

from which follows that

$$!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A})\,,\,!\mathcal{A}^{\perp} \vdash ?\mathcal{A}$$

and thus

$$!((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A})\,,\,!\mathcal{A}^{\perp} \vdash \perp$$

and we are done since $?\mathcal{A} = (!\mathcal{A}^{\perp})^{\perp}$. □

Proposition G.6 (Weak Completeness (Ex. 5.17.(ii))). *Given regular automata $\mathcal{A}, \mathcal{B} : \Sigma$, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then there is an effective winning P-strategy in $\Sigma \vdash !\mathcal{A} \multimap (!(\mathcal{B}^{\perp}))^{\perp}$.*

Proof. By Cor. G.4, if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then $\mathcal{L}(!\mathcal{A}) \cap \mathcal{L}(!(\mathcal{B}^{\perp})) = \emptyset$, and we conclude by Prop. E.3. □

The proof of the next Proposition (Ex. 5.17.(iii)) is defered to §H. This is a uniform version of [7, Thm. 1].

Proposition G.7 (Ex. 5.17.(iii)). *For each regular $\mathcal{L} \subseteq \Sigma^{\mathfrak{D}^*}$, there is a non-deterministic automaton $\mathcal{B}$ with $\mathcal{L}(\mathcal{B}) = \mathcal{L}$, and such that for every non-deterministic parity $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}$, there is a winning strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ induced by a function $Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \to V$.*

H Further Examples

This Appendix is devoted to the proof of Prop. G.7 (Ex. 5.17.(iii)). It relies on the existence of position winning P-strategies in games of the form $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$, for non-deterministic parity automata $\mathcal{A}, \mathcal{B} : \Sigma$ such that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

Further more, we show in §H.4 that such strategies, when combined with our internalized linear implication can handle a construction for the separation property of [33, Thm. 2.7].

H.1 On Positional Strategies

Consider non-deterministic parity automata $\mathcal{A}, \mathcal{B} : \Sigma$. It follows from §G.1 that if P has a winning strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$, then P has a positional winning strategy. But the game graph of $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$ is equivalent to the graph G with vertices:

$$(A_{\mathsf{P}} \times B_{\mathsf{P}}) + (A_{\mathsf{O}} \times B_{\mathsf{P}}) + (A_{\mathsf{O}} \times B_{\mathsf{O}}) + (A_{\mathsf{O}} \times B_{\mathsf{P}} \times \mathfrak{D})$$

where

$$A_{\mathsf{P}} := Q_{\mathcal{A}} \qquad A_{\mathsf{O}} := \Sigma \times Q_{\mathcal{A}} \times U \qquad B_{\mathsf{P}} := Q_{\mathcal{B}} \qquad B_{\mathsf{O}} := \Sigma \times Q_{\mathcal{B}} \times V$$

and with edges

- from $(A_{\mathsf{P}} \times B_{\mathsf{P}})$ to $(A_{\mathsf{O}} \times B_{\mathsf{P}})$:
$$(q_{\mathcal{A}}\ ,\ q_{\mathcal{B}}) \quad \xrightarrow{\mathsf{O}} \quad ((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ q_{\mathcal{B}}) \qquad\qquad \text{for all } \mathsf{a} \in \Sigma \text{ and all } u \in U$$
- from $(A_{\mathsf{O}} \times B_{\mathsf{P}})$ to $(A_{\mathsf{O}} \times B_{\mathsf{O}})$:
$$((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ q_{\mathcal{B}}) \quad \xrightarrow{\mathsf{P}} \quad ((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ (\mathsf{a}, q_{\mathcal{B}}, v)) \qquad\qquad \text{for all } v \in V$$
- from $(A_{\mathsf{O}} \times B_{\mathsf{O}})$ to $(A_{\mathsf{O}} \times B_{\mathsf{P}} \times \mathfrak{D})$:
$$((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ (\mathsf{a}, q_{\mathcal{B}}, v)) \quad \xrightarrow{\mathsf{O}} \quad ((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ q'_{\mathcal{B}}\ ,\ d)$$
where $d \in \mathfrak{D}$ and $q'_{\mathcal{B}} := \delta_{\mathcal{B}}(q_{\mathcal{B}}, \mathsf{a}, v, d)$,
- from $(A_{\mathsf{O}} \times B_{\mathsf{P}} \times \mathfrak{D})$ to $(A_{\mathsf{P}} \times B_{\mathsf{P}})$:
$$((\mathsf{a}, q_{\mathcal{A}}, u)\ ,\ q'_{\mathcal{B}}\ ,\ d) \quad \xrightarrow{\mathsf{P}} \quad (q'_{\mathcal{A}}\ ,\ q'_{\mathcal{B}})$$
where $q'_{\mathcal{A}} := \delta_{\mathcal{A}}(q_{\mathcal{A}}, \mathsf{a}, u, d)$.

Since a positional P-strategy in G is given by a function

$$g \quad : \quad Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \quad \longrightarrow \quad V$$

we thus have:

Lemma H.1. *Given non-deterministic parity automata $\mathcal{A}, \mathcal{B} : \Sigma$, if P has a winning strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$, then P has a winning strategy induced by a function $Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \to V$.*

H.2 On Positional Strategies for Separation

Consider now non-deterministic parity automata $\mathcal{A}, \mathcal{B} : \Sigma$ such that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$. Then by Prop. E.3 there is a winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \bot$. It follows from Lem. H.1 that P has winning strategy induced by a function

$$g \quad : \quad Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \mathbb{B} \times \Sigma \times U \times V \quad \longrightarrow \quad \mathfrak{D}$$

The game $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \bot$ is won by P if $\bot$ goes to state $\mathbb{t}$, since it can not switch back to $\mathbb{f}$. It follows that it is sufficient to have the values of g above with $\bot$ in state $\mathbb{f}$. It follows that P has a winning strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \bot$ induced by a map of the form

$$h \quad : \quad Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \times V \quad \longrightarrow \quad \mathfrak{D}$$

H.3 Proof of Prop. G.7 (Ex. 5.17.(iii))

Example 5.17.(iii) is the following uniform version of [7, Thm. 1]:

Proposition H.2. *For each regular language $\mathcal{L} \subseteq \Sigma^{\mathfrak{D}^*}$, there is a non-deterministic $\mathcal{B} : \Sigma$ such that for every non-deterministic parity automaton $\mathcal{A} : \Sigma$ with $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, there is a map $g : Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \to V$ which induces a winning* P*-strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$.*

The proof of Prop. H.2 follows the lines of [7], itself based on the complementation construction used in [35, Proof of Thm. 6.9].

Fix a regular $\mathcal{L} \subseteq \Sigma^{\mathfrak{D}^*}$, and consider a non-deterministic parity $\mathcal{C} = (Q_{\mathcal{C}}, q^{\imath}_{\mathcal{C}}, W, \delta_{\mathcal{C}}, \Omega_{\mathcal{C}})$ recognizing the complement of $\mathcal{L}$. Using the closure properties of ω-regular languages, there is a deterministic parity ω-word automaton $\mathcal{D} : \Sigma \times V \times \mathfrak{D}$ where

$$V \quad := \quad (Q_{\mathcal{C}} \times W \quad \longrightarrow \quad \mathfrak{D})$$

such that $\mathcal{D}$ accepts $(\mathsf{a}_k, f_k, d_k)_k$ iff for all $(u_k)_k \in U^\omega$ and all $(q_k)_k \in Q^\omega_{\mathcal{C}}$, we have $(q_k)_k \notin \Omega_{\mathcal{C}}$ whenever $q_0 := q^{\imath}_{\mathcal{C}}$, $q_{k+1} := \delta_{\mathcal{C}}(q_k, \mathsf{a}_k, u_k, f_k(q_k, \mathsf{a}_k, u_k))$, and $d_k = f_k(q_k, \mathsf{a}_k, u_k)$.

Write $\mathcal{D} := (Q_{\mathcal{D}}, q^{\imath}_{\mathcal{D}}, \Omega_{\mathcal{D}})$. Let now $\mathcal{B} : \Sigma$ be a parity non-deterministic automaton with P-moves V and such that an infinite play $((\mathsf{a}_k, f_k) \cdot d_k)_k$ is winning iff $(\mathsf{a}_k, f_k, d_k)_k$ is accepted by $\mathcal{D}$. Explicitly, we let

$$\mathcal{B} \quad = \quad (Q_{\mathcal{D}}, q^{\imath}_{\mathcal{D}}, V, \delta_{\mathcal{B}}, \Omega_{\mathcal{D}})$$

where

$$\delta_{\mathcal{B}}(q, \mathsf{a}, f, d) \quad := \quad \delta_{\mathcal{D}}(q, (\mathsf{a}, f, d))$$

Lemma H.3 ([35]). $\mathcal{L}(\mathcal{B}) = \mathcal{L}$.

Proof (of the Lemma). We show that $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{C}^{\perp})$. Let $T : \mathfrak{D}^* \to \Sigma$. Assume first that $T \in \mathcal{L}(\mathcal{C}^{\perp})$, so that P has winning strategy in $\mathcal{C}^{\perp}(T)$. Since $\mathcal{C}$ is a parity automaton, this strategy can be assumed to be positional, hence to be determined by a function $\mathfrak{D}^* \to (Q_{\mathcal{C}} \times W \to \mathfrak{D})$. But this determines a P-strategy in $\mathcal{B}(T)$, which is winning by definition of $\mathcal{B}$. Conversely, assume that $T \in \mathcal{L}(\mathcal{B})$. Since $\mathcal{B}$ is non-deterministic, a winning P-strategy in $\mathcal{B}(T)$ is given by a function $\mathfrak{D}^* \to V = \mathfrak{D}^* \to (Q_{\mathcal{C}} \times W \to \mathfrak{D})$. □

Going back to the proof of Prop. H.2, consider a non-deterministic parity $\mathcal{A} : \Sigma$ with $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}$. Since $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{C}) = \emptyset$, it follows from §H.2 that there is a function

$$g \quad : \quad Q_{\mathcal{A}} \times Q_{\mathcal{C}} \times \Sigma \times U \times W \quad \longrightarrow \quad \mathfrak{D}$$

which generates a winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{C} \multimap \perp$. But g can be seen as a map

$$Q_{\mathcal{A}} \times \Sigma \times U \quad \longrightarrow \quad V$$

and this map generates a winning P-strategy in $\Sigma \vdash \mathcal{A} \multimap \mathcal{B}$. □

H.4 A Separation Property from [33]

Our internalized linear arrow can handle a construction for the separation property of [33, Thm. 2.7].

Consider non-deterministic parity automata $\mathcal{A}, \mathcal{B} : \Sigma$ such that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) = \emptyset$. Assume moreover that both $\mathcal{A}$ and $\mathcal{B}$ are parity with colorings of range $\{0, \ldots, n\}$ for some even n. Theorem 2.7 of [33] say that there is a parity automaton $\mathcal{C}$ such that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$ and such that $\Omega_{\mathcal{C}}$ is generated by a coloring $c_{\mathcal{B}} : Q_{\mathcal{C}} \to \mathbb{N}$ of range $\subseteq \{0, \ldots, n\}$ and such that in each reachable strongly connected component of $\mathcal{C}$ (for $q \to q'$ iff $q' = \delta_{\mathcal{C}}(q, \mathsf{a}, f, v, d)$ for some a, f, v, d), $c_{\mathcal{C}}$ has range either $\{1, \ldots, n\}$ or $\{0, \ldots, n-1\}$.

We build $\mathcal{C}$ by restricting $\mathcal{B} \multimap \mathcal{A}$ along a winning strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$. By §H.2, there is a function

$$g \quad : \quad Q_{\mathcal{A}} \times Q_{\mathcal{B}} \times \Sigma \times U \times V \quad \longrightarrow \quad \mathfrak{D}$$

wich generates a winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$.

We restrict the automaton $\mathcal{B} \multimap \mathcal{A} : \Sigma$ along g as follows. Recall that $Q_{\mathcal{B} \multimap \mathcal{A}} = Q_{\mathcal{B}} \times Q_{\mathcal{A}}$. Define $\mathcal{C} : \Sigma$ as follows:

$$\mathcal{C} \ := \ (Q_{\mathcal{B} \multimap \mathcal{A}} + \{\mathbb{t}\}, q^{\imath}_{\mathcal{B} \multimap \mathcal{A}}, U^V, V, \delta_{\mathcal{C}}, \Omega_{\mathcal{C}})$$

where $\delta_{\mathcal{C}}(\mathbb{t}, \_, \_, \_, \_) := \mathbb{t}$, and

$$\delta_{\mathcal{C}}((q_{\mathcal{B}}, q_{\mathcal{A}}), \mathsf{a}, f, v, d) \quad := \quad \begin{cases} \mathbb{t} & \text{if } g(q_{\mathcal{A}}, q_{\mathcal{B}}, \mathsf{a}, f(v), v) \neq d \\ \delta_{\mathcal{B} \multimap \mathcal{A}}((q_{\mathcal{A}}, q_{\mathcal{B}}), \mathsf{a}, f, v, d) & \text{otherwise} \end{cases}$$

The coloring $c_{\mathcal{C}}$ of $\mathcal{C}$ is then defined as in [33, §2.2.2]. We define it explicitly as follows. Consider a reachable strongly connected component C of $\mathcal{C}$. Note that if C contains $\mathbb{t}$, then $C = \{\mathbb{t}\}$, and we put $c_{\mathcal{C}}(\mathbb{t}) := n$. Otherwise, C contains only states of $\mathcal{B} \multimap \mathcal{A}$, that is states in $Q_{\mathcal{B}} \times Q_{\mathcal{A}}$. Assume that C is non-trivial and contains two states $(\_, q_{\mathcal{A}})$ and $(q_{\mathcal{B}}, \_)$ with $c_{\mathcal{A}}(q_{\mathcal{A}}) = c_{\mathcal{B}}(q_{\mathcal{B}}) = n$. By definition of $\delta_{\mathcal{C}}$, the set of states

$$\{(q'_{\mathcal{A}}, q'_{\mathcal{B}}, \mathbb{f}) \mid (q'_{\mathcal{B}}, q'_{\mathcal{A}}) \in C\}$$

is reached infinitely often in an infinite play of the strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$ induced by g. But this contradicts the fact that this strategy is winning. It follows that either (a) $c_{\mathcal{A}}$ never takes the value n in C or (b) $c_{\mathcal{B}}$ never takes the value n in C. In the case (a), for each state $(q_{\mathcal{B}}, q_{\mathcal{A}})$ of C we put $c_{\mathcal{C}}(q_{\mathcal{B}}, q_{\mathcal{A}}) := c_{\mathcal{A}}(q_{\mathcal{A}})$, and in the case (b) we put $c_{\mathcal{C}}(q_{\mathcal{B}}, q_{\mathcal{A}}) := c_{\mathcal{B}}(q_{\mathcal{B}}) + 1$.

Consider now an infinite sequence of the form $\rho := (q'_k, q_k)_k \in Q^{\omega}_{\mathcal{B} \multimap \mathcal{A}}$ and let C be a strongly connected component of $\mathcal{C}$ such that $\mathrm{Inf}_k(q'_k, q_k) \subseteq C$. Let $m = \max(\mathrm{Inf}_k c_{\mathcal{C}}(q'_k, q_k))$.

Claim. If m is even, then $\rho \in \Omega_{\mathcal{B} \multimap \mathcal{A}}$

Proof (of the Claim). In case (a) above, we have $m = \max(\mathrm{Inf}_k c_{\mathcal{A}}(q_k))$ hence $(q_k)_k \in \Omega_{\mathcal{A}}$ and $\rho \in \Omega_{\mathcal{B} \multimap \mathcal{A}}$. In case (b), $m = \max((\mathrm{Inf}_k c_{\mathcal{B}}(q'_k) + 1))$, hence $\max(\mathrm{Inf}_k c_{\mathcal{B}}(q'_k))$ is odd, so that $(q'_k)_k \notin \Omega_{\mathcal{B}}$ and $\rho \in \Omega_{\mathcal{B} \multimap \mathcal{A}}$. □

Lemma H.4. *$\mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B}^{\perp})$.*

Proof. Consider a winning P-strategy σ in $\mathcal{C}(T)$. Recall that the P-moves of $\mathcal{B}^{\perp}$ are $\mathfrak{D}^V$ and that its O-moves are V, and that the P-moves of $\mathcal{C}$ are U^V and that its O-moves are V. Let τ be the winning P-strategy τ on $\mathcal{A} \otimes \mathcal{B} \multimap \perp$ (whose P-moves are $\mathfrak{D}$ and O-moves are $U \times V$) induced by g. We define a P-strategy θ by combining σ and τ as follows: modulo Currying, θ plays from $v \in V$ the tree direction $d \in \mathfrak{D}$ proposed by $T^*(\tau)$ from v and the $u \in U$ given by σ on v. Hence the strategies σ and θ play the same moves in $\mathcal{B}$ (provided by O). So the sequences of $Q_{\mathcal{B}}$-states produced by σ and θ are the same, unless O plays in $\mathcal{B}^{\perp}$ a tree direction $d \in \mathfrak{D}$ different from the one proposed by θ, *i.e.* different from the one proposed by τ. In this case, the play on $\mathcal{B}^{\perp}(T)$ is P-winning and we are done. Assume now that the sequences of $Q_{\mathcal{B}}$-states agree. We show that they can not be in $\Omega_{\mathcal{B}}$. Assume toward a contradiction that they are. By the claim above, since σ is winning, the sequence of states in $\mathcal{C}$ belongs to $\Omega_{\mathcal{B} \multimap \mathcal{A}}$ The play respects σ, so the sequence of $Q_{\mathcal{A}}$-states must belong to $\Omega_{\mathcal{A}}$ since σ is winning. But the play also respects $T^*(\tau)$, which is winning in $\mathcal{A}(T) \otimes \mathcal{B}(T) \multimap \perp$, so the sequence of $Q_{\mathcal{A}}$-states can not belong to $\Omega_{\mathcal{A}}$. It follows that the sequence of $Q_{\mathcal{B}}$-states can not belong to $\Omega_{\mathcal{B}}$, and we are done since the play in $\mathcal{B}^{\perp}(T)$ is then P-winning. □

In order to complete the proof of the separation property, it remains to show the following

Lemma H.5. *$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{C})$.*

Proof. Let $T : \mathfrak{D}^* \to \Sigma$ such that $T \in \mathcal{L}(\mathcal{A})$. Consider a winning positional P-strategy τ in $\mathcal{A}(T)$ induced by a function $\mathfrak{D}^* \to (Q_{\mathcal{A}} \to U)$. This gives a function $\mathfrak{D}^* \to (Q_{\mathcal{C}} \times V \to U)$ which induces a strategy σ in $\mathcal{C}(T)$. Consider an infinite play ϖ of σ induced by an infinite play ϖ_τ of τ. Let $\rho \in Q_{\mathcal{C}}^{\omega}$ be the sequence of states produced by ϖ. If ρ contains $\mathbb{t}$, then $\rho \in Q^*_{\mathcal{B} \multimap \mathcal{A}}.\mathbb{t}^{\omega} \subseteq \Omega_{\mathcal{C}}$ and we are done. Otherwise, let $\rho = (q'_k, q_k)_k \in Q_{\mathcal{B} \multimap \mathcal{A}}$. If we are in case (a) above, then $\max(\mathrm{Inf}_k(c_{\mathcal{C}}(\rho))) = \max(\mathrm{Inf}_k(c_{\mathcal{A}}(q_k)))$, hence $\rho \in \Omega_{\mathcal{C}}$. Assume that we are in case (b), so that $\max(\mathrm{Inf}_k(c_{\mathcal{C}}(\rho))) = \max(\mathrm{Inf}_k(c_{\mathcal{B}}(q'_k))+1)$. Let θ be the winning P-strategy in $\Sigma \vdash \mathcal{A} \otimes \mathcal{B} \multimap \perp$ induced by g. Then, by combining ϖ_τ and $\varpi_{\restriction \mathcal{B}}$, we obtain an infinite play ϖ' of θ. Note that in this play, $\perp$ never switches to $\mathbb{t}$ since we assumed $\rho \in \Omega_{\mathcal{B} \multimap \mathcal{A}}$. It follows that ϖ' produces the same sequence of states $(q'_k)_k \in Q_{\mathcal{B}}$ as ϖ, and we must have $(q'_k)_k \notin \Omega_{\mathcal{B}}$ since $(q_k)_k \in \Omega_{\mathcal{A}}$. It follows that $\max(\mathrm{Inf}_k(c_{\mathcal{C}}(\rho))) = \max(\mathrm{Inf}_k(c_{\mathcal{B}}(q'_k)) + 1)$ is even. □

I Simple Self Dualization

In this appendix, we present some aspects of the construction called *simple self dualization* in [18]. We begin by basic definitions and facts, and then give a general method to construct (lax) symmetric monoidal monads and oplax symmetric monoidal comonads in this setting, which will be used later on in §J to explain the monoidal structure of **DZ**.

I.1 Some Basic Definitions and Facts

We recall here some basic material about Dialectica-like categories from [8, 18]. Given a category $\mathbb{C}$, its *simple self-dualization* is $\mathbf{G}(\mathbb{C}) := \mathbb{C} \times \mathbb{C}^{\mathrm{op}}$ (also written $\mathbb{C}^{\mathrm{d}}$ in [18]). Its objects are pairs U, X of objects of $\mathbb{C}$, and a morphism from (U, X) to (V, Y) is given by a pair of maps (f, F), denoted

$$(f, F) \quad : \quad (U, X) \quad \multimap\!\!\!\!\!\!\rightarrow \quad (V, Y)$$

where $f : U \to V$ and $F : Y \to X$. If $\mathbb{C}$ is symmetric monoidal, then $\mathbf{G}(\mathbb{C})$ is an instance of a *Girard* category, in the sense of de Paiva [8, 18].

Assume now that the monoidal structure $(\otimes, \mathbf{I}) = (\times, \mathbf{1})$ of $\mathbb{C}$ is Cartesian. Then $\mathbf{G}(\mathbb{C})$ is symmetric monoidal closed w.r.t.

$$(U, X) \otimes_{\mathbf{G}} (V, Y) \;:=\; (U \otimes V, X^V \otimes Y^U) \qquad \text{with unit } (\mathbf{I}, \mathbf{I})$$

The linear exponentials are given by

$$(U, X) \multimap_{\mathbf{G}} (V, Y) \quad := \quad (V^U \times X^Y \;,\; U \times Y)$$

Moreover, $\mathbf{G}(\mathbb{C})$ can be equipped with a comonad (T, δ, ϵ) where the action on objects of T is

$$T(U, X) \quad := \quad (U, X^U)$$

and the maps δ and ϵ are given by

$$\begin{array}{rclcl} (f_\epsilon, F_\epsilon) & : & (U, X^U) & \multimap\!\!\!\!\!\!\rightarrow & (U, X) \\ (f_\delta, F_\delta) & : & (U, X^U) & \multimap\!\!\!\!\!\!\rightarrow & (U, X^{U \times U}) \end{array}$$

where $f_\epsilon = f_\delta = \mathrm{id}_U$, $F_\epsilon(u, x) = x$ and $F_\delta(h, u) = h(u, u)$ (see e.g. [8, Def. 15, §4.2]).

The co-Kleiseli category $\mathbf{D}(\mathbb{C}) := \mathbf{Kl}(T)$ is a Dialectica category in the sense of [8, 16] (see e.g. [8, Prop. 52, §4.3]). Explicitely, its objects are pairs $A = (U, X)$ of objects of $\mathbb{C}$, and a map from A to (V, Y) is a $\mathbf{G}(\mathbb{C})$-morphism (f, F) from TA to (V, Y), that is

$$(f, F) \quad : \quad (U, X^U) \quad \multimap\!\!\!\!\!\!\rightarrow \quad (V, Y)$$

$\mathbf{D}(\mathbb{C})$ is symmetric monoidal closed w.r.t. the product

$$(U, X) \otimes (V, Y) \;:=\; (U \times V, X \times Y) \qquad \text{with unit } (\mathbf{1}, \mathbf{1})$$

Note that with $A = (U, X)$ and $B = (V, Y)$,

$$\begin{array}{rcl} T(A \otimes B) & = & (U \times V, (X \times Y)^{U \times V}) \\ & \simeq & (U \times V, X^{U^V} \times Y^{V^U}) \\ & = & TA \otimes_{\mathbf{G}} TB \end{array}$$

The linear exponentials of $\mathbf{D}(\mathbb{C})$ are given by

$$(U, X) \multimap (V, Y) \quad := \quad (V^U \times X^{U \times Y} \;,\; U \times Y)$$

Note that $A \multimap B \simeq TA \multimap_{\mathbf{G}} B$, so the monoidal closure of $\mathbf{D}(\mathbb{C})$ actually follows from that of $\mathbf{G}(\mathbb{C})$:

$$\begin{array}{rcl} \mathbf{D}(\mathbb{C})[A \otimes B, C] & = & \mathbf{G}(\mathbb{C})[T(A \otimes B), C] \\ & \simeq & \mathbf{G}(\mathbb{C})[TA \otimes_{\mathbf{G}} TB, C] \\ & \simeq & \mathbf{G}(\mathbb{C})[TA, TB \multimap_{\mathbf{G}} C] \\ & \simeq & \mathbf{D}(\mathbb{C})[A, B \multimap C] \end{array}$$

I.2 Self Duality

The category $\mathbf{G}(\mathbb{C})$ is equipped with an isomorphism

$$(-)^{\perp} \quad : \quad \mathbf{G}(\mathbb{C}) \quad \stackrel{\simeq}{\longrightarrow} \quad \mathbf{G}(\mathbb{C})^{\mathrm{op}}$$

mapping the $\mathbf{G}(\mathbb{C})$-object (U,X) to (X,U) and taking $(f,F) : (U,X) \longrightarrow\!\!\!\!\!\!+\; (V,Y)$ to $(F,f) : (X,U) \longrightarrow_{\mathbf{G}(\mathbb{C})^{\mathrm{op}}} (Y,V)$ (that is $(F,f) : (Y,V) \longrightarrow\!\!\!\!\!\!+\; (X,U)$). Note that $(-)^{\perp}$ is a strict involution: $\mathbf{G}(\mathbb{C})^{\perp\perp} = \mathbf{G}(\mathbb{C})$.

I.3 Monoidal Structure

Consider an SMC $\mathbb{C}$. Note that $\mathbb{C}^{\mathrm{op}}$ is also an SMC, and recall from §I.1 the tensor product $\otimes$ of $\mathbf{G}(\mathbb{C})$ given by

$$(U,X) \otimes (V,Y) \quad = \quad (U \otimes V, X \otimes Y) \qquad \text{with unit} \quad \mathbf{I} = (\mathbf{I},\mathbf{I})$$

Assuming the following structure maps of $\mathbb{C}$

$$\alpha : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \qquad \lambda : \mathbf{I} \otimes A \longrightarrow A \qquad \rho : A \otimes \mathbf{I} \longrightarrow A \qquad \gamma : A \otimes B \longrightarrow B \otimes A$$

the structure maps of $(\mathbf{G}(\mathbb{C}), \otimes, \mathbf{I})$ are given by:

$$\begin{array}{lclcl} \alpha & := & (\alpha, \alpha^{-1}) & : & ((U,X) \otimes (V,Y)) \otimes (W,Z) \quad \longrightarrow \quad (U,X) \otimes ((V,Y) \otimes (W,Z)) \\ \lambda & := & (\lambda, \lambda^{-1}) & : & (\mathbf{I},\mathbf{I}) \otimes (U,X) \quad \longrightarrow \quad (U,X) \\ \rho & := & (\rho, \rho^{-1}) & : & (U,X) \otimes (\mathbf{I},\mathbf{I}) \quad \longrightarrow \quad (U,X) \\ \gamma & := & (\gamma, \gamma^{-1}) & : & (U,X) \otimes (V,Y) \quad \longrightarrow \quad (V,Y) \otimes (U,X) \end{array}$$

Proposition I.1 ([18]). *Equipped with the above data, the category $\mathbf{G}(\mathbb{C})$ is symmetric monoidal.*

I.4 (Commutative) Monoids

Proposition I.2. *Consider an SMC $\mathbb{C}$. Given a comutative monoid (M,u,m) and a commutative comonoid (K,e,d) in $\mathbb{C}$, the $\mathbf{G}(\mathbb{C})$-object (M,K) is a commutative monoid in $\mathbf{G}(\mathbb{C})$ with structure maps*

$$\begin{array}{rclcll} u_{(M,K)} & := & (u,e) & : & (\mathbf{I},\mathbf{I}) & \longrightarrow\!\!\!\!\!\!+\; (M,K) \\ m_{(M,K)} & := & (m,d) & : & (M \otimes M, K \otimes K) & \longrightarrow\!\!\!\!\!\!+\; (M,K) \end{array}$$

Proof. The proof is trivial since (1) commutation of the required diagrams amounts to componentwise commutation of the corresponding diagrams in $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$, and (2) the second components of commutative monoids diagrams in $\mathbf{G}(\mathbb{C})$ are commutative comonoids diagrams in $\mathbb{C}^{\mathrm{op}}$.

I.5 (Commutative) Comonoids

Recall that a (commutative) comonoid in a category is a (commutative) monoid in the oppostive category. Since $\mathbf{G}(\mathbb{C})^{\mathrm{op}} \simeq \mathbf{G}(\mathbb{C})^{\perp}$, it follows that Prop. I.2 dualizes to:

Corollary I.3. *Consider an SMC $\mathbb{C}$. Given a comonoid (K, e, d) and a monoid (M, u, m) in $\mathbb{C}$, the $\mathbf{G}(\mathbb{C})$-object (K, M) is a commutative comonoid in $\mathbf{G}(\mathbb{C})$ with structure maps*

$$\begin{array}{lclcccl} e_{(K,M)} & := & (e, u) & : & (K, M) & \multimap & (\mathbf{I}, \mathbf{I}) \\ d_{(M,K)} & := & (d, m) & : & (K, M) & \multimap & (K \otimes K, M \otimes M) \end{array}$$

I.6 A (Lax) Symmetric Monoidal Monad

Assume now that $\mathbb{C}$ is Cartesian closed, and fix a functor $H : \mathbb{C} \to \mathbb{C}$. Recall (from e.g. [27, §5.2]) that H lifts in a unique way to an oplax symmetric monoidal functor, with strength

$$\mathsf{t}^2_{A,B} \quad := \quad \langle H(\pi_1), H(\pi_2) \rangle \quad : \quad H(A \times B) \quad \longrightarrow \quad HA \times HB$$

and

$$\mathsf{t}^0 \quad := \quad \mathbf{1}_{H\mathbf{1}} \quad : \quad H\mathbf{1} \quad \longrightarrow \quad \mathbf{1}$$

Note that the naturality of $\mathsf{t}^2_{(-),(-)}$, that is

$$(H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2) \rangle \quad = \quad \langle H(\pi_1), H(\pi_2) \rangle \circ H(f \times g)$$

follows from the universality property of the Cartesian product since (say)

$$\pi_1 \circ (H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2) \rangle \quad = \quad H(f \circ \pi_1) \quad = \quad H(\pi_1 \circ (f \times g))$$

Consider now the functor

$$(-)^H \quad : \quad \mathbf{G}(\mathbb{C}) \quad \longrightarrow \quad \mathbf{G}(\mathbb{C})$$

defined as

$$(U, X)^H \quad := \quad (U^{HX}, X)$$

and

$$(f, F)^H \quad := \quad (\boldsymbol{\lambda} h. f \circ h \circ H(F) \ , \ F) \quad : \quad (U^{HX}, X) \multimap (V^{HY}, Y)$$

(where $(f, F) : (U, X) \multimap (V, Y)$), and the maps

$$\begin{array}{lclclcrcl} \eta_{(U,X)} & = & (f_\eta, F_\eta) & := & (\boldsymbol{\lambda} u. \boldsymbol{\lambda} \_ . u \ , \ \mathrm{id}_X) & : & (U, X) & \multimap & (U^{HX}, X) \\ \mu_{(U,X)} & = & (f_\mu, F_\mu) & := & (\boldsymbol{\lambda} h. \boldsymbol{\lambda} x. h(x, x) \ , \ \mathrm{id}_X) & : & (U^{HX \times HX}, X) & \multimap & (U^{HX}, X) \end{array}$$

Proposition I.4. *$((-)^H, \mu, \eta)$ is a (lax) symmetric monoidal monad, with strength*

$$\begin{array}{lcl} m^2_{A,B} \quad = \quad (f^2_{A,B}, F^2_{A,B}) & := & (\boldsymbol{\lambda}(h, k).(h \times k) \circ \mathsf{t}^2_{X,Y} \ , \ \mathrm{id}_{X \times Y}) \quad : \\ (U^{HX} \times V^{HY} \ , \ X \times Y) & \longrightarrow & ((U \times V)^{H(X \times Y)} \ , \ X \times Y) \end{array}$$

(where $A = (U, X)$ and $B = (V, Y)$), and

$$m^0 \quad := \quad (\mathbf{1}, \mathbf{1}) \quad : \quad (\mathbf{1}, \mathbf{1}) \quad \longrightarrow \quad (\mathbf{1}^{H\mathbf{1}}, \mathbf{1})$$

The proof of Prop. I.4 is defered to §K.

I.7 An Oplax Symmetric Monoidal Comonad

Proposition I.4 can be dualized thanks to the self duality $\mathbf{G}(\mathbb{C})^{\mathrm{op}} = \mathbf{G}(\mathbb{C})^{\perp}$:

Corollary I.5. *Assume $\mathbb{C}$ is a CCC and $H : \mathbb{C} \to \mathbb{C}$ is a funtor. Then $({}^{H}(-), \delta, \epsilon)$ is an oplax symmetric monoidal comonad on $\mathbb{C}$, where*

$$ {}^{H}(U,X) \quad := \quad (U, X^{HU}) \qquad \text{and} \qquad {}^{H}(f,F) \quad := \quad (f \;,\; \boldsymbol{\lambda} h.F{\circ}h{\circ}H(f)) \quad : \quad (U,X^{HU}) \;{-\!\!\!\!+\!\!\!\!\to}\; (V,Y^{HV}) $$

(for $(f,F) : (U,X) \;{-\!\!\!\!+\!\!\!\!\to}\; (V,Y)$), and

$$ \begin{array}{lllllllll} \epsilon_{(U,X)} & = & (f_\epsilon, F_\epsilon) & := & (\mathrm{id}_U \;,\; \boldsymbol{\lambda} x.\boldsymbol{\lambda}\_.x) & : & (U,X^{HU}) & {-\!\!\!\!+\!\!\!\!\to} & (U,X) \\ \delta_{(U,X)} & = & (f_\delta, F_\delta) & := & (\mathrm{id}_U \;,\; \boldsymbol{\lambda} h.\boldsymbol{\lambda} u.h(u,u)) & : & (U,X^{HU}) & {-\!\!\!\!+\!\!\!\!\to} & (U,X^{HU \times HU}) \end{array} $$

and where the oplax strength of ${}^{H}(-)$ is given by

$$ \begin{array}{rl} n^2_{A,B} \quad = \quad (f^2_{A,B}, F^2_{A,B}) \quad := & (\mathrm{id}_{U\times V} \;,\; \boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{U,V}) \quad : \\ & (U \times V \;,\; (X \times Y)^{H(U\times V)}) \quad \longrightarrow \quad (U \times V \;,\; X^{HU} \times Y^{HV}) \end{array} $$

where $A = (U,X)$, $B = (V,Y)$ and $\mathsf{t}^2_{U,V}$ is defined as in I.6, and

$$ n^0 \quad := \quad (\mathbf{1},\mathbf{1}) \quad : \quad (\mathbf{1},\mathbf{1}^{H\mathbf{1}}) \quad \longrightarrow \quad (\mathbf{1},\mathbf{1}) $$

J A Dialectica-Like Interpretation of Zig-Zag Strategies

We give here a Dialectica-like presentation of total zig-zag strategies $\sigma : A \multimap B$ for A and B positive full games. It relies on a distributive law ζ in an instance of Dialectica called *simple self-dualization* in [18]. We will perform it in the topos of trees $\mathscr{S}$.

We first instantiate the constructions and results of §K to the case of $\mathbf{G}(\mathscr{S})$. We then show in §J.4 that the category **DZ** of simple zig-zag games can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$. In §J.5 we present the distributive law ζ based on the constructions of §K. Finally, using the fact that **DZ** can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$ described as the Kleisli category of the distributive law ζ, we discuss the monoidal structure of **DZ** and $\mathbf{DZ}_{\mathfrak{D}}$.

J.1 The Topos of Trees

The *topos of trees* $\mathscr{S}$ is the presheaf category over the order $(\mathbb{N}, \leq)$ seen as a category, see e.g. [5].

An object X of $\mathscr{S}$ is given by a family of sets $(X_n)_{n\in\mathbb{N}}$ equipped with *restriction maps* $r^X_n : X_{n+1} \to X_n$. A morphism f from X to Y is a family of functions $f_n : X_n \to Y_n$ compatible with restriction: $r^Y_n \circ f_{n+1} = f_n \circ r^X_n$.

As a topos, $\mathscr{S}$ is Cartesian closed w.r.t. to the Cartesian product of presheaves, which is given by $(X \times Y)_n := X_n \times Y_n$. Exponentials are defined as usual for presheaves (see e.g. [25]) by

$$(X^Y)_n \quad := \quad \mathrm{Nat}[\mathbb{N}[-, n] \times Y, X]$$

Explicitly, $(X^Y)_n$ consists of sequences $(\xi_k : Y_k \to X_k)_{k \leq n}$ which are compatible with r^X and r^Y. The restriction map of X^Y takes $(\xi_k)_{k \leq n+1} \in (X^Y)_{n+1}$ to $(\xi_k)_{k \leq n} \in (X^Y)_n$.

We will use the functor $\blacktriangleright : \mathscr{S} \to \mathscr{S}$ of [5]. On objects, it maps X to $((\blacktriangleright X)_n)_{n \in \mathbb{N}}$ where $(\blacktriangleright X)_{n+1} := X_n$ and $(\blacktriangleright X)_0 := \mathbf{1}$, with $r^{\blacktriangleright X}_{n+1} := r^X_n$ and $r^{\blacktriangleright X}_0 := \mathbf{1} : X_0 \to \mathbf{1}$. On morphisms, $(\blacktriangleright f)_{n+1} := f_n$ and $(\blacktriangleright f)_0 := \mathbf{1} : \mathbf{1} \to \mathbf{1}$. Note that $\blacktriangleright(X \times Y) \simeq \blacktriangleright X \times \blacktriangleright Y$.

Define the family of maps $\mathsf{pred}^X : X \Rightarrow \blacktriangleright X$, natural in X, as $\mathsf{pred}^X_0 := \mathbf{1} : X_0 \to \mathbf{1}$ and $\mathsf{pred}^X_{n+1} := r^X_n$.

The functor $\blacktriangleright$ allows $\mathscr{S}$ to be equipped with fixpoint operators $\mathrm{fix}^X : X^{\blacktriangleright X} \Rightarrow X$, defined as

$$\mathrm{fix}^X_n((f_m)_{m \leq n}) \quad := \quad (f_n \circ \ldots \circ f_0)(\bullet)$$

The maps fix^X are natural in X. Given $f : \blacktriangleright X \times Y \Rightarrow X$, writing $f^t : Y \Rightarrow X^{\blacktriangleright X}$ for the exponential transpose of f, $\mathrm{fix}^X \circ f^t$ is the unique map $h : Y \Rightarrow X$ satisfying $f \circ \langle \mathsf{pred}^X \circ h, \mathrm{id}_Y \rangle = h$ (see [5, Thm. 2.4]).

Given a sequence of sets $\overline{M} = (M_n)_n$, we also denote by $\overline{M}$ the $\mathscr{S}$-object with $\overline{M}_n := \prod_{i=0}^n M_i$ and restriction maps $r^{\overline{M}}_n(\overline{m}.m) := \overline{m}$. ($r^{\overline{M}}$ is an epi). Note that $\overline{M} \times \overline{N} \simeq \overline{M \times N}$, where $\overline{M \times N}_n := \prod_{i=0}^n M_i \times N_i$. If $M_n = M$ for all n, then we write $M^\star$ for the $\mathscr{S}$-object $\overline{M}$.

J.2 The Monoidal Structure of $\mathbf{G}(\mathscr{S})$

Following §J.1, we take for $\mathscr{S}$ the monoidal structure given by its Cartesian product (so that $\otimes := \times$ with $\mathbf{I} := \mathbf{1}$). Since $(A_n)_n \times (B_n)_n = (A_n \times B_n)_n$ the structure maps of $(\mathscr{S}, \otimes, \mathbf{I})$ (induced from its Cartesian structure) have as components the corresponding structure maps of $\mathbf{Set}$:

$$\begin{array}{lllllll} \alpha_n & := & \alpha : (A_n \times B_n) \times C_n \to A_n \times (B_n \times C_n) & \quad & \lambda_n & := & \lambda : \mathbf{1} \times A_n \to A_n \\ \rho_n & := & \rho : A_n \times \mathbf{1} \to A_n & & \gamma_n & := & \gamma : A_n \times B_n \to B_n \times A_n \end{array}$$

The required diagrams follow as usual from the fact that Cartesian categories are monoidal (using the universal property of the Cartesian product).

J.3 Monoids and Comonoids in $\mathbf{G}(\mathscr{S})$

Prop. I.2 and Cor. I.3 (on monoid and comonoid objects in categories of the form $\mathbf{G}(\mathbb{C})$) specialize to:

Proposition J.1. *Let X be an object of $\mathscr{S}$.*

(i) The $\mathbf{G}(\mathscr{S})$*-object* $(\mathbf{1}, X)$ *is a commutative monoid of* $\mathbf{G}(\mathscr{S})$*, with structure maps*

$$\begin{array}{rcrcrcl} u & := & (\mathbf{1}, \mathbf{1}) & : & (\mathbf{1}, \mathbf{1}) & \longrightarrow\!\!\!\!\!\!+ & (\mathbf{1}, X) \\ m & := & (\mathbf{1}, \langle \mathrm{id}, \mathrm{id} \rangle) & : & (\mathbf{1} \times \mathbf{1}, X \times X) & \longrightarrow\!\!\!\!\!\!+ & (\mathbf{1}, X) \end{array}$$

(ii) The $\mathbf{G}(\mathscr{S})$*-object* $(X, \mathbf{1})$ *is a commutative comonoid of* $\mathbf{G}(\mathscr{S})$*, with structure maps*

$$\begin{array}{rcrcrcl} e & := & (\mathbf{1}, \mathbf{1}) & : & (X, \mathbf{1}) & \longrightarrow\!\!\!\!\!\!+ & (\mathbf{1}, \mathbf{1}) \\ d & := & (\langle \mathrm{id}, \mathrm{id} \rangle, \mathbf{1}) & : & (X, \mathbf{1}) & \longrightarrow\!\!\!\!\!\!+ & (X \times X, \mathbf{1} \times \mathbf{1}) \end{array}$$

Proof. By Prop. I.2 and Cor. I.3, since the terminal object $\mathbf{1}$ of a Cartesian category is a commutative monoid, and since any object of a Cartesian category is a commutative comonoid.

J.4 A Dialectica-Like Interpretation of Zig-Zag Strategies

We now show that $\mathbf{DZ}$ is equivalent to a category obtained from a distributive law in $\mathbf{G}(\mathscr{S})$. We first show (Prop. J.2) that total zig-zag strategies are in 1-1 correspondence with $\mathbf{G}(\mathscr{S})$ morphisms

$$(f, F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \longrightarrow\!\!\!\!\!\!+ \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

We then describe a composition of these morphisms respecting composition of strategies. The distributive law ζ is presented in §J.5.

Total Zig-Zag Strategies in $\mathbf{G}(\mathscr{S})$. Consider a positive full game $A = (U, X)$. Recall from App. A the bijection

$$\partial \quad = \quad \langle \partial_U, \partial_X \rangle \quad : \quad \wp_A^{\mathsf{even}} \quad \longrightarrow \quad \cup_{n \in \mathbb{N}} (U^n \times X^n)$$

with $\partial(\varepsilon) = (\bullet, \bullet)$ and $\partial(s.u.x) = (\partial_U(s).u, \partial_X(s).x)$. Recall also from Prop. A.1 the faithfull functor $\mathrm{HS} : \mathbf{SG} \longrightarrow \mathbf{Rel}$.

Consider now another positive full game $B = (V, X)$ and let $\sigma : A \multimap B$ be a total zig-zag strategy. By induction on $n \in \mathbb{N}$, it is easy to see that for all $(\overline{u}, \overline{y}) \in U^n \times Y^n$, there is a unique $(s, t) \in \mathrm{HS}(\sigma)$ such that $\overline{u} = \partial_U(s)$ and $\overline{y} = \partial_Y(t)$.

The property vacuously holds for $n = 0$. Assuming it for n, given $(\overline{u}.u, \overline{y}.y) \in U^{n+1} \times Y^{n+1}$, by induction hypothesis, there is a unique $(s, t) \in \mathrm{HS}(\sigma)$ such that $\overline{u} = \partial_U(s)$ and $\overline{y} = \partial_Y(t)$. Now, since σ is total and zig-zag, there is a unique $v \in V$ such that $(s.u, t.v) \in \mathrm{HS}(\sigma)$. Similarly, there is a unique $x \in X$ such that $(s.u.x, t.v.y) \in \mathrm{HS}(\sigma)$, and the property follows.

Furthermore, since $\overline{u}.u$ and $\overline{y}$ uniquely determine $\overline{v} = \partial_V(t)$ and v, and since $\overline{u}.u$ and $\overline{y}.y$ uniquely determine $\overline{x} = \partial_X(s)$ and x, we obtain functions

$$\begin{array}{rcrcl} f_{n+1} & : & U^{n+1} \times Y^n & \longrightarrow & V^{n+1} \\ F_{n+1} & : & U^{n+1} \times Y^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

It follows that σ uniquely determine a $\mathbf{G}(\mathscr{S})$-morphism

$$\sigma_{\mathbf{G}(\mathscr{S})} \quad = \quad (f,F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \longrightarrow\!\!\!\!\!\!+ \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

Conversely, each (f,F) uniquely determine a total zig-zag strategy σ, with, for all $\overline{u}.u \in U^{n+1}$, and all $\overline{y} \in Y^n$,

$$(\partial^{-1}(\overline{u},\overline{x}).u \ , \ \partial^{-1}(\overline{v},\overline{y}).v) \in \mathrm{HS}(\sigma)$$

where $\overline{v}.v = f_{n+1}(\overline{u}.u,\overline{y})$ and $\overline{x} = F_n(\overline{u},\overline{y})$; and for all y,

$$(\partial^{-1}(\overline{u},\overline{x}).u.x \ , \ \partial^{-1}(\overline{v},\overline{y}).v.y) \in \mathrm{HS}(\sigma)$$

where $\overline{x}.x = F_{n+1}(\overline{u}.u,\overline{y}.y)$.

We therefore have shown:

Proposition J.2. *Given positive full games $A = (U,X)$ and $B = (V,Y)$, the map $(-)_{\mathbf{G}(\mathscr{S})}$ is a bijection from total zig-zag strategies $\sigma : A \multimap B$ to $\mathbf{G}(\mathscr{S})$-morphisms*

$$(f,F) \quad : \quad (U^\star, X^{\star U^\star}) \quad \longrightarrow\!\!\!\!\!\!+ \quad (V^{\star \blacktriangleright Y^\star}, Y^\star)$$

Composition of Total Zig-Zag Strategies in $\mathbf{G}(\mathscr{S})$. Note that given $(\overline{u},\overline{x},\overline{v},\overline{y}) \in (U \times X \times V \times Y)^n$, we have $((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$ if and only if $\overline{v} = f_n(\overline{u},\blacktriangleright(\overline{y}))$ and $\overline{x} = F_n(\overline{u},\overline{y})$. Here, we have written $((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$ for $(\partial^{-1}(\overline{u},\overline{x}),\partial^{-1}(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma)$. We adopt the same convention in the following.

Consider positive full games $A = (U,X)$, $B = (V,Y)$ and $C = (W,Z)$, and $\mathbf{G}(\mathscr{S})$-morphisms

$$\begin{array}{rcccl} (f,F) & : & (U^\star, X^{\star U^\star}) & \longrightarrow\!\!\!\!\!\!+ & (V^{\star \blacktriangleright Y^\star}, Y^\star) \\ (g,G) & : & (V^\star, Y^{\star V^\star}) & \longrightarrow\!\!\!\!\!\!+ & (W^{\star \blacktriangleright Z^\star}, Z^\star) \end{array}$$

We want to define their composite

$$(h,H) \quad : \quad (U^\star, X^{\star U^\star}) \quad \longrightarrow\!\!\!\!\!\!+ \quad (W^{\star \blacktriangleright Z^\star}, Z^\star)$$

Write σ and τ for the total zig-zag strategies corresponding to resp. (f,F) and (g,G). Then the relational composite

$$\mathrm{HS}(\tau \circ \sigma) \quad = \quad \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$$

must be such that $((\overline{u},\overline{x}),(\overline{w},\overline{z})) \in \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$ if and only if there are $(\overline{v},\overline{y})$ such that

$$((\overline{u},\overline{x}),(\overline{v},\overline{y})) \in \mathrm{HS}(\sigma) \qquad \text{and} \qquad ((\overline{v},\overline{y}),(\overline{w},\overline{z})) \in \mathrm{HS}(\tau)$$

But this is possible iff the following equations are satisfied:

$$\begin{array}{rclcrcl} \overline{v} & = & f_n(\overline{u},\blacktriangleright(\overline{y})) & \qquad & \overline{w} & = & g_n(\overline{v},\blacktriangleright(\overline{z})) \\ \overline{x} & = & F_n(\overline{u},\overline{y}) & \qquad & \overline{y} & = & G_n(\overline{v},\overline{z}) \end{array}$$

The derived equation

$$\overline{y} \quad = \quad G_n(f_n(\overline{u}, \blacktriangleright(\overline{y})), \overline{z})$$

uniquely defines $\overline{y}$ from $\overline{u}$ and $\overline{z}$ as

$$\overline{y} \quad = \quad y(\overline{u}, \overline{z}) \quad = \quad \mathrm{fix}_n^Y(\boldsymbol{\lambda} y. G_n(f_n(\overline{u}, y), \overline{z}))$$

(We have here tacitly used the fact that $\xi \in (M^{\star \blacktriangleright M^\star})_n$ is completely determined by its last component ξ_n.) Now, since $\blacktriangleright(y(\overline{u}, \overline{z})) = y(\blacktriangleright\overline{u}, \blacktriangleright\overline{z})$, we can define

$$\begin{array}{rcl} h_{n+1}(\overline{u}u, \overline{z}) & := & g_{n+1}(f_{n+1}(\overline{u}u, y(\overline{u}, \overline{z}))) \ , \ \overline{z}) \\ H_{n+1}(\overline{u}u, \overline{z}z) & := & F_{n+1}(\overline{u}u, y(\overline{u}u, \overline{z}z)) \end{array}$$

More generally, given $\mathbf{G}(\mathscr{S})$-objects (U, X), (V, Y), (W, Z), and $\mathbf{G}(\mathscr{S})$-morphisms

$$\begin{array}{rcccl} (f, F) & : & (U, X^U) & \twoheadrightarrow & (V^{\blacktriangleright Y}, Y) \\ (g, G) & : & (V, Y^V) & \twoheadrightarrow & (W^{\blacktriangleright Z}, Z) \end{array}$$

we can define their composite

$$(g, G) \circ (f, F) \ = \ (h, H) \quad : \quad (U, X^U) \quad \twoheadrightarrow \quad (W^{\blacktriangleright Z}, Z)$$

as, modulo exponential transpose and again using the internal $\boldsymbol{\lambda}$-calculus of $\mathscr{S}$:

$$\begin{array}{rrcl} & h(u, z) & := & g(f(u, y(\blacktriangleright u, z)), z) \\ & H(z, u) & := & F(u, y(u, z)) \\ \text{where} & y(u, z) & := & \mathrm{fix}^Y(\boldsymbol{\lambda} y. G(f(u, y), z)) \end{array}$$

J.5 The Distributive Law ζ

It is possible to directly check that the composition described in the previous paragraph is associative and preserves identities. We can actually do better: The category **DZ** of simple zig-zag games can be obtained as a full subcategory of some category of zig-zag games in $\mathbf{G}(\mathscr{S})$ described as the Kleisli category of a distributive law ζ.

The law ζ is based on the constructions of §K. It distributes an oplax symmetric monoidal comonad obtained from Cor. I.5 over a (lax) symmetric monoidal monad obtained from Prop. I.4:

- The oplax symmetric monoidal comonad, denoted $T = (T, \epsilon, \delta)$, is obtained from Cor. I.5 by taking $H := \mathrm{Id}_{\mathscr{S}}$.
  Explicitely, $T(U, X) := (U, X^U)$ and the action of T on morphisms is given by:

  $$(f, F) : (U, X) \twoheadrightarrow (V, Y) \qquad \overset{T}{\longmapsto} \qquad (f \ , \ \boldsymbol{\lambda} h. F \circ h \circ f) : (U, X^U) \twoheadrightarrow (V, Y^V)$$

  The maps ϵ and δ are given by:

  $$\begin{array}{rclcrcl} (f_\epsilon, F_\epsilon) & := & (\mathrm{id}_U \ , \ \boldsymbol{\lambda} x. \boldsymbol{\lambda} \_. x) & : & (U, X^U) & \twoheadrightarrow & (U, X) \\ (f_\delta, F_\delta) & := & (\mathrm{id}_U \ , \ \boldsymbol{\lambda} h. \boldsymbol{\lambda} u. h(u, u)) & : & (U, X^U) & \twoheadrightarrow & (U, X^{U \times U}) \end{array}$$

– The (lax) symmetric monoidal monad, denoted $(-)^{\blacktriangleright} = ((-)^{\blacktriangleright}, \epsilon, \delta)$, is obtained from Prop. I.4 by taking $H(-) := \blacktriangleright(-)$ (see §J.1 and [5]).
Explicitely, $(U, X)^{\blacktriangleright} := (U^{\blacktriangleright X}, X)$ and the action of $(-)^{\blacktriangleright}$ on morphisms is given by:

$$(f, F) : (U, X) \longrightarrow\!\!\!\!\!\!+\;\, (V, Y) \quad \stackrel{(-)^{\blacktriangleright}}{\longmapsto} \quad (\boldsymbol{\lambda} h. f \circ h \circ \blacktriangleright F \ , \ F) : (U^{\blacktriangleright X}, X) \longrightarrow\!\!\!\!\!\!+\;\, (V^{\blacktriangleright Y}, Y)$$

The maps η and μ are given by:

$$\begin{array}{lcrcrcl} (f_\eta, F_\eta) & := & (\boldsymbol{\lambda} u. \boldsymbol{\lambda}\_.u \ , \ \mathrm{id}_X) & : & (U, X) & \longrightarrow\!\!\!\!\!\!+\;\, & (U^{\blacktriangleright X}, X) \\ (f_\mu, F_\mu) & := & (\boldsymbol{\lambda} h. \boldsymbol{\lambda} x. h(x, x) \ , \ \mathrm{id}_X) & : & (U^{\blacktriangleright X \times \blacktriangleright X}, X) & \longrightarrow\!\!\!\!\!\!+\;\, & (U^{\blacktriangleright X}, X) \end{array}$$

The distributive law

$$\zeta \quad : \quad T((-)^{\blacktriangleright}) \quad \Longrightarrow \quad (T(-))^{\blacktriangleright}$$

is given by

$$\zeta_A \quad = \quad (f^\zeta, F^\zeta) \quad : \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \quad \longrightarrow\!\!\!\!\!\!+\;\, \quad (U^{\blacktriangleright (X^U)}, X^U)$$

where the maps

$$f^\zeta \quad : \quad U^{\blacktriangleright X} \times \blacktriangleright(X^U) \quad \longrightarrow \quad U \qquad \text{and} \qquad F^\zeta \quad : \quad U^{\blacktriangleright X} \times X^U \quad \longrightarrow \quad X$$

are defined as follows. Let $f^\zeta_0(\theta_0, \bullet) := \theta_0$. Given $\xi \in (X^U)_n$, $\theta \in (U^{\blacktriangleright X})_n$ and $\theta' \in (U^{\blacktriangleright X})_{n+1}$,

$$\begin{array}{rcl} F^\zeta_n(\theta, \xi) & := & \mathrm{fix}^X_n(\xi \circ \theta) \\ f^\zeta_{n+1}(\theta', \xi) & := & \theta'_{n+1}(\mathrm{fix}^X_n(\xi \circ r_n(\theta'))) \\ & = & \theta'_{n+1}(F_n(r_n(\theta'), \xi)) \end{array}$$

The maps ζ_A form a distributive law of T over $(-)^{\blacktriangleright}$, which is moreover monoidal in the sense of Prop. M.7. These facts are summarized in the following Proposition whose proof is defered to §L.

Proposition J.3.

(i) The family of maps $\zeta_A : T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\!\!+\;\, (TA)^{\blacktriangleright}$ *forms a distributive law.*
(ii) Moreover, $\zeta_{(-)}$ *is monoidal in the sense of Prop. M.7, that is:*

$$\begin{array}{ccc} T(A^{\blacktriangleright} \otimes B^{\blacktriangleright}) & \xrightarrow{T(m^2_{A,B})} & T((A \otimes B)^{\blacktriangleright}) \\ \downarrow {\scriptstyle g^2_{A^{\blacktriangleright}, B^{\blacktriangleright}}} & & \downarrow {\scriptstyle \zeta_{A \otimes B}} \\ T(A^{\blacktriangleright}) \otimes T(B^{\blacktriangleright}) & & (T(A \otimes B))^{\blacktriangleright} \\ \downarrow {\scriptstyle \zeta_A \otimes \zeta_B} & & \downarrow {\scriptstyle (g^2_{A,B})^{\blacktriangleright}} \\ (TA)^{\blacktriangleright} \otimes (TB)^{\blacktriangleright} & \xrightarrow[m^2_{TA,TB}]{} & (TA \otimes TB)^{\blacktriangleright} \end{array} \qquad (3)$$

where (m^2, m^0) *is the (lax) strength of* $(-)^{\blacktriangleright}$ *defined as in Prop. I.4, and* (g^2, g^0) *is the oplax strength of* T *defined as in Cor. I.5, so that:*

– *For* $(-)^{\blacktriangleright}$*:*

$$
\begin{array}{rcl}
m^2_{A,B} & := & (\boldsymbol{\lambda}(h,k).(h \times k) \circ \langle \blacktriangleright(\pi_1), \blacktriangleright(\pi_2) \rangle \ , \ \mathrm{id}_{X \times Y}) \quad : \\
& & (U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \ , \ X \times Y) \quad \longrightarrow \quad ((U \times V)^{\blacktriangleright (X \times Y)} \ , \ X \times Y)
\end{array}
$$

(where $A = (U,X)$ *and* $B = (V,Y)$*), and* $m^0 := (\mathbf{1},\mathbf{1}) : (\mathbf{1},\mathbf{1}) \longrightarrow (\mathbf{1}^{\blacktriangleright \mathbf{1}}, \mathbf{1})$.

– *For* T*:*

$$
g^2_{A,B} \quad := \quad (\mathrm{id}_{U \times V} \ , \ \boldsymbol{\lambda}(h,k).(h \times k)) \quad : \quad (U \times V \ , \ (X \times Y)^{U \times V}) \quad \longrightarrow \quad (U \times V \ , \ X^U \times Y^V)
$$

(where $A = (U,X)$ *and* $B = (V,Y)$*), and* $g^0 := (\mathbf{1},\mathbf{1}) : (\mathbf{1},\mathbf{1}^{\mathbf{1}}) \longrightarrow (\mathbf{1},\mathbf{1})$.

It then follows from Prop. J.3 and Cor. M.8 that $\mathbf{Kl}(\zeta)$ is symmetric monoidal.

– Its monoidal product is that of $\mathbf{G}(\mathscr{S})$ on objects, so that

$$
(U,X) \otimes_{\mathbf{Kl}(\zeta)} (V,Y) = (U,X) \otimes (V,Y) = (U \times V, X \times Y) \qquad \text{and} \qquad \mathbf{I} = (\mathbf{1},\mathbf{1})
$$

and on maps, given $(f,F) \in \mathbf{Kl}(\zeta)[A_0,B_0]$ and $(g,G) \in \mathbf{Kl}(\zeta)[A_1,B_1]$, we let

$$
(f,F) \otimes_{\mathbf{Kl}(\zeta)} (g,G) \quad := \quad m^2_{B_0,B_1} \circ ((f,F) \otimes (g,G)) \circ g^2_{A_0,A_1}
$$

– The structure maps are the image under $\boldsymbol{\lambda} h^{A \to B}.\eta_B \circ h \circ \epsilon_A$ of the structure maps of $\mathbf{G}(\mathscr{S})$.

From now on, if no ambiguity arises, we write $\otimes$ for the monoidal product of $\mathbf{Kl}(\zeta)$.

We write $\mathbf{Kl}(\zeta^\star)$ for the full subcategory of $\mathbf{Kl}(\zeta)$ whose objects are of the form $(U^\star, X^\star)$. Together with §J.4, Prop. J.3 gives:

Proposition J.4. *The category* **DZ** *is equivalent to* $\mathbf{Kl}(\zeta^\star)$*.*

J.6 The Symmetric Monoidal Structure of DZ

Recall from Prop. J.4 that **DZ** is isomorphic to $\mathbf{Kl}(\zeta^\star)$ the full subcategory of $\mathbf{Kl}(\zeta)$ whose objects are of the form $(U^\star, X^\star)$.

Note that $\mathbf{I}$ is an object of $\mathbf{Kl}(\zeta^\star)$, as well as $A \otimes B$ as soon as A and B are objects of $\mathbf{Kl}(\zeta^\star)$. It thus follows from Prop. J.4, Prop. J.3 and Cor. M.8 that:

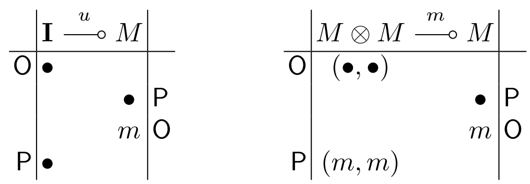
Proposition J.5. *Equipped with the above data, the category* $\mathbf{Kl}(\zeta^\star)$ *(and thus* **DZ***) is symmetric monoidal.*

J.7 Monoids and Comonoids in DZ

Thanks to Prop. M.11, we therefore get from Prop. J.3 and Prop. J.1:

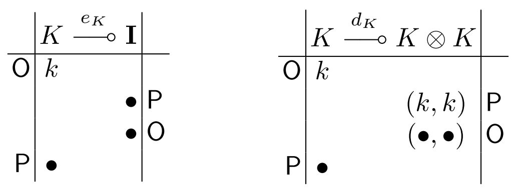
Proposition J.6 (Prop. A.9).

(i) *Objects of the form* $M = (\mathbf{1}, M)$ *equipped with structure maps*



are monoids in **DZ**.

(ii) *Objects of the form* $K = (K, \mathbf{1})$ *equipped with structure maps*



are comonoids in **DZ**.

J.8 The Base Category T

Proposition J.7 (Prop. B.1). *The category* $\mathbf{T}$ *embeds to* $\mathbf{Comon}(\mathbf{DZ}_{\mathfrak{D}})$ *via the functor* $\mathsf{E}_{\mathbf{T}}$ *mapping an object* Σ *of* $\mathbf{T}$ *to the comonoid* $(\Sigma, e_\Sigma, d_\Sigma)$ *and a morphism* $M : \mathbf{T}[\Gamma, \Sigma]$ *to itself.*

Proof (Proof of Proposition J.7). Fix $M \in \mathbf{T}[\Sigma, \Gamma]$, so that

$$M \quad \simeq \quad (f_M, \mathbf{1}) \quad : \quad (\Sigma, \mathbf{1}^{\Sigma}) \quad \twoheadrightarrow \quad (\Gamma^{\blacktriangleright(\mathbf{1}\times\mathfrak{D})}, \mathbf{1}\times\mathfrak{D})$$

The comonoid structure maps can be explicitelly defined as

$$e_\Sigma \quad \simeq \quad (\mathbf{1}, \mathbf{1}) \quad : \quad (\Sigma, \mathbf{1}^{\Sigma}) \quad \twoheadrightarrow \quad (\mathbf{1}^{\blacktriangleright(\mathbf{1}\times\mathfrak{D})}, \mathbf{1}\times\mathfrak{D})$$

and

$$d_\Sigma \quad \simeq \quad (\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a},\mathsf{a})}\,,\, \mathbf{1}) \quad : \quad (\Sigma, \mathbf{1}^{\Sigma}) \quad \twoheadrightarrow \quad ((\Sigma\times\Sigma)^{\blacktriangleright(\mathbf{1}\times\mathbf{1}\times\mathfrak{D})}\,,\, \mathbf{1}\times\mathbf{1}\times\mathfrak{D})$$

We check the required diagrams:

– First,

$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad M \quad} & \Gamma \\ {\scriptstyle d_\Sigma}\downarrow & & \downarrow{\scriptstyle d_\Gamma} \\ \Sigma \otimes \Sigma & \xrightarrow{\ M\otimes M\ } & \Gamma \otimes \Gamma \end{array}$$

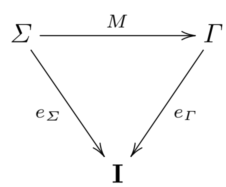
Note that all maps involved are $\mathbf{1}$ on the second component, so we only check the first one.
We then compute (leaving implicit the monad maps used for composition in $\mathbf{DZ}_{\mathfrak{D}}$):

$$(f_M \times f_M) \circ (\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a},\mathsf{a})}) \quad = \quad \boldsymbol{\lambda}\blacktriangleright(p).\boldsymbol{\lambda}\bar{\mathsf{a}}.\langle f_M(\blacktriangleright(p),\bar{\mathsf{a}})\ ,\ f_M(\blacktriangleright(p),\bar{\mathsf{a}})\rangle$$

and we are done since on the other hand

$$(\boldsymbol{\lambda}\_.\boldsymbol{\lambda}\bar{\mathsf{a}}.\overline{(\mathsf{a},\mathsf{a})}) \circ f_M \quad = \quad \boldsymbol{\lambda}\blacktriangleright(p).\boldsymbol{\lambda}\bar{\mathsf{a}}.(f_M(\blacktriangleright(p),\bar{\mathsf{a}}), f_M(\blacktriangleright(p),\bar{\mathsf{a}}))$$

– Second, the coherence diagram



trivially holds since all involved maps are in the second component are $\mathbf{1}$, and, for the first component, since $\mathbf{1}$ is terminal in $\mathscr{S}$.

K Proof of Proposition I.4

In this appendix we give a proof of Prop. I.4. We first recall its statment.

Assume that $\mathbb{C}$ is Cartesian closed, and fix a functor $H : \mathbb{C} \to \mathbb{C}$. Recall (from e.g. [27, §5.2]) that H lifts in a unique way to an oplax symmetric monoidal functor, with strength

$$\mathsf{t}^2_{A,B} \quad := \quad \langle H(\pi_1), H(\pi_2)\rangle \quad : \quad H(A \times B) \quad \longrightarrow \quad HA \times HB$$

and

$$\mathsf{t}^0 \quad := \quad \mathbf{1}_{H\mathbf{1}} \quad : \quad H\mathbf{1} \quad \longrightarrow \quad \mathbf{1}$$

Note that the naturality of $\mathsf{t}^2_{(-),(-)}$, that is

$$(H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2)\rangle \quad = \quad \langle H(\pi_1), H(\pi_2)\rangle \circ H(f \times g)$$

follows from the universality property of the Cartesian product since (say)

$$\pi_1 \circ (H(f) \times H(g)) \circ \langle H(\pi_1), H(\pi_2)\rangle \quad = \quad H(f \circ \pi_1) \quad = \quad H(\pi_1 \circ (f \times g))$$

Consider now the functor

$$(-)^H \quad : \quad \mathbf{G}(\mathbb{C}) \quad \longrightarrow \quad \mathbf{G}(\mathbb{C})$$

defined as

$$(U, X)^H \quad := \quad (U^{HX}, X)$$

and

$$(f,F)^H \quad := \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)\ ,\ F) \quad : \quad (U^{HX},X) \multimap (V^{HY},Y)$$

(where $(f,F) : (U,X) \multimap (V,Y)$), and the maps

$$\begin{array}{lllllll} \eta_{(U,X)} & = & (f_\eta, F_\eta) & := & (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u\ ,\ \mathrm{id}_X) & : & (U,X) \multimap (U^{HX},X) \\ \mu_{(U,X)} & = & (f_\mu, F_\mu) & := & (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)\ ,\ \mathrm{id}_X) & : & (U^{HX\times HX},X) \multimap (U^{HX},X) \end{array}$$

Proposition K.1 (Prop. I.4). *$((-)^H,\eta,\mu)$ is a (lax) symmetric monoidal monad, with strength*

$$\begin{array}{llll} m^2_{A,B} & = & (f^2_{A,B}, F^2_{A,B}) \quad := \quad (\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathrm{t}^2_{X,Y}\ ,\ \mathrm{id}_{X\times Y}) & : \\ & & (U^{HX}\times V^{HY}\ ,\ X\times Y) \quad \longrightarrow \quad ((U\times V)^{H(X\times Y)}\ ,\ X\times Y) & \end{array}$$

(where $A = (U,X)$ and $B = (V,Y)$), and

$$m^0 \quad := \quad (\mathbf{1},\mathbf{1}) \quad : \quad (\mathbf{1},\mathbf{1}) \quad \longrightarrow \quad (\mathbf{1}^{H\mathbf{1}},\mathbf{1})$$

K.1 $(-)^H$ is a lax symmetric monoidal functor

$(-)^H$ is a functor. First, given $A = (U,X)$ we have

$$(\mathrm{id}_A)^H \quad = \quad (\boldsymbol{\lambda} h.\mathrm{id}_U \circ h \circ H(\mathrm{id}_X)\ ,\ \mathrm{id}_X) \quad = \quad (\boldsymbol{\lambda} h.h\ ,\ \mathrm{id}_X) \quad = \quad \mathrm{id}_{A^H}$$

Moreover, given $(f,F) : (U,X) \multimap (V,Y)$ and $(g,G) : (V,Y) \multimap (W,Z)$, we have

$$\begin{array}{lll} ((g,G)\circ(f,F))^H & = & (g\circ f\ ,\ F\circ G)^H \quad = \quad (\boldsymbol{\lambda} h.g\circ f\circ h\circ H(F\circ G)\ ,\ F\circ G) \\ & & \quad = \quad (\boldsymbol{\lambda} h.g\circ h\circ HG, G)\circ(\boldsymbol{\lambda} h.f\circ h\circ HF, F) \end{array}$$

since

$$\boldsymbol{\lambda} h.g\circ f\circ h\circ H(F\circ G) \quad = \quad \boldsymbol{\lambda} h.g\circ f\circ h\circ H(F)\circ H(G) \quad = \quad \boldsymbol{\lambda} h.(\boldsymbol{\lambda} k.g\circ k\circ H(G))(f\circ h\circ H(F))$$

The maps $m^2_{(-),(-)}$ are natural. We have to check that given $(f,F) : (U,X) \multimap (V,Y)$ and $(g,G) : (U',X') \multimap (V',Y')$ we have

$$m^2_{B,B'}\circ((f,F)^H\otimes(g,G)^H) \quad = \quad ((f,F)\otimes(g,G))^H\circ m^2_{A,A'}$$

(where $A=(U,X)$, $B=(V,Y)$, $A'=(U',X')$ and $B'=(V',Y')$). We compute

$$
\begin{array}{rcl}
m^2_{B,B'}\circ((f,F)^H\otimes(g,G)^H) & = & m^2_{B,B'}\circ((\boldsymbol{\lambda}h.f\circ h\circ H(F),F)\otimes(\boldsymbol{\lambda}k.g\circ k\circ H(G),G))\\
& = & m^2_{B,B'}\circ((\boldsymbol{\lambda}h.f\circ h\circ H(F))\times(\boldsymbol{\lambda}k.g\circ k\circ H(G))\ ,\ F\times G)\\
& = & ((\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{Y,Y'})\circ((\boldsymbol{\lambda}h.f\circ h\circ H(F))\times(\boldsymbol{\lambda}k.g\circ k\circ H(G)))\ ,\ F\times G\\
& = & ((\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{Y,Y'})\circ(\boldsymbol{\lambda}(h,k).\langle f\circ h\circ H(F)\ ,\ g\circ k\circ H(G)\rangle)\ ,\ F\times G\\
& = & (\boldsymbol{\lambda}(h,k).((f\circ h\circ H(F))\times(g\circ k\circ H(G)))\circ \mathsf{t}^2_{Y,Y'}\ ,\ F\times G)\\
& = & (\boldsymbol{\lambda}(h,k).(f\times g)\circ(h\times k)\circ(H(F)\times H(G))\circ \mathsf{t}^2_{Y,Y'}\ ,\ F\times G)\\
& = & (\boldsymbol{\lambda}(h,k).(f\times g)\circ(h\times k)\circ \mathsf{t}^2_{X,X'}\circ H(F\times G)\ ,\ F\times G)\\
& = & (\boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda}p.(f\times g)\circ p\circ H(F\times G))\circ((h\times k)\circ \mathsf{t}^2_{X,X'})\ ,\ F\times G)\\
& = & ((f,F)\otimes(g,G))^H\circ m^2_{A,A'}
\end{array}
$$

$(-)^H$ *is lax symmetric monoidal.* Note that $(-)^H$ is the identity on the second components, so we only have to check diagrams for the first components.

– The associativity diagram leads to check

$$
\begin{array}{ccc}
(U^{HX}\times V^{HY})\times W^{HZ} & \xrightarrow{\alpha_{U^{HX},V^{HY},W^{HZ}}} & U^{HX}\times(V^{HY}\times W^{HZ})\\
\big\downarrow{\scriptstyle (\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y})\times \mathrm{id}_{W^{HZ}}} & & \big\downarrow{\scriptstyle \mathrm{id}_{U^{HX}}\times(\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{Y,Z})}\\
(U\times V)^{H(X\times Y)}\times W^{HZ} & & U^{HX}\times(V\times W)^{H(Y\times Z)}\\
\big\downarrow{\scriptstyle \boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X\times Y,Z}} & & \big\downarrow{\scriptstyle \boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{X,Y\times Z}}\\
((U\times V)\times W)^{H((X\times Y)\times Z)} & \xrightarrow[\boldsymbol{\lambda}h.\alpha_{U,V,W}\circ h\circ H(\alpha^{-1}_{X,Y,Z})]{} & (U\times(V\times W))^{H(X\times(Y\times Z))}
\end{array}
$$

(where $A=(U,X)$, $B=(V,Y)$ and $C=(W,Z)$). Note that since $\mathbb{C}$ is Cartesian closed:

$$\alpha \quad = \quad \langle \pi_1\circ\pi_1\ ,\ \langle\pi_2\circ\pi_1,\pi_2\rangle\rangle \quad = \quad \boldsymbol{\lambda}((u,v),w).(u,(v,w))$$

We have to check

$$\boldsymbol{\lambda}((h,k),l).\alpha_{U,V,W}\circ(((h\times k)\circ \mathsf{t}^2_{X,Y})\times l)\circ \mathsf{t}^2_{X\times Y,Z}\circ H(\alpha^{-1}_{X,Y,Z}) \quad = \quad \boldsymbol{\lambda}((h,k),l).(h\times((k\times l)\circ \mathsf{t}^2_{Y,Z}))\circ \mathsf{t}^2_{X,Y\times Z}$$

But we are done since it follows from the universal property of the Cartesian product of $\mathbb{C}$ that we have

$$
\begin{array}{ccc}
(HX\times HY)\times HZ & \xrightarrow{\alpha_{HX,HY,HZ}} & HX\times(HY\times HZ)\\
\big\uparrow{\scriptstyle \mathsf{t}^2_{X,Y}\times \mathrm{id}_{HZ}} & & \big\uparrow{\scriptstyle \mathrm{id}_{HX}\times \mathsf{t}^2_{Y,Z}}\\
H(X\times Y)\times HZ & & HX\times H(Y\times Z)\\
\big\uparrow{\scriptstyle \mathsf{t}^2_{X\times Y,Z}} & & \big\uparrow{\scriptstyle \mathsf{t}^2_{X,Y\times Z}}\\
H((X\times Y)\times Z) & \xleftarrow[H(\alpha^{-1}_{X,Y,Z})]{} & H(X\times(Y\times Z))
\end{array}
$$

- The unit diagrams are dealt-with similarly. We only check the diagram for the unit $\lambda_{(-)}$, which lead to check

$$
\begin{array}{ccc}
\mathbf{1} \times U^{HX} & \xrightarrow{\lambda_{U^{HX}}} & U^{HX} \\
{\scriptstyle \mathbf{1}\times \mathrm{id}_{U^{HX}}} \downarrow & & \uparrow {\scriptstyle \boldsymbol{\lambda} h.\lambda_U \circ h \circ H(\lambda_X^{-1})} \\
\mathbf{1}^{\mathbf{1}} \times U^{HX} & \xrightarrow[\boldsymbol{\lambda}(h,k).(h\times k)\circ \mathsf{t}^2_{\mathbf{1},X}]{} & (\mathbf{1} \times U)^{H(\mathbf{1}\times X)}
\end{array}
$$

Since $\lambda_{(-)} = \pi_2$, we have to show

$$\boldsymbol{\lambda}(\bullet, h).h \quad = \quad \boldsymbol{\lambda}(\bullet, h).\lambda_U \circ (\bullet \times h) \circ \mathsf{t}^2_{\mathbf{1},X} \circ H(\lambda_X^{-1})$$

It follows from the unversal property of the Cartesian product of $\mathbb{C}$ that we have have

$$
\begin{array}{ccc}
\mathbf{1} \times HX & \xleftarrow{\lambda^{-1}_{HX}} & HX \\
{\scriptstyle \mathbf{1}\times \mathrm{id}_{HX}} \uparrow & & \downarrow {\scriptstyle H(\lambda_X^{-1})} \\
H\mathbf{1} \times HX & \xleftarrow[\mathsf{t}^2_{\mathbf{1},X}]{} & H(\mathbf{1} \times X)
\end{array}
$$

We are therefore lead to check

$$\boldsymbol{\lambda}(\bullet, h).h \quad = \quad \boldsymbol{\lambda}(\bullet, h).\lambda_U \circ (\bullet \times h) \circ \lambda^{-1}_{HX}$$

and we are done since $\lambda^{-1}_{(-)} = \langle \mathbf{1}, \mathrm{id}_{(-)} \rangle$.
- The symmetry diagram is dealt-with similarly.

K.2 $((-)^{\boldsymbol{H}}, \boldsymbol{\eta}, \boldsymbol{\mu})$ is a monad

The maps $\eta_{(-)}$ are natural. Let $(f, F) : (U, X) \longrightarrow\!\!\!\!\!\!+\;\; (V, Y)$. We have to check

$$\eta_{(V,Y)} \circ (f, F) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)\ ,\ F) \circ \eta_{(U,X)}$$

which amounts to

$$(\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u) \circ f \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)) \circ (\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.u)$$

that is

$$\boldsymbol{\lambda} u.\boldsymbol{\lambda}\_.f(u) \quad = \quad \boldsymbol{\lambda} u.f \circ (\boldsymbol{\lambda}\_.u) \circ H(F)$$

and we are done.

The maps $\mu_{(-)}$ are natural. Let $(f, F) : (U, X) \longrightarrow\!\!\!\!\!\!\!\!\longrightarrow (V, Y)$. We have to check

$$\mu_{(V,Y)} \circ (\boldsymbol{\lambda} h.(\boldsymbol{\lambda} k.f \circ k \circ H(F)) \circ h \circ H(F)\ ,\ F) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)\ ,\ F) \circ \mu_{(U,X)}$$

which amounts to

$$(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.f \circ (h(H(F)(x))) \circ H(F)) \quad = \quad (\boldsymbol{\lambda} h.f \circ h \circ H(F)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x))$$

that is

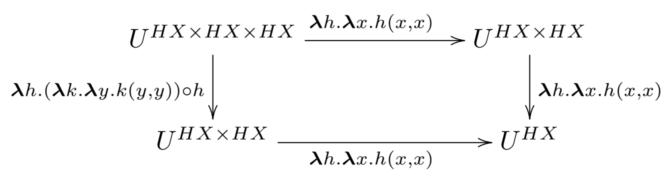
$$(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \circ (\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.\boldsymbol{\lambda} y.f(h(H(F)(x)), H(F)(y))) \quad = \quad \boldsymbol{\lambda} h.f \circ (\boldsymbol{\lambda} x.h(x,x)) \circ H(F)$$

which reduces to

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} x.\boldsymbol{\lambda} y.f(h(H(F)(x)), H(F)(y)))(x,x) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.f(h(H(F)(x), H(F)(x)))$$

and we are done.

Associativity Law. Since $\mu_{(-)}$ is the identity on the second component, we only have to check



that is

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} y.(\boldsymbol{\lambda} x.h(x,x))(y,y) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.((\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y)) \circ h)(x,x)$$

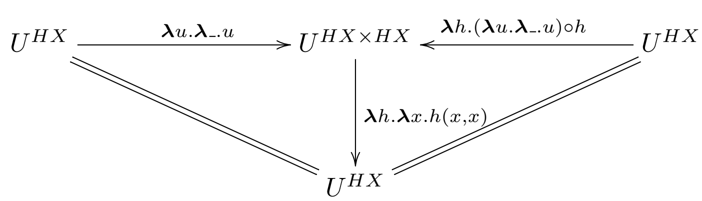
We compute

$$\boldsymbol{\lambda} h.\boldsymbol{\lambda} y.(\boldsymbol{\lambda} x.h(x,x))(y,y) \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} y.h(y,y,y)$$

and we are done since

$$\begin{aligned} \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.((\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y)) \circ h)(x,x) \quad &= \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} z.\boldsymbol{\lambda} y.h(z)(y,y))(x,x) \\ &= \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.(\boldsymbol{\lambda} y.h(x)(y,y))x \quad = \quad \boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x,x) \end{aligned}$$

Unit Laws. Since $\eta_{(-)}$ and $\mu_{(-)}$ are the identity on the second component, we only have to check



We are done since

$$(\boldsymbol{\lambda}h.\boldsymbol{\lambda}x.h(x,x))\circ(\boldsymbol{\lambda}u.\boldsymbol{\lambda}\_.u) \quad = \quad \boldsymbol{\lambda}u.\boldsymbol{\lambda}x.(\boldsymbol{\lambda}\_.u)(x,x) \quad = \quad \boldsymbol{\lambda}u.\boldsymbol{\lambda}x.ux \quad = \quad \mathrm{id}_{U^{HX}}$$

and

$$\begin{array}{rcl}
(\boldsymbol{\lambda}h.\boldsymbol{\lambda}x.h(x,x)) \circ (\boldsymbol{\lambda}h.(\boldsymbol{\lambda}u.\boldsymbol{\lambda}_.u) \circ h) & = & \boldsymbol{\lambda}h.\boldsymbol{\lambda}x.((\boldsymbol{\lambda}u.\boldsymbol{\lambda}_.u) \circ h)(x,x) \\
& = & \boldsymbol{\lambda}h.\boldsymbol{\lambda}x.(\boldsymbol{\lambda}y.\boldsymbol{\lambda}_.h(y))(x,x) \\
& = & \boldsymbol{\lambda}h.\boldsymbol{\lambda}x.(\boldsymbol{\lambda}_.h(x))x \\
& = & \boldsymbol{\lambda}h.\boldsymbol{\lambda}x.hx \\
& = & \mathrm{id}_{U^{HX}}
\end{array}$$

K.3 $((-)^{H}, \eta, \mu)$ is lax symmetric monoidal

It remains to show that η and μ are lax monoidal natural transformations. Once again, we only check the second components, which amount to the following.

$\eta_{(-)}$ *is lax monoidal.* We have to check

$$\begin{array}{ccc}
U \times V & \xrightarrow{(\boldsymbol{\lambda}u.\boldsymbol{\lambda}_.u)\times(\boldsymbol{\lambda}v.\boldsymbol{\lambda}_.v)} & U^{HX} \times V^{HY} \\
\| & & \downarrow{\scriptstyle \boldsymbol{\lambda}(h,k).(h\times k)\circ \mathrm{t}^2_{X,Y}} \\
U \times V & \xrightarrow[\boldsymbol{\lambda}p.\boldsymbol{\lambda}_.p]{} & (U \times V)^{H(X\times Y)}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
 & \mathbf{1} & \\
 /\!/ & & \searrow{\scriptstyle \mathbf{1}} \\
\mathbf{1} & \xrightarrow[\boldsymbol{\lambda}u.\boldsymbol{\lambda}_.u]{} & \mathbf{1}^{H\mathbf{1}}
\end{array}$$

The second diagram is obvious. The first one amounts to

$$\boldsymbol{\lambda}p.\boldsymbol{\lambda}\_.p \quad = \quad \boldsymbol{\lambda}(u,v).((\boldsymbol{\lambda}\_.u) \times (\boldsymbol{\lambda}\_.v)) \circ \langle H(\pi_1), H(\pi_2)\rangle$$

and we are done since

$$\boldsymbol{\lambda}(u,v).((\boldsymbol{\lambda}\_.u)\times(\boldsymbol{\lambda}\_.v))\circ\langle H(\pi_1), H(\pi_2)\rangle \quad = \quad \boldsymbol{\lambda}(u,v).\langle \boldsymbol{\lambda}\_.u, \boldsymbol{\lambda}\_.v\rangle \quad = \quad \boldsymbol{\lambda}(u,v).\boldsymbol{\lambda}\_.\langle u,v\rangle \quad = \quad \boldsymbol{\lambda}p.\boldsymbol{\lambda}\_.p$$

$\mu_{(-)}$ *is lax monoidal.*

– Preservation of the binary strength amounts to

$$\begin{array}{ccc}
U^{HX\times HX} \times V^{HY\times HY} & \xrightarrow{(\boldsymbol{\lambda}h.\boldsymbol{\lambda}x.h(x,x))\times(\boldsymbol{\lambda}k.\boldsymbol{\lambda}y.k(y,y))} & U^{HX} \times V^{HY} \\
{\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\lambda}(h,k).(h\times k)\circ \mathrm{t}^2_{X,Y}} \\
(U \times V)^{H(X\times Y)\times H(X\times Y)} & \xrightarrow[\boldsymbol{\lambda}h.\boldsymbol{\lambda}x.h(x,x)]{} & (U \times V)^{H(X\times Y)}
\end{array}$$

where n is the first component of $(m^2_{A,B})^H \circ m^2_{A^H,B^H}$ (for $A = (U, X)$ and $B = (V, Y)$), so that

$$
\begin{aligned}
n &= (\boldsymbol{\lambda} l.((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \circ l) \circ ((\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \\
&= \boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda}(h',k').(h' \times k') \circ \mathsf{t}^2_{X,Y}) \circ ((h \times k) \circ \mathsf{t}^2_{X,Y}) \\
&= \boldsymbol{\lambda}(h,k).(\boldsymbol{\lambda}(h',k').(h' \times k') \circ \mathsf{t}^2_{X,Y}) \circ \langle h \circ H(\pi_1)\ ,\ k \circ H(\pi_2) \rangle \\
&= \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.((h(H(\pi_1)p)) \times (k(H(\pi_2)p))) \circ \mathsf{t}^2_{X,Y} \\
&= \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.((h(H(\pi_1)p)) \times (k(H(\pi_2)p))) \circ \langle H(\pi_1)\ ,\ H(\pi_2) \rangle \\
&= \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda}(p,q).\langle h(H(\pi_1)p), H(\pi_1)q)\ ,\ k(H(\pi_2)p, H(\pi_2)q) \rangle
\end{aligned}
$$

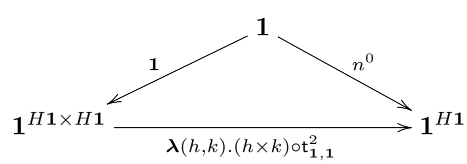
and therefore

$$
\begin{aligned}
(\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \circ n &= \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} x.n(h,k)(x,x) \\
&= \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} x.\langle h(H(\pi_1)x, H(\pi_1)x)\ ,\ k(H(\pi_2)x, H(\pi_2)x) \rangle
\end{aligned}
$$

But now we are done since on the other hand,

$$
\begin{aligned}
& (\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{X,Y}) \circ ((\boldsymbol{\lambda} h.\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} k.\boldsymbol{\lambda} y.k(y,y))) \\
= {} & \boldsymbol{\lambda}(h,k).((\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} y.k(y,y))) \circ \mathsf{t}^2_{X,Y} \\
= {} & \boldsymbol{\lambda}(h,k).((\boldsymbol{\lambda} x.h(x,x)) \times (\boldsymbol{\lambda} y.k(y,y))) \circ \langle H(\pi_1), H(\pi_2) \rangle \\
= {} & \boldsymbol{\lambda}(h,k).\boldsymbol{\lambda} p.\langle h(H(\pi_1)p, H(\pi_1)p)\ ,\ k(H(\pi_2)p, H(\pi_2)p) \rangle
\end{aligned}
$$

– Preservation of the unit strength amounts to



where n^0 is the first component of $(m^0)^H \circ m^0$, so that $n^0 = (\boldsymbol{\lambda} h.\mathbf{1} \circ h) \circ \mathbf{1} = \mathbf{1}$ and we are done since

$$
(\boldsymbol{\lambda}(h,k).(h \times k) \circ \mathsf{t}^2_{\mathbf{1},\mathbf{1}}) \circ \mathbf{1} \quad = \quad \mathbf{1}
$$

L Proof of Proposition J.3

This appendix is devoted to the proof of Prop. J.3. We first recall its statment.

Proposition L.1 (Prop. J.3).

(i) The family of maps $\zeta_A : T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\rightarrow (TA)^{\blacktriangleright}$ forms a distributive law.

(ii) Moreover, $\zeta_{(-)}$ is monoidal in the sense of Prop. M.7, that is:

$$\begin{array}{ccc} T(A^{\blacktriangleright} \otimes B^{\blacktriangleright}) & \xrightarrow{T(m^2_{A,B})} & T((A \otimes B)^{\blacktriangleright}) \\ \downarrow{\scriptstyle g^2_{A^{\blacktriangleright},B^{\blacktriangleright}}} & & \downarrow{\scriptstyle \zeta_{A\otimes B}} \\ T(A^{\blacktriangleright}) \otimes T(B^{\blacktriangleright}) & & (T(A \otimes B))^{\blacktriangleright} \\ \downarrow{\scriptstyle \zeta_A \otimes \zeta_B} & & \downarrow{\scriptstyle (g^2_{A,B})^{\blacktriangleright}} \\ (TA)^{\blacktriangleright} \otimes (TB)^{\blacktriangleright} & \xrightarrow[m^2_{TA,TB}]{} & (TA \otimes TB)^{\blacktriangleright} \end{array} \tag{4}$$

where (m^2, m^0) is the (lax) strength of $(-)^{\blacktriangleright}$ defined as in Prop. I.4, and (g^2, g^0) is the oplax strength of T defined as in Cor. I.5, so that:

– *For $(-)^{\blacktriangleright}$:*

$$\begin{array}{rcl} m^2_{A,B} & := & (\boldsymbol{\lambda}(h,k).(h \times k) \circ \langle \blacktriangleright(\pi_1), \blacktriangleright(\pi_2)\rangle \ , \ \mathrm{id}_{X\times Y}) \quad : \\ & & (U^{\blacktriangleright X} \times V^{\blacktriangleright Y} \ , \ X \times Y) \quad \longrightarrow \quad ((U \times V)^{\blacktriangleright (X \times Y)} \ , \ X \times Y) \end{array}$$

(where $A = (U,X)$ and $B = (V,Y)$), and $m^0 := (\mathbf{1},\mathbf{1}) : (\mathbf{1},\mathbf{1}) \longrightarrow (\mathbf{1}^{\blacktriangleright \mathbf{1}}, \mathbf{1})$.

– *For T:*

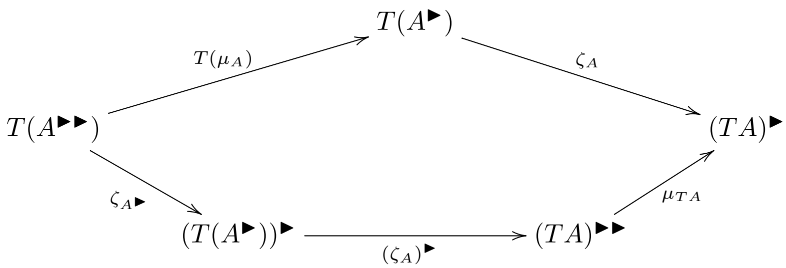
$$g^2_{A,B} \quad := \quad (\mathrm{id}_{U\times V} \ , \ \boldsymbol{\lambda}(h,k).(h\times k)) \quad : \quad (U\times V \ , \ (X\times Y)^{U\times V}) \quad \longrightarrow \quad (U\times V \ , \ X^U \times Y^V)$$

(where $A = (U,X)$ and $B = (V,Y)$), and $g^0 := (\mathbf{1},\mathbf{1}) : (\mathbf{1},\mathbf{1}^{\mathbf{1}}) \longrightarrow (\mathbf{1},\mathbf{1})$.

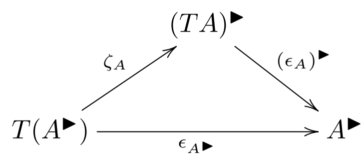
L.1 Proof of Proposition L.1.(i)

We have to check that $\zeta : T((-)^{\blacktriangleright}) \to (T-)^{\blacktriangleright}$ is natural and that the following four coherence diagrams commute (see e.g. [13]):

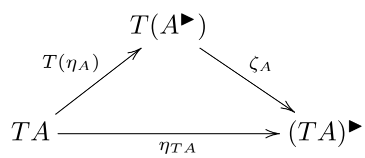
$$\begin{array}{ccccc} & & (TA)^{\blacktriangleright} & & \\ & \nearrow^{\zeta_A} & & \searrow^{(\delta_A)^{\blacktriangleright}} & \\ T(A^{\blacktriangleright}) & & & & (TTA)^{\blacktriangleright} \\ \searrow_{\delta_{A^{\blacktriangleright}}} & & & & \nearrow_{\zeta_{TA}} \\ & TT(A^{\blacktriangleright}) & \xrightarrow[T\zeta_A]{} & T((TA)^{\blacktriangleright}) & \end{array} \tag{5}$$



(6)



(7)



(8)

Recall that T is the comonad $T = (T, \epsilon, \delta)$ and that $(-)^{\blacktriangleright}$ is the monad $((-)^{\blacktriangleright}, \eta, \mu)$ on $\mathbf{G}(\mathscr{S})$. We repeat the definitions of the functors T and $(-)^{\blacktriangleright}$:

$$(f,F):(U,X) \multimap (V,Y) \quad \overset{T}{\longmapsto} \quad (f\ ,\ \boldsymbol{\lambda} h.F\circ h\circ f):\quad (U,X^U) \multimap \quad (V,Y^V)$$
$$(f,F):(U,X) \multimap (V,Y) \quad \overset{(-)^{\blacktriangleright}}{\longmapsto} \quad (\boldsymbol{\lambda} h.f\circ h\circ {\blacktriangleright}F\ ,\ F):(U^{\blacktriangleright X},X) \ \multimap (V^{\blacktriangleright Y},Y)$$

and of the natural maps η and μ:

$$\begin{array}{rcrcl} (f_\eta, F_\eta) & : & (U,X) & \multimap & (U^{\blacktriangleright X}, X) \\ (f_\mu, F_\mu) & : & (U^{\blacktriangleright X \times \blacktriangleright X}, X) & \multimap & (U^{\blacktriangleright X}, X) \end{array}$$

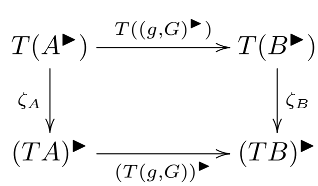
where $F_\eta = F_\mu = \mathrm{id}_X$, $f_\eta(u,x) = u$ and $f_\mu(h,x) = h(x,x)$.
Moreover, the natural maps ϵ and δ are given by

$$\begin{array}{rcrcl} (f_\epsilon, F_\epsilon) & : & (U,X^U) & \multimap & (U,X) \\ (f_\delta, F_\delta) & : & (U,X^U) & \multimap & (U,X^{U\times U}) \end{array}$$

where $f_\epsilon = f_\delta = \mathrm{id}_U$, $F_\epsilon(u,x) = x$ and $F_\delta(h,u) = h(u,u)$.

We check in turn the required diagrams.

Lemma L.2. *ζ is natural, that is, given $(g,G) : A \multimap B$, we have*



Proof. Let $A = (U, X)$ and $B = (V, Y)$, and consider $(g, G) : (U, X) \longrightarrow\!\!\!\!\!\mapsto (V, Y)$. Note that

$$\begin{array}{rclcrcl}
(g,G)^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}G\ ,\ G) & : & (U^{\blacktriangleright X}, X) & \longrightarrow\!\!\!\!\!\mapsto & (V^{\blacktriangleright Y}, Y) \\
T((g,G)^{\blacktriangleright}) & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}G\ ,\ \boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G)) & : & (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) & \longrightarrow\!\!\!\!\!\mapsto & (V^{\blacktriangleright Y}, Y^{V^{\blacktriangleright Y}}) \\
T(g,G) & = & (g\ ,\ \boldsymbol{\lambda} h.Ghg) & : & (U, X^U) & \longrightarrow\!\!\!\!\!\mapsto & (V, Y^V) \\
(T(g,G))^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.gh{\blacktriangleright}(\boldsymbol{\lambda} h.Ghg)\ ,\ \boldsymbol{\lambda} h.Ghg) & : & (U^{\blacktriangleright (X^U)}, X^U) & \longrightarrow\!\!\!\!\!\mapsto & (V^{\blacktriangleright (Y^V)}, Y^V)
\end{array}$$

We have to show that

$$(T(g,G))^{\blacktriangleright} \circ \zeta_A \quad = \quad \zeta_B \circ T((g,G)^{\blacktriangleright})$$

that is

$$(\boldsymbol{\lambda} h.gh{\blacktriangleright}(\boldsymbol{\lambda} h.Ghg)){\circ}f^{\zeta_A} \quad = \quad f^{\zeta_B}{\circ}(\boldsymbol{\lambda} h.gh{\blacktriangleright}G) \qquad \text{and} \qquad F^{\zeta_A}{\circ}(\boldsymbol{\lambda} h.Ghg) \quad = \quad \boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G){\circ}F^{\zeta_B}$$

For the first equation, which has type $U^{\blacktriangleright X} \to V^{\blacktriangleright (Y^V)}$, given $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$ and $\xi_n \in (Y^V)_n$, one has to show the following (where some $\circ$ are replaced by juxtaposition)

$$((\boldsymbol{\lambda} h.g_{n+1}h{\blacktriangleright}(\boldsymbol{\lambda} h.G_{n+1}hg_{n+1})){\circ}f^{\zeta_A}_{n+1})(\theta_{n+1})(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1}{\circ}(\boldsymbol{\lambda} h.g_{n+1}h{\blacktriangleright}G_{n+1}))(\theta_{n+1})(\xi_n)$$

that is

$$((\boldsymbol{\lambda} h.g_{n+1}{\circ}h{\circ}(\boldsymbol{\lambda} h.G_nhg_n))(f^{\zeta_A}_{n+1}(\theta_{n+1})))(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1}((\boldsymbol{\lambda} h.g_{n+1}{\circ}h{\circ}G_n)(\theta_{n+1})))(\xi_n)$$

that is

$$(g_{n+1} \circ (f^{\zeta_A}_{n+1}(\theta_{n+1})) \circ (\boldsymbol{\lambda} h.G_nhg_n))(\xi_n) \quad = \quad (f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n))(\xi_n)$$

that is

$$g_{n+1}(f^{\zeta_A}_{n+1}(\theta_{n+1})((\boldsymbol{\lambda} h.G_nhg_n)\xi_n)) \quad = \quad f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n\ ,\ \xi_n)$$

that is

$$g_{n+1}(f^{\zeta_A}_{n+1}(\theta_{n+1}\ ,\ G_n\xi_ng_n)) \quad = \quad f^{\zeta_B}_{n+1}(g_{n+1}\theta_{n+1}G_n\ ,\ \xi_n)$$

that is

$$g_{n+1} \circ \theta_{n+1} \circ \mathrm{fix}_n(G_n\xi_n\theta_n) \quad = \quad g_{n+1} \circ \theta_{n+1} \circ G_n \circ \mathrm{fix}_n(\xi_ng_n\theta_nG_{n-1})$$

which is easily seens to hold, when unfolding the fixpoints, thanks to associativity of composition.

The second equation, of type $Y^V \to X^{U^{\blacktriangleright X}}$, amounts, for $\xi_n \in (Y^V)_n$ and $\theta_n \in (U^{\blacktriangleright X})_n$, to the following (where some $\circ$ are replaced by juxtaposition)

$$F^{\zeta_A}_n(G_n\xi_ng_n\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} h.Gh(\boldsymbol{\lambda} h.gh{\blacktriangleright}G))(F^{\zeta_B}(\xi_n)))(\theta_n)$$

that is

$$F_n^{\zeta_A}(G_n \xi_n g_n \ , \ \theta_n) \quad = \quad (G_n \circ (F_n^{\zeta_B}(\xi_n)) \circ (\boldsymbol{\lambda} h. g_n h \blacktriangleright G_n))(\theta_n)$$

that is

$$F_n^{\zeta_A}(G_n \xi_n g_n \ , \ \theta_n) \quad = \quad G_n(F_n^{\zeta_B}(\xi_n)((\boldsymbol{\lambda} h. g_n h \blacktriangleright G_n)(\theta_n)))$$

that is

$$F_n^{\zeta_A}(G_n \xi_n g_n \ , \ \theta_n) \quad = \quad G_n(F_n^{\zeta_B}(\xi_n \ , \ g_n \theta_n \blacktriangleright G_n))$$

which also holds thanks to associativity of composition (when unfolding the fixpoints).

Lemma L.3. *Diagram (5) commutes.*

Proof. Let $A = (U, X)$, so that

$$T(A^{\blacktriangleright}) \quad = \quad T(U^{\blacktriangleright X}, X) \quad = \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \qquad \text{and} \qquad (TA)^{\blacktriangleright} \quad = \quad (U, X^U)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright (X^U)}, X^U)$$

The diagram has type

$$T(A^{\blacktriangleright}) \longrightarrow (TTA)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X} \ , \ X^{U^{\blacktriangleright X}}) \longrightarrow (U^{\blacktriangleright (X^{U \times U})} \ , \ X^{U \times U})$$

Moreover,

$$\begin{array}{rclcl} (\delta_A)^{\blacktriangleright} & = & (\mathrm{id}_U \ , \ \boldsymbol{\lambda} hu.h(u,u))^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.h \blacktriangleright (\boldsymbol{\lambda} hu.h(u,u)) \ , \ \boldsymbol{\lambda} hu.h(u,u)) \\ T\zeta_A & = & T(f^{\zeta_A}, F^{\zeta_A}) & = & (f^{\zeta_A} \ , \ \boldsymbol{\lambda} h.F^{\zeta_A} h f^{\zeta_A}) \end{array}$$

We have to check the following two equations:

$$f_{\delta_A \blacktriangleright} \circ f^{\zeta_A} \quad = \quad f^{\zeta_{TA}} \circ f_{T\zeta_A} \circ f_{\delta_A \blacktriangleright} \qquad \text{and} \qquad F^{\zeta_A} \circ F_{\delta_A \blacktriangleright} \quad = \quad F_{\delta_A \blacktriangleright} \circ F_{T\zeta_A} \circ F^{\zeta_{TA}}$$

The first one, of type $U^{\blacktriangleright X} \to U^{\blacktriangleright (X^{U \times U})}$, amounts, for $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$ and $\xi_{n+1} \in X_{n+1}^{U \times U}$, to the following

$$((\boldsymbol{\lambda} h.h \blacktriangleright (\boldsymbol{\lambda} hu.h(u,u))) \circ f_{n+1}^{\zeta_A})(\theta_{n+1})(\xi_{n+1}) \quad = \quad (f_{n+1}^{\zeta_{TA}} f_{n+1}^{\zeta_A})(\theta_{n+1})(\blacktriangleright \xi_{n+1})$$

that is

$$(f_{n+1}^{\zeta_A}(\theta_{n+1}) \circ \blacktriangleright (\boldsymbol{\lambda} hu.h(u,u)))(\xi_{n+1}) \quad = \quad f_{n+1}^{\zeta_{TA}}(f_{n+1}^{\zeta_A}(\theta_{n+1}) \ , \ \xi_n)$$

that is

$$f_{n+1}^{\zeta_A}(\theta_{n+1} \ , \ \boldsymbol{\lambda} u.\xi_n(u,u)) \quad = \quad f_{n+1}^{\zeta_A}(\theta_{n+1} \ , \ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n)))$$

Write

$$l_n \quad := \quad f_{n+1}^{\zeta_A}(\theta_{n+1} \ , \boldsymbol{\lambda} u.\xi_n(u,u)) \qquad \text{and} \qquad r_n \quad := \quad f_{n+1}^{\zeta_A}(\theta_{n+1} \ , \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n)))$$

The proof is then by induction on n. In the base case $n = 0$, both sides unfold to $\theta_1(\bullet)$. For the induction step, assuming the property for $r_n = l_n$, we show $l_{n+1} = r_{n+1}$.

First, note that Note that

$$\begin{array}{rcl} \mathrm{fix}^U_{n+1}(\boldsymbol{\lambda} u.\xi_{n+1}(u,u) \circ \theta_{n+1}) & = & \mathrm{fix}^U_{n+1}(\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x), \theta_{n+1}(x))) \\ & = & (\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x)\ ,\ \theta_{n+1}(x)))(\mathrm{fix}^U_n(\boldsymbol{\lambda} x.\xi_n(\theta_n(x), \theta_n(x)))) \\ & = & (\boldsymbol{\lambda} u.\xi_{n+1}(u,u))(\theta_{n+1}(\mathrm{fix}^U_n((\boldsymbol{\lambda} u.\xi_n(u,u)) \circ \theta_n))) \\ & = & \xi_{n+1}(l_n, l_n) \end{array}$$

so that

$$l_{n+1} \quad = \quad \theta_{n+2}(\xi_{n+1}(l_n, l_n))$$

On the other hand, note that

$$\begin{array}{rcl} \mathrm{fix}^{X^U}_{n+1}(\xi_{n+1} \circ f^{\zeta_A}_{n+1}(\theta_{n+1})) & = & \xi_{n+1}(f^{\zeta_A}_{n+1}(\theta_{n+1}\ ,\ \mathrm{fix}^{X^U}_n(\xi_n \circ f^{\zeta_A}_n(\theta_n)))) \\ & = & \xi_{n+1}(r_n) \end{array}$$

and so in particular

$$\begin{array}{rcl} r_n & = & \theta_{n+1}(\mathrm{fix}_n(\mathrm{fix}^{X^U}_n(\xi_n \circ f^{\zeta_A}(\theta_n)) \circ \theta_n)) \\ & = & \theta_{n+1}(\mathrm{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)) \end{array}$$

We thus have

$$\begin{array}{rcl} r_{n+1} & = & \theta_{n+2}(\mathrm{fix}_{n+1}(\mathrm{fix}^{X^U}_{n+1}(\xi_{n+1} \circ f^{\zeta_A}_{n+1}(\theta_{n+1})) \circ \theta_{n+1})) \\ & = & \theta_{n+2}(\mathrm{fix}_{n+1}(\xi_{n+1}(r_n) \circ \theta_{n+1}) \\ & = & \theta_{n+2}(\xi_{n+1}(r_n)(\theta_{n+1}(\mathrm{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)))) \\ & = & \theta_{n+2}(\xi_{n+1}(r_n)(r_n)) \end{array}$$

and we conclude by induction hypothesis.

The second equation, of type $X^{U \times U} \to X^{U^{\blacktriangleright X}}$, amounts, for $\xi_n \in (X^{U\times U})_n$ and $\theta_n \in (U^{\blacktriangleright X})_n$, to the following:

$$F^{\zeta_A}_n \circ (\boldsymbol{\lambda} hu.h(u,u))(\xi_n)(\theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k)) \circ (\boldsymbol{\lambda} h.F^{\zeta_A}_n h f^{\zeta_A}_n) \circ F^{\zeta_{TA}}_n)(\xi_n)(\theta_n)$$

that is

$$F^{\zeta_A}_n((\boldsymbol{\lambda} hu.h(u,u))\xi_n\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k))((\boldsymbol{\lambda} h.F^{\zeta_A}_n h f^{\zeta_A}_n)(F^{\zeta_{TA}}_n(\xi_n))))(\theta_n)$$

that is

$$F^{\zeta_A}_n(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k))((F^{\zeta_A}_n \circ F^{\zeta_{TA}}_n(\xi_n) \circ f^{\zeta_A}_n)))(\theta_n)$$

that is

$$F^{\zeta_A}_n(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad (\boldsymbol{\lambda} k.(F^{\zeta_A}_N \circ F^{\zeta_{TA}}_n(\xi_n) \circ f^{\zeta_A}_n)(k,k))\theta_n$$

that is

$$F^{\zeta_A}_n(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad (F^{\zeta_A}_n \circ F^{\zeta_{TA}}_n(\xi_n) \circ f^{\zeta_A}_n)(\theta_n)(\theta_n)$$

that is

$$F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\ ,\ \theta_n) \quad = \quad F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n\ ,\ f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)$$

Reasonning as for the first equation, write

$$l_n \quad := \quad F_n^{\zeta_A}(\boldsymbol{\lambda} u.\xi_n(u,u)\,,\,\theta_n) \qquad \text{and} \qquad r_n \quad := \quad F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n\,,\,f_n^{\zeta_A}(\theta_n))\,,\,\theta_n)$$

with

$$\begin{array}{rcl} l_{n+1} & = & \mathrm{fix}_{n+1}((\boldsymbol{\lambda} u.\xi_{n+1}(u,u)) \circ \theta_{n+1}) \\ & = & \xi_{n+1}(\theta_{n+1}(l_n)\ ,\ \theta_{n+1}(l_n)) \end{array}$$

and on the other hand

$$\begin{array}{rcl} F_{n+1}^{\zeta_{TA}}(\xi_{n+1}\ ,\ f_{n+1}^{\zeta_A}(\theta_{n+1})) & = & \mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) \\ & = & \xi_{n+1}(f_{n+1}^{\zeta_A}(\theta_{n+1}\ ,\ \mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n))) \\ & = & \xi_{n+1}(\theta_{n+1}(F_n^{\zeta_A}(\mathrm{fix}_n^{X^U}(\xi_n \circ f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)) \\ & = & \xi_{n+1}(\theta_{n+1}(F_n^{\zeta_A}(F_n^{\zeta_{TA}}(\xi_n, f_n^{\zeta_A}(\theta_n))\ ,\ \theta_n)) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)) \end{array}$$

We thus have

$$\begin{array}{rcl} r_{n+1} & = & \mathrm{fix}_{n+1}(\mathrm{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\zeta_A}(\theta_{n+1})) \circ \theta_{n+1}) \\ & = & \mathrm{fix}_{n+1}(\xi_{n+1}(\theta_{n+1}(r_n)) \circ \theta_{n+1}) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)\ ,\ \theta_{n+1}(\mathrm{fix}_n(\xi_n(\theta_n(r_{n-1})) \circ \theta_n))) \\ & = & \xi_{n+1}(\theta_{n+1}(r_n)\ ,\ \theta_{n+1}(r_n)) \end{array}$$

and we conclude by induction hypothesis.

Lemma L.4. *Diagram (6) commutes.*

Proof. Let $A = (U, X)$ so that the diagram has type

$$T(A^{\blacktriangleright\blacktriangleright}) \longrightarrow (TA)^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X \times \blacktriangleright X}, X^{U^{\blacktriangleright X \times \blacktriangleright X}}) \longrightarrow (U^{\blacktriangleright (X^U)}, X^U)$$

Note that

$$\begin{array}{rcccl} T(\mu_A) & = & T(\boldsymbol{\lambda} hx.h(x,x)\ ,\ \mathrm{id}_X) & = & (\boldsymbol{\lambda} hx.h(x,x)\ ,\ \boldsymbol{\lambda} k.(k \circ \boldsymbol{\lambda} hx.h(x,x))) \\ (\zeta_A)^{\blacktriangleright} & = & (f^{\zeta_A}, F^{\zeta_A})^{\blacktriangleright} & = & (\boldsymbol{\lambda} h.f^{\zeta_A} \circ h \circ \blacktriangleright F^{\zeta_A}\ ,\ F^{\zeta_A}) \end{array}$$

We have to check the following two equations:

$$f^{\zeta_A} \circ f_{T\mu_A} \quad = \quad f_{\mu_{TA}} \circ f_{(\zeta_A)^{\blacktriangleright}} \circ f^{\zeta_A^{\blacktriangleright}} \qquad \text{and} \qquad F_{T\mu_A} \circ F^{\zeta_A} \quad = \quad F^{\zeta_A^{\blacktriangleright}} \circ F_{(\zeta_A)^{\blacktriangleright}} \circ F_{\mu_{TA}}$$

The first equation, of type $U^{\blacktriangleright X \times \blacktriangleright X} \to U^{\blacktriangleright (X^U)}$, amounts, for $\theta_{n+1} \in (U^{\blacktriangleright X \times \blacktriangleright X})_{n+1}$ and $\xi_n \in (X^U)_n$, to the following:

$$(f_{n+1}^{\zeta_A} \circ (\boldsymbol{\lambda} hx.h(x,x)))(\theta_{n+1})(\xi_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k)) \circ (\boldsymbol{\lambda} h.f_{n+1}^{\zeta_A} h \blacktriangleright F_{n+1}^{\zeta_A}) \circ f_{n+1}^{\zeta_A^{\blacktriangleright}})(\theta_{n+1})(\xi_n)$$

that is

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad ((\boldsymbol{\lambda} hk.h(k,k))\circ(\boldsymbol{\lambda} h.f^{\zeta_A}_{n+1} h F^{\zeta_A}_n)\circ f^{\zeta_A \blacktriangleright}_{n+1})(\theta_{n+1})(\xi_n)$$

that is

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad (\boldsymbol{\lambda} hk.h(k,k))(f^{\zeta_A}_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}) \circ F^{\zeta_A}_n)(\xi_n)$$

that is

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad (f^{\zeta_A}_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}) \circ F^{\zeta_A}_n)(\xi_n)(\xi_n)$$

that is

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \quad = \quad f^{\zeta_A}_{n+1}(f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n))\ ,\ \xi_n)$$

Let

$$l_n \quad := \quad f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.\theta_{n+1}(x,x)\ ,\ \xi_n) \qquad \text{and} \qquad r_n \quad := \quad f^{\zeta_A}_{n+1}(f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n))\ ,\ \xi_n)$$

Note that for all n we have

$$\begin{aligned} l_{n+1} &= (\boldsymbol{\lambda} x.\theta_{n+2}(x,x))\mathrm{fix}_{n+1}(\xi_{n+1} \circ \boldsymbol{\lambda} x.\theta_{n+1}(x,x)) \\ &= (\boldsymbol{\lambda} x.\theta_{n+2}(x,x))((\boldsymbol{\lambda} x.\xi_{n+1}(\theta_{n+1}(x,x))\mathrm{fix}_n(\xi_n \circ \boldsymbol{\lambda} x.\theta_n(x,x)))) \\ &= \theta_{n+2}(\xi_{n+1}(l_n),\xi_{n+1}(l_n)) \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{n+1} &= f^{\zeta_A}_{n+2}(f^{\zeta_A \blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))\ ,\ \xi_{n+1}) \\ &= f^{\zeta_A \blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))(\mathrm{fix}^X_{n+1}(\xi_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \\ &= \theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1})\ ,\ \mathrm{fix}^X_{n+1}(\xi_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \end{aligned}$$

So we show by induction on n that

$$\xi_{n+1}(r_n) \quad = \quad \mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1})\circ\theta_{n+1}) \quad = \quad \mathrm{fix}^X_{n+1}(\xi_{n+1}\circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))$$

The base case is trivial. For the induction step, on the one hand we have

$$\begin{aligned} & \mathrm{fix}_{n+2}(F^{\zeta_A}_{n+2}(\xi_{n+2}) \circ \theta_{n+2}) \\ = \quad & F^{\zeta_A}_{n+2}(\xi_{n+2}\ ,\ \theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1}))) \\ = \quad & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1})\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}\ ,\ \theta_{n+1}(\mathrm{fix}_n(F^{\zeta_A}_n(\xi_n) \circ \theta_n)))) \\ = \quad & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1})\ ,\ \mathrm{fix}_{n+1}(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1})) \end{aligned}$$

and we conclude by induction hypothesis, and on the other hand

$$\begin{aligned} & \mathrm{fix}^X_{n+2}(\xi_{n+2} \circ f^{\zeta_A \blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))) \\ = \quad & \xi_{n+2} \circ f^{\zeta_A \blacktriangleright}_{n+2}(\theta_{n+2}\ ,\ F^{\zeta_A}_{n+1}(\xi_{n+1}))(\mathrm{fix}^X_{n+1}(\xi_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n)))) \\ = \quad & \xi_{n+2}(\theta_{n+2}(\mathrm{fix}_n(F^{\zeta_A}_{n+1}(\xi_{n+1}) \circ \theta_{n+1})\ ,\ \mathrm{fix}^X_{n+1}(\xi_{n+1} \circ f^{\zeta_A \blacktriangleright}_{n+1}(\theta_{n+1}\ ,\ F^{\zeta_A}_n(\xi_n))))) \end{aligned}$$

and we also conclude by induction hypothesis.

The second equation, of type $X^U \to X^{U^{\blacktriangleright X \times \blacktriangleright X}}$, amounts, for $\xi_n \in (X^U)_n$ and $\theta_n \in (U^{\blacktriangleright X \times \blacktriangleright X})_n$, to the following

$$((\boldsymbol{\lambda} k.(k \circ \boldsymbol{\lambda} hx.h(x,x))) \circ F_n^{\zeta_A})(\xi_n)(\theta_n) \quad = \quad (F_n^{\zeta_A \blacktriangleright} \circ F_n^{\zeta_A})(\xi_n)(\theta_n)$$

that is

$$(F_n^{\zeta_A}(\xi_n) \circ \boldsymbol{\lambda} hx.h(x,x))(\theta_n) \quad = \quad F_n^{\zeta_A \blacktriangleright}(F_n^{\zeta_A}(\xi_n)\ ,\ \theta_n)$$

that is

$$F_n^{\zeta_A}(\xi_n\ ,\ \boldsymbol{\lambda} x.\theta_n(x,x)) \quad = \quad F_n^{\zeta_A \blacktriangleright}(F_n^{\zeta_A}(\xi_n)\ ,\ \theta_n)$$

This is dealt-with similarly to (but in a much simpler way than) the first equation.

Lemma L.5. *Diagram (7) commutes.*

Proof. Let $A = (U, X)$, so that the diagram has type

$$T(A^{\blacktriangleright}) \longrightarrow\!\!\!\!\!\!\!+\ \ A^{\blacktriangleright} \quad = \quad (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \longrightarrow\!\!\!\!\!\!\!+\ \ (U^{\blacktriangleright X}, X)$$

Note that

$$(\epsilon_A)^{\blacktriangleright} \quad = \quad (\mathrm{id}_U, \boldsymbol{\lambda} xu.x)^{\blacktriangleright} \quad = \quad (\boldsymbol{\lambda} h.(h \circ \blacktriangleright(\boldsymbol{\lambda} xu.x)), \boldsymbol{\lambda} xu.x)$$

We have to show

$$\boldsymbol{\lambda} h.(h \circ \blacktriangleright(\boldsymbol{\lambda} xu.x)) \circ f^{\zeta_A} \quad = \quad \mathrm{id}_{U^{\blacktriangleright X}} \qquad \text{and} \qquad F^{\zeta_A} \circ \boldsymbol{\lambda} xu.x \quad = \quad \boldsymbol{\lambda} xu.x$$

For the first equation, given $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$, we have to show

$$f_{n+1}^{\zeta_A}(\theta_{n+1}) \circ \blacktriangleright(\boldsymbol{\lambda} xu.x) \quad = \quad \theta_{n+1}$$

The result is trivial since the left-hand side unfolds to

$$\boldsymbol{\lambda}\blacktriangleright x.f_{n+1}^{\zeta_A}(\theta_{n+1}, \boldsymbol{\lambda}\_.x) \quad = \quad \boldsymbol{\lambda}\blacktriangleright x.\theta_{n+1}(\mathrm{fix}_n(\boldsymbol{\lambda}\_.x)) \quad = \quad \boldsymbol{\lambda}\blacktriangleright x.\theta_{n+1}(x)$$

The second equation is simpler and omitted.

Lemma L.6. *Diagram (8) commutes.*

Proof. Let $A = (U, X)$, so that the diragram has type

$$TA \longrightarrow\!\!\!\!\!\!\!+\ \ (TA)^{\blacktriangleright} \quad = \quad (U, X^U) \longrightarrow\!\!\!\!\!\!\!+\ \ (U^{\blacktriangleright (X^U)}, X^U)$$

Note that

$$T(\eta_A) \quad = \quad T(\boldsymbol{\lambda} ux.u, \mathrm{id}_X) \quad = \quad (\boldsymbol{\lambda} ux.u, \boldsymbol{\lambda} h.h \circ (\boldsymbol{\lambda} ux.u))$$

We have to show

$$f^{\zeta_A} \circ (\boldsymbol{\lambda} ux.u) \quad = \quad \boldsymbol{\lambda} ux.u \qquad \text{and} \qquad (\boldsymbol{\lambda} h.h \circ (\boldsymbol{\lambda} ux.u)) \circ F^{\zeta_A} \quad = \quad \mathrm{id}_{X^U}$$

For the first equation, given $u \in U_{n+1}$ and $\xi_n \in (X^U)_n$, we have to show

$$f^{\zeta_A}_{n+1}(\boldsymbol{\lambda} x.u\ ,\ \xi_n) \quad = \quad u$$

which is trivial. For the second equation, given $\xi_n \in X_n$ and $u \in U_n$ we have to show

$$F^{\zeta_A}(\xi_n\ ,\ \boldsymbol{\lambda} x.u) \quad = \quad \xi_n(u)$$

which is also trivial.

L.2 Proof of Proposition L.1.(ii)

Fix $\mathbf{G}(\mathscr{S})$-objects $A = (U, X)$ and $B = (V, Y)$. Diagram (4) amounts, to the following two diagrams, for resp. the first and second component of $\mathbf{G}(\mathscr{S})$:

$$\begin{array}{ccc}
U^{\blacktriangleright X} \times V^{\blacktriangleright Y} & \xrightarrow{\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle \blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle} & (U \times V)^{\blacktriangleright(X\times Y)} \\
\Big\downarrow \mathrm{id}_{U^{\blacktriangleright X}\times V^{\blacktriangleright Y}} & & \Big\downarrow f^{\zeta_{A\otimes B}} \\
U^{\blacktriangleright X} \times V^{\blacktriangleright Y} & & (U \times V)^{\blacktriangleright((X\times Y)^{U\times V})} \\
\Big\downarrow f^{\zeta_A}\times f^{\zeta_B} & & \Big\downarrow \boldsymbol{\lambda} h.h\circ\blacktriangleright(\boldsymbol{\lambda}(h,k).h\times k) \\
U^{\blacktriangleright(X^U)} \times V^{\blacktriangleright(Y^V)} & \xrightarrow[\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle \blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle]{} & (U \times V)^{\blacktriangleright(X^U\times Y^V)}
\end{array} \tag{9}$$

$$\begin{array}{ccc}
(X \times Y)^{U^{\blacktriangleright X}\times V^{\blacktriangleright Y}} & \xleftarrow{\boldsymbol{\lambda} h.h\circ(\boldsymbol{\lambda}(h,k).(h\times k)\circ\langle \blacktriangleright(\pi_1),\blacktriangleright(\pi_2)\rangle)} & (X \times Y)^{(U\times V)^{\blacktriangleright(X\times Y)}} \\
\Big\uparrow \boldsymbol{\lambda}(h,k).h\times k & & \Big\uparrow F^{\zeta_{A\otimes B}} \\
X^{U^{\blacktriangleright X}} \times Y^{V^{\blacktriangleright Y}} & & (X \times Y)^{U\times V} \\
\Big\uparrow F^{\zeta_A}\times F^{\zeta_B} & & \Big\uparrow \boldsymbol{\lambda}(h,k).h\times k \\
X^U \times Y^V & \xleftarrow[\mathrm{id}_{X^U\times Y^V}]{} & X^U \times Y^V
\end{array} \tag{10}$$

Commutation of (10). We reason modulo $((-)\times(-))_n \simeq (-)_n \times (-)_n$. Consider $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$, $\theta'_{n+1} \in (V^{\blacktriangleright Y})$, and $\xi_{n+1} \in (X^U)_{n+1}$, $\xi'_{n+1} \in (Y^V)_{n+1}$.

We have to show that

$$\langle F^{\zeta_A}_{n+1}(\xi_{n+1}, \theta_{n+1})\ ,\ F^{\zeta_B}_{n+1}(\xi'_{n+1}, \theta'_{n+1})\rangle \quad = \quad F^{\zeta_{A\otimes B}}_{n+1}(\xi_{n+1}\times\xi'_{n+1}, \boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle)$$

which amounts to

$$\langle \mathrm{fix}_{n+1}(\xi_{n+1}\circ\theta_{n+1})\ ,\ \mathrm{fix}_{n+1}(\xi'_{n+1}\circ\theta'_{n+1})\rangle \quad = \quad \mathrm{fix}_{n+1}((\xi_{n+1}\times\xi'_{n+1})\circ(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x), \theta'_{n+1}(y)\rangle))$$

that is

$$\langle((\xi_{n+1}\circ\theta_{n+1}\circ\xi_n\circ\theta_n\circ\ldots\circ\xi_0\circ\theta_0)(\bullet)\ ,\ (\xi'_{n+1}\circ\theta'_{n+1}\circ\xi'_n\circ\theta'_n\circ\ldots\circ\xi'_0\circ\theta'_0)(\bullet)\rangle \quad = \quad ((\xi_{n+1}\times\xi'_{n+1})\circ(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x),\theta'_{n+1}(y)\rangle)\circ\ldots\circ(\xi_0\times\xi'_0)\circ(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_0(x),\theta'_0(y)\rangle))(\bullet,\bullet)$$

which follows from an easy induction on $n\in\mathbb{N}$.

Commutation of (9). We reason modulo $((-)\times(-))_n \simeq (-)_n\times(-)_n$. Consider $\theta_{n+1}\in(U^{\blacktriangleright X})_{n+1}$, $\theta'_{n+1}\in(V^{\blacktriangleright Y})$, and $\xi_n\in(X^U)_n$, $\xi'_n\in(Y^V)_n$.

We have to show that

$$\langle f^{\zeta_A}_{n+1}(\theta_{n+1},\xi_n)\ ,\ f^{\zeta_B}_{n+1}(\theta'_{n+1},\xi'_n)\rangle \quad = \quad f^{\zeta_{A\otimes B}}_{n+1}(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x),\theta'_{n+1}(y)\rangle\ ,\ \xi_n\times\xi'_n)$$

which amounts to (leaving implicit the restriction map r_n):

$$\begin{aligned}&\langle\theta_{n+1}(F^{\zeta_A}_{n+1}(\theta_n,\xi_n))\ ,\ \theta'_{n+1}(F^{\zeta_B}_{n+1}(\theta'_n,\xi'_n))\rangle\\ =\quad&(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x),\theta'_{n+1}(y)\rangle)(F^{\zeta_{A\otimes B}}_n(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_n(x),\theta'_n(y)\rangle\ ,\ \xi_n\times\xi'_n))\end{aligned}$$

that is

$$\begin{aligned}&(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x),\theta'_{n+1}(y)\rangle)\langle F^{\zeta_A}_{n+1}(\theta_n,\xi_n)\ ,\ F^{\zeta_B}_{n+1}(\theta'_n,\xi'_n)\rangle\\ =\quad&(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_{n+1}(x),\theta'_{n+1}(y)\rangle)(F^{\zeta_{A\otimes B}}_n(\boldsymbol{\lambda}\blacktriangleright(x,y).\langle\theta_n(x),\theta'_n(y)\rangle\ ,\ \xi_n\times\xi'_n))\end{aligned}$$

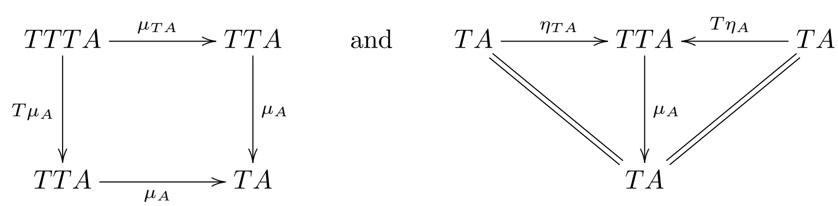
and we are done by (10).

M Monoids, Monads and Monoidal Categories

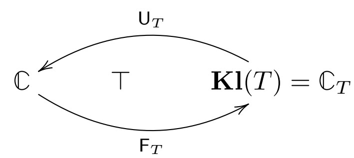
This appendix gathers easy and possibly well-known facts about monoidal categories. We refer to [27, 24] for missing details.

M.1 Monads and Comonads

Monads. A monad on a category $\mathbb{C}$ is a triple $T=(T,\mu,\eta)$ consisting of a functor $T:\mathbb{C}\to\mathbb{C}$ and two natural transformations $\mu_A: TTA\to TA$ and $\eta_A: A\to TA$ satisfying:



The Kleisli category $\mathbf{Kl}(T) = \mathbb{C}_T$ of T has the same objects as $\mathbb{C}$ and $\mathbf{Kl}(T)[A, B] := \mathbb{C}[A, TB]$. The categories $\mathbb{C}$ and $\mathbf{Kl}(T) = \mathbb{C}_T$ are related by an adjunction



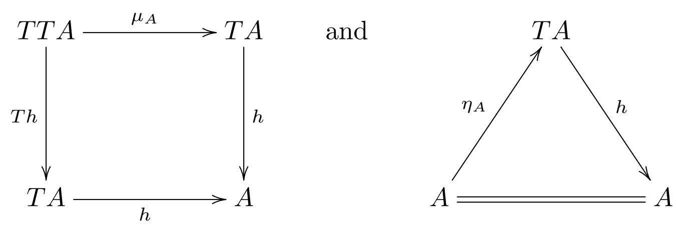
where:

- The right adjoint $\mathsf{U}_T : \mathbf{Kl}(T) \to \mathbb{C}$ maps objects A of $\mathbf{Kl}(T)$ to TA and takes $f \in \mathbf{Kl}(T)[A, B] = \mathbb{C}[A, TB]$ to

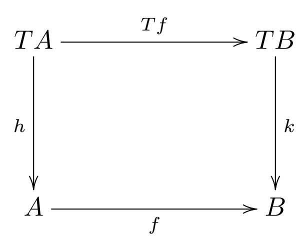
$$\mu_B \circ T(f) \in \mathbb{C}[\mathsf{U}_T A, \mathsf{U}_T B] = \mathbb{C}[TA, TB]$$

- The left adjoint $\mathsf{F}_T : \mathbb{C} \to \mathbf{Kl}(T)$ is the identity on objects and takes $f \in \mathbb{C}[A, B]$ to $\mathsf{F}_T(f) := \eta_B \circ f \in \mathbf{Kl}(T)[A, B] = \mathbb{C}[A, TB]$.

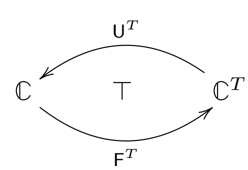
The category $\mathbb{C}^T$ of *Eilenberg-Moore* algebras has, as objects, T-algebras $h : TA \to A$ such that



and as morphisms from $h : TA \to A$ to $k : TB \to B$, maps $f : A \to B$ such that



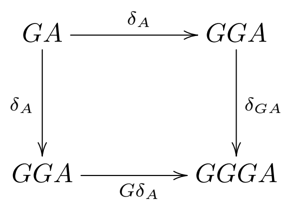
The categories $\mathbb{C}$ and $\mathbb{C}^T$ are related by an adjunction



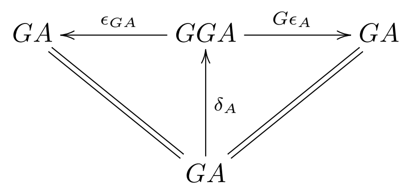
where:

- The forgetful functor $\mathsf{U}^T : \mathbb{C}^T \to \mathbb{C}$ maps $h : TA \to A$ to A and $f : (A, h) \to (B, k)$ to $f : A \to B$.
- The free functor $\mathsf{F}^T : \mathbb{C} \to \mathbb{C}^T$ maps A to (TA, μ_A) and $f : A \to B$ to $Tf : TA \to TB$.

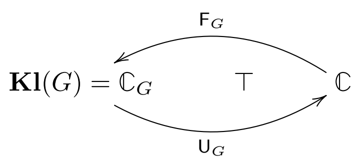
Comonads. Dually a comonad on $\mathbb{C}$ is a monad on $\mathbb{C}^{\mathrm{op}}$. It is therefore given by a triple $G = (G, \delta, \epsilon)$ where the functor $G : \mathbb{C} \to \mathbb{C}$ and the natural transformations $\delta_A : GA \to GGA$ and $\epsilon_A : GA \to A$ satisfy:



and



The Kleisli category $\mathbf{Kl}(G) = \mathbb{C}_G$ of G has the same objects as $\mathbb{C}$ and $\mathbf{Kl}(G)[A, B] := \mathbb{C}[GA, B]$. The categories $\mathbb{C}$ and $\mathbf{Kl}(G) = \mathbb{C}_G$ are related by an adjunction



where:

- The left adjoint $\mathsf{U}_G : \mathbf{Kl}(G) \to \mathbb{C}$ maps objects A of $\mathbf{Kl}(G)$ to GA and takes $f \in \mathbf{Kl}(G)[A, B] = \mathbb{C}[GA, B]$ to

$$G(f) \circ \delta_A \in \mathbb{C}[\mathsf{U}_G A, \mathsf{U}_G B] = \mathbb{C}[GA, GB]$$

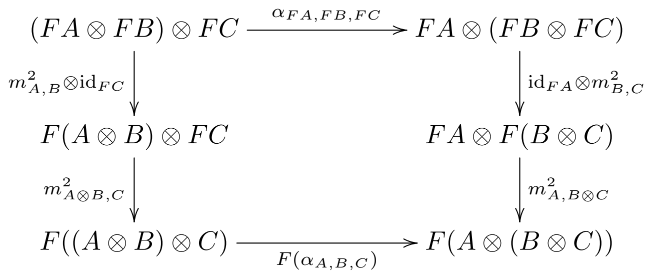
- The right adjoint $\mathsf{F}_G : \mathbb{C} \to \mathbf{Kl}(G)$ is the identity on objects and takes $f \in \mathbb{C}[A, B]$ to $\mathsf{F}_G(f) := f \circ \epsilon_A \in \mathbf{Kl}(G)[A, B] = \mathbb{C}[GA, B]$.

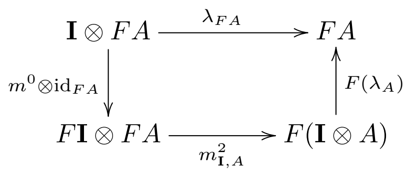
(Lax) (Symmetric) Monoidal Monads. There are different notions of monoidal functor (see e.g. [27]). Here we use *lax* monoidal functors (as the functor part of *lax* monoidal *monads*), and the dual notion of *oplax* monoidal functor (as the functor part of *oplax* monoidal *comonads*).

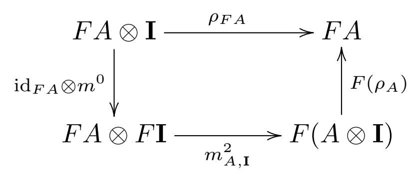
(Lax) Symmetric Monoidal Functors. A (lax) symmetric monoidal functor on a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is a functor F equipped with natural transformations

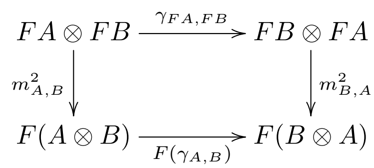
$$m^2_{A,B} : FA \otimes FB \to F(A \otimes B) \qquad \text{and} \qquad m^0 : \mathbf{I} \to F(\mathbf{I})$$

making the following diagrams commute:

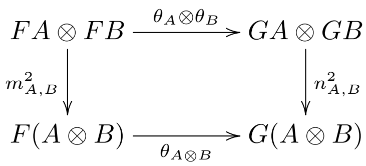




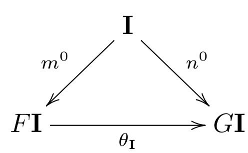




(Lax) Monoidal Natural Transformations. A monoidal natural transformation between (lax) monoidal functors $\theta : (F, m^2, m^0) \Longrightarrow (G, n^2, n^0)$ is a natural transformation $\theta : F \Longrightarrow G$ making the following diagrams commute:



and



The following is [27, Prop. 10]:

Proposition M.1. *Symmetric monoidal categories, (lax) symmetric monoidal functors, and monoidal natural transformations form a* 2*-category* **SymMonCat**.

Proof.

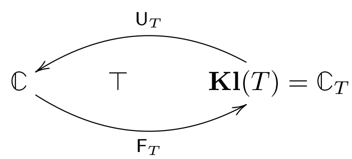
- The identity functor $\mathrm{Id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is monoidal (actually strict monoidal), with $m^2_{A,B} = \mathrm{id}_{A\otimes B}$ and $m^0 = \mathrm{id}_{\mathbf{I}}$.
- If (F, m^2, m^0) and (G, n^2, n^0) are lax monoidal, then so is FG, with structure maps

$$\begin{aligned} F(n^2_{A,B}) \circ m^2_{GA,GB} \quad &: \quad FGA \otimes FGB \to F(GA \otimes GB) \to FG(A \otimes B) \\ F(n^0) \circ m^0 \quad &: \quad \mathbf{I} \to F\mathbf{I} \to FG\mathbf{I} \end{aligned}$$

□

(Lax) (Symmetric) Monoidal Monads. A (lax) symmetric monoidal monad on a monoidal category $\mathbb{C}$ is a monad (T, μ, η) such that T is a (lax) symmetric monoidal functor and the transformations μ, η are monoidal (see e.g. [27]). It then follows from [27, §6.10] that:

Proposition M.2. *If $T = (T, \mu, \eta)$ is a (lax) symmetric monoidal monad on $(\mathbb{C}, \otimes, \mathbf{I})$ then its Kleisely category $\mathbf{Kl}(T) = \mathbb{C}_T$ is symmetric monoidal. Moreover, the functor $\mathsf{F}_T : \mathbb{C} \to \mathbf{Kl}(T) = \mathbb{C}_T$ is strict and the adjunction*



is (lax) symmetric monoidal (i.e. is an adjunction in **SymMonCat***).*

Proof.

– The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(T)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$. On morphisms, given $f \in \mathbf{Kl}(T)[A_0, B_0] = \mathbb{C}[A_0, TB_0]$ and $g \in \mathbf{Kl}(T)[A_1, B_1] = \mathbb{C}[A_1, TB_1]$, we let $f \otimes_{\mathbf{Kl}} g$ be the composite

$$A_0 \otimes A_1 \quad \xrightarrow{f \otimes g} \quad TB_0 \otimes TB_1 \quad \xrightarrow{m^2_{B_0,B_1}} \quad T(B_0 \otimes B_1)$$

where m^2 is the binary strength of T.

– The functor F_T is strict, since its strength is given by:

$$f^2_{A,B} \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{A \otimes B} \quad = \quad \eta_{A \otimes B} \quad \in \quad \mathbf{Kl}(T)[A \otimes_{\mathbf{Kl}} B \,,\, A \otimes_{\mathbf{Kl}} B] \quad = \quad \mathbb{C}[A \otimes B \,,\, T(A \otimes B)]$$

and

$$f^0 \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{\mathbf{I}} \quad = \quad \eta_{\mathbf{I}} \quad \in \quad \mathbf{Kl}(T)[\mathbf{I} \,,\, \mathbf{I}] \quad = \quad \mathbb{C}[\mathbf{I} \,,\, T\mathbf{I}]$$

– The functor U_T is lax symmetric monoidal. Its strength is given by:

$$u^2_{A,B} \quad := \quad m^2_{A,B} \quad \in \quad \mathbb{C}[\mathsf{U}_T A \otimes \mathsf{U}_T B \,,\, \mathsf{U}_T(A \otimes B)] \quad = \quad \mathbb{C}[TA \otimes TB \,,\, T(A \otimes B)]$$

and

$$u^0 \quad := \quad m^0 \quad \in \quad \mathbb{C}[\mathbf{I} \,,\, \mathsf{U}_T \mathbf{I}] \quad = \quad \mathbb{C}[\mathbf{I} \,,\, T\mathbf{I}]$$

where m^2, m^0 is the strength of T.

– The structure maps of $\mathbf{Kl}(T)$ are taken to be the image under F_T of the structure maps of $\mathbb{C}$. It thus directly follows that the coherence conditions are met on $\mathbb{C}$.

– It remains to check the naturality of the structural maps of $\mathbf{Kl}(T)$, which amounts to the following diagrams:

- For the associativity structure map $\alpha_{(-),(-),(-)}$:

$$\begin{array}{ccc} (A\otimes B)\otimes C & \xrightarrow{(f\otimes g)\otimes h} & (A'\otimes B')\otimes C' \\ {\scriptstyle \eta_{A\otimes(B\otimes C)}\circ\alpha_{A,B,C}}\downarrow & & \downarrow{\scriptstyle \eta_{A'\otimes(B'\otimes C')}\circ\alpha_{A',B',C'}} \\ T(A\otimes(B\otimes C)) & \xrightarrow{T(f\otimes(g\otimes h))} & T(A'\otimes(B'\otimes C')) \end{array}$$

Proof. By naturality of η and α, we have

$$\eta_{A'\otimes(B'\otimes C')}\circ\alpha_{A',B',C'}\circ((f\otimes g)\otimes h) \quad = \quad T(f\otimes(g\otimes h))\circ\eta_{A\otimes(B\otimes C)}\circ\alpha_{A,B,C}$$

and we are done. □

- For the unit structure maps $\lambda_{(-)}$ and $\rho_{(-)}$:

$$\begin{array}{ccc} \mathbf{I}\otimes A & \xrightarrow{\mathrm{id}_{\mathbf{I}}\otimes f} & \mathbf{I}\otimes A' \\ {\scriptstyle \eta_A\circ\lambda_A}\downarrow & & \downarrow{\scriptstyle \eta_{A'}\circ\lambda_{A'}} \\ TA & \xrightarrow{T(f)} & TA' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A\otimes\mathbf{I} & \xrightarrow{f\otimes\mathrm{id}_{\mathbf{I}}} & A'\otimes\mathbf{I} \\ {\scriptstyle \eta_A\circ\rho_A}\downarrow & & \downarrow{\scriptstyle \eta_{A'}\circ\rho_{A'}} \\ TA & \xrightarrow{T(f)} & TA' \end{array}$$

Proof. By naturality of η, λ and ρ we have

$$\eta_{A'}\circ\lambda_{A'}\circ(\mathrm{id}_{\mathbf{I}}\otimes f) \quad = \quad T(f)\circ\eta_A\circ\lambda_A \qquad \text{and} \qquad \eta_{A'}\circ\lambda_{A'}\circ(f\otimes\mathrm{id}_{\mathbf{I}}) \quad = \quad T(f)\circ\eta_A\circ\lambda_A$$

and we are done. □

- For the symmetry structure map $\gamma_{(-),(-)}$:

$$\begin{array}{ccc} A\otimes B & \xrightarrow{f\otimes g} & A'\otimes B' \\ {\scriptstyle \eta_{B\otimes A}\circ\gamma_{A,B}}\downarrow & & \downarrow{\scriptstyle \eta_{B'\otimes A'}\circ\gamma_{A',B'}} \\ T(B\otimes A) & \xrightarrow{T(g\otimes f)} & T(B'\otimes A') \end{array}$$

Proof. By naturality of η and γ, we have

$$\eta_{B'\otimes A'}\circ\gamma_{A',B'}\circ(f\otimes g) \quad = \quad T(g\otimes f)\circ\eta_{B\otimes A}\circ\gamma_{A,B}$$

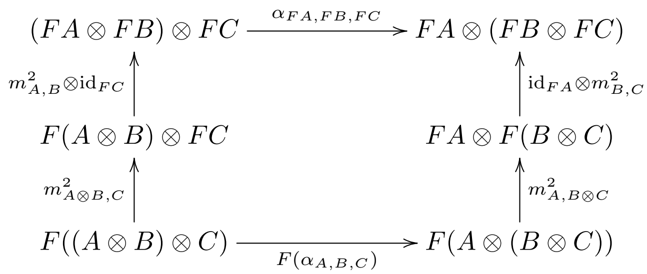
and we are done. □

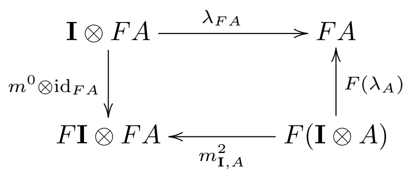
Oplax (Symmetric) Monoidal Comonads. We sketch the dual notion of *oplax* (symmetric) monoidal *comonad*. All constructions and results follow by duality from the case of lax monads.

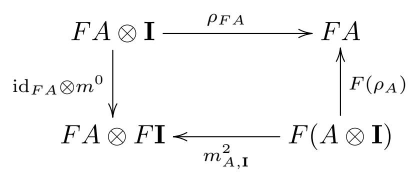
Oplax Monoidal Functors. An oplax symmetric monoidal functor F on a symmetric monoidal category $(\mathbb{C}, \otimes, \mathbf{I})$ is equipped with natural transformations

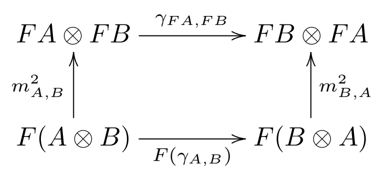
$$m^2_{A,B} : F(A \otimes B) \to FA \otimes FB \qquad \text{and} \qquad m^0 : F(\mathbf{I}) \to \mathbf{I}$$

making the following diagrams commute:

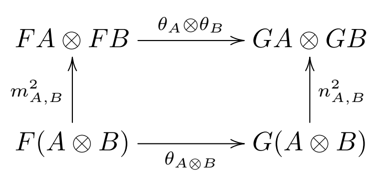




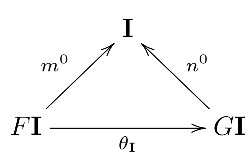




(Oplax) Monoidal Natural Transformations. A monoidal natural transformation between oplax monoidal functors $\theta : (F, m^2, m^0) \Longrightarrow (G, n^2, n^0)$ is a natural transformation $\theta : F \Longrightarrow G$ making the following diagrams commute:



and

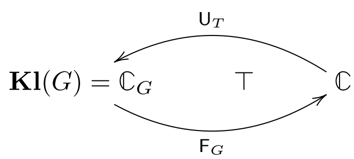


The following is [27, Prop. 11]:

Proposition M.3. *Symmetric monoidal categories, oplax symmetric monoidal functors, and monoidal natural transformations form a* 2*-category* **SymOplaxMonCat**.

Oplax Monoidal Comonads. An oplax monoidal comonad on a monoidal category $\mathbb{C}$ is a comonad (G, δ, ϵ) such that G is an oplax monoidal functor and the transformations δ, ϵ are monoidal (see e.g. [27]). It then follows from [27, §6.10] that:

Proposition M.4. *If $G = (G, \delta, \epsilon)$ is an oplax symmetric monoidal comonad on $\mathbb{C}$ then the Kleisely category $\mathbf{Kl}(G) = \mathbb{C}_G$ is symmetric monoidal. Moreover, the functor $\mathsf{F}_G : \mathbb{C} \to \mathbf{Kl}(G) = \mathbb{C}_G$ is strict and and the adjunction*



is oplax symmetric monoidal (i.e. is an adjunction in **SymOplaxMonCat***).*

Proof. By Prop. M.2, since an oplax comonad on $\mathbb{C}$ is a lax monad on $\mathbb{C}^{\mathrm{op}}$, and since $\mathbb{C}^{\mathrm{op}}$ is symmetric monoidal iff $\mathbb{C}$ is symmetric monoidal.

We record for future use the monoidal structure of $\mathbf{Kl}(G)$:

- The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(G)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$. On morphisms, given $f \in \mathbf{Kl}(G)[A_0, B_0] = \mathbb{C}[GA_0, B_0]$ and $g \in \mathbf{Kl}(G)[A_1, B_1] = \mathbb{C}[GA_1, B_1]$, we let $f \otimes_{\mathbf{Kl}} g$ be the composite

$$G(A_0 \otimes A_1) \quad \xrightarrow{g^2_{A_0,A_1}} \quad GA_0 \otimes GA_1 \quad \xrightarrow{f \otimes g} \quad B_0 \otimes B_1$$

  where g^2 is the binary strength of G.
- The functor F_G is strict, since its strength is given by:

$$f^2_{A,B} \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{A\otimes B} \quad = \quad \epsilon_{A\otimes B} \quad \in \quad \mathbf{Kl}(G)[A\otimes_{\mathbf{Kl}}B \ , \ A\otimes_{\mathbf{Kl}}B] \quad = \quad \mathbb{C}[G(A\otimes B) \ , \ A\otimes B]$$

  and

$$f^0 \quad := \quad \mathrm{id}^{\mathbf{Kl}}_{\mathbf{I}} \quad = \quad \epsilon_{\mathbf{I}} \quad \in \quad \mathbf{Kl}(G)[\mathbf{I} \ , \ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I} \ , \ \mathbf{I}]$$

- The functor U_G is oplax symmetric monoidal. Its strength is given by:

$$u^2_{A,B} \quad := \quad g^2_{A,B} \quad \in \quad \mathbb{C}[\mathsf{U}_G(A\otimes B) \ , \ \mathsf{U}_G A\otimes \mathsf{U}_G B)] \quad = \quad \mathbb{C}[G(A\otimes B) \ , \ GA\otimes GB]$$

  and

$$u^0 \quad := \quad g^0 \quad \in \quad \mathbb{C}[\mathsf{U}_G\mathbf{I} \ , \ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I} \ , \ \mathbf{I}]$$

  where g^2, g^0 is the oplax strength of G.
- The structure maps of $\mathbf{Kl}(G)$ are taken to be the image under F_G of the structure maps of $\mathbb{C}$.

□

M.2 Distributive Laws of a Comonad over a Monad

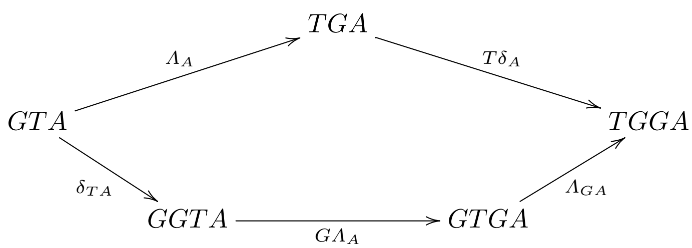
Consider a category $\mathbb{C}$ equipped with a comonad (G, δ, ϵ) and monad (T, μ, η).

A *distributive law* of G over T is a natural tranformation

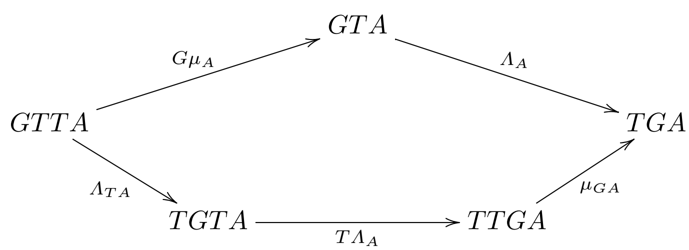
$$\Lambda \quad : \quad G \circ T \quad \Longrightarrow \quad T \circ G$$

such that the following diagrams commute (see e.g. [13]):

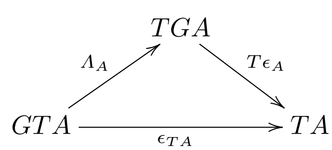
(11)



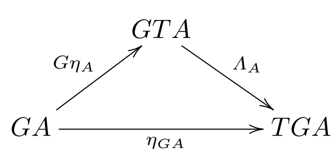
(12)



(13)



(14)



The Kleisli Category Kl(Λ). The category $\mathbf{Kl}(\Lambda)$ has the same objects as $\mathbb{C}$, and its morphisms are given by $\mathbf{Kl}(\Lambda)[A, B] := \mathbb{C}[GA, TB]$. Identity and composition laws follow from that of $\mathbb{C}$ using the monad T and comonad G and the coherence properties of $\Lambda : GT \Rightarrow TG$.

Lifting of a Comonad to the Kleiseli Category of a Monad. Given a distributive law $\Lambda : GT \Rightarrow TG$ as above, the comonad (G, δ, ϵ) on $\mathbb{C}$ lifts to a comonad $(G_T, \delta_T, \epsilon_T)$ on $\mathbb{C}_T = \mathbf{Kl}(T)$, where:

– $G_T(A) := G(A)$ and given $f \in \mathbf{Kl}(T)[A,B] = \mathbb{C}[A,TB]$,

$$G_T(f) \quad := \quad \Lambda_B \circ G(f) \quad \in \mathbf{Kl}(T)[G_T A, G_T B] \quad = \quad \mathbb{C}[GA, TGB]$$

– $\delta_{T,A} := \mathsf{F}_T(\delta_A) \in \mathbf{Kl}(T)[GA, GGA] = \mathbb{C}[GA, TGGA]$ is explictitely given by

$$\delta_{T,A} \quad := \quad \eta_{GGA} \circ \delta_A$$

– $\epsilon_{T,A} := \mathsf{F}_T(\epsilon_A) \in \mathbf{Kl}(T)[GA, A] = \mathbb{C}[GA, TA]$ is explicitely given by

$$\epsilon_{T,A} \quad := \quad \eta_A \circ \epsilon_A$$

Proposition M.5. *The category* $\mathbf{Kl}(\Lambda)$ *is equivalent to the Kleisli category* $\mathbf{Kl}(G_T)$.

Of course, one may alternatively consider the equivalent dual operation of lifting the monad T to the Kleiseli category $\mathbf{Kl}(G)$.

Remark M.6. The above definition of the lift G_T of G to $\mathbf{Kl}(T)$ satisfies the properties asked in [**?**, Def. 3.10].

(Oplax) Monoidal Lifting. Assume now that G is an oplax (symmetric) monoidal comonad and that T is a (lax) (symmetric) monoidal monad on a symmetric monoidal category $\mathbb{C}$. It follows from Prop. M.2 that the Kleiseli category $\mathbf{Kl}(T)$ is symmetric monoidal. Moreover,

Proposition M.7. *If* $\Lambda : GT \Rightarrow TG$ *is monoidal, in the sense that*

$$\begin{array}{ccc} G(TA \otimes TB) & \xrightarrow{G(m^2_{A,B})} & GT(A \otimes B) \\ \downarrow{\scriptstyle g^2_{TA,TB}} & & \downarrow{\scriptstyle \Lambda_{A\otimes B}} \\ GTA \otimes GTB & & TG(A \otimes B) \\ \downarrow{\scriptstyle \Lambda_A \otimes \Lambda_B} & & \downarrow{\scriptstyle T(g^2_{A,B})} \\ TGA \otimes TGB & \xrightarrow[m^2_{GA,GB}]{} & T(GA \otimes GB) \end{array} \tag{15}$$

where (m^2, m^0) *is the strength of* T *and* (g^2, g^0) *is the strength of* G*, then* $(G_T, \delta_T, \epsilon_T)$ *is an oplax (symmetric) monoidal comonad on* $\mathbf{Kl}(T)$*. The oplax monoidal strength of* G_T *is given by*

$$g^2_{T,A,B} \quad := \quad \mathsf{F}_T(g^2_{A,B}) \quad = \quad \eta_{GA\otimes GB} \circ g^2_{A,B} \quad \in \quad \mathbf{Kl}(T)[G_T(A \otimes_{\mathbf{Kl}} B)\ ,\ G_T A \otimes_{\mathbf{Kl}} G_T B]$$
$$(= \quad \mathbb{C}[G(A \otimes B)\ ,\ T(GA \otimes GB)])$$

and

$$g^0_T \quad := \quad \mathsf{F}_T(g^0) \quad = \quad \eta_{\mathbf{I}} \circ g^0 \quad \in \quad \mathbf{Kl}(T)[G_T \mathbf{I}\ ,\ \mathbf{I}] \quad = \quad \mathbb{C}[G\mathbf{I}\ ,\ T\mathbf{I}]$$

where

$$g^2_{A,B} : G(A \otimes B) \to GA \otimes GB \qquad \text{and} \qquad g^0 : G\mathbf{I} \to \mathbf{I}$$

since g^2*,* g^0 *is an* oplax *monoidal strength.*

By applying now Prop. M.4 together with Prop. M.7, we thus get:

Corollary M.8. *With the same assumptions,* $\mathbf{Kl}(\Lambda)$ *is symmetric monoidal.*

Proof. We record for future use the monoidal structure of $\mathbf{Kl}(\Lambda) = \mathbf{Kl}(G_T)$:

- The monoidal product $\otimes_{\mathbf{Kl}}$ of $\mathbf{Kl}(\Lambda)$ is on objects the same as that of $\mathbb{C}$ and has the same unit $\mathbf{I}$.
  On morphisms, given

  $$\begin{array}{ll} & f \in \mathbf{Kl}(\Lambda)[A_0, B_0] = \mathbf{Kl}(G_T)[A_0, B_0] = \mathbf{Kl}(T)[GA_0, B_0] = \mathbb{C}[GA_0, TB_0] \\ \text{and} & g \in \mathbf{Kl}(\Lambda)[A_1, B_1] = \mathbb{C}[GA_1, TB_1] \end{array}$$

  we let $f \otimes_{\mathbf{Kl}} g$ be the composite

  $$G(A_0 \otimes A_1) \quad \overset{g^2_{A_0,A_1}}{\longrightarrow} \quad GA_0 \otimes GA_1 \quad \overset{f\otimes g}{\longrightarrow} \quad TB_0 \otimes TB_1 \quad \overset{m^2_{B_0,B_1}}{\longrightarrow} \quad T(B_0 \otimes B_1)$$

  where g^2 is the binary strength of G and m^2 that of T. Note that we could equivalently have taken the following composite (corresponding to composition in $\mathbf{Kl}(T)$):

  $$G(A_0 \otimes A_1) \quad \overset{g^2_{T,A_0,A_1}}{\longrightarrow} \quad T(GA_0 \otimes GA_1) \quad \overset{T(f\otimes_{\mathbf{Kl}(T)} g)}{\longrightarrow} \quad TT(B_0 \otimes B_1) \quad \overset{\mu_{B_0\otimes B_1}}{\longrightarrow} \quad T(B_0 \otimes B_1)$$

  since $g^2_{T,A_0,A_1} = \eta_{GA_0,GA_1} \circ g^2_{A_0,A_1}$ and by the monad laws:

  $$\mu_B \circ T(h) \circ \eta_A \quad = \quad \mu_B \circ \eta_B \circ h \quad = \quad h$$

- The structure maps of $\mathbf{Kl}(\Lambda)$ are taken to be the image under F_{G_T} of the structure maps of $\mathbf{Kl}(T)$, itself beeing the image under F_T of the structure maps of $\mathbb{C}$. Note that on maps,

  $$\mathsf{F}_{G_T}(\mathsf{F}_T(h)) \quad = \quad \eta_B \circ h \circ \epsilon_A \qquad \text{for } h : A \to B$$

□

Proof of Proposition M.7.

Naturality of $g^2_{T,A,B}$*.* The naturality of $g^2_{T,A,B}$, that is, in $\mathbf{Kl}(T)$:

$$\begin{array}{ccc} G_T(A \otimes_{\mathbf{Kl}} B) & \xrightarrow{G_T(f \otimes_{\mathbf{Kl}} g)} & G_T(A' \otimes_{\mathbf{Kl}} B') \\ {\scriptstyle g^2_{T,A,B}} \downarrow & & \downarrow {\scriptstyle g^2_{T,A',B'}} \\ G_T A \otimes_{\mathbf{Kl}} G_T B & \xrightarrow[G_T(f) \otimes_{\mathbf{Kl}} G_T(g)]{} & G_T A' \otimes_{\mathbf{Kl}} G_T B' \end{array}$$

(where $f \in \mathbf{Kl}(T)[A,B] = \mathbb{C}[A,TB]$ and $g \in \mathbf{Kl}(T)[A',B'] = \mathbb{C}[A',TB']$), amounts to, in $\mathbb{C}$:

$$\begin{array}{ccc} G(A\otimes B) & \xrightarrow{\Lambda_{A'\otimes B'}\circ G(m^2_{A',B'}\circ(f\otimes g))} & TG(A'\otimes B') \\ {\scriptstyle \eta_{GA\otimes GB}\circ g^2_{A,B}}\downarrow & & \downarrow{\scriptstyle \mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'}\circ g^2_{A',B'})} \\ T(GA\otimes GB) & \xrightarrow[\mu_{GA'\otimes GB'}\circ T(m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))]{} & T(GA'\otimes GB') \end{array}$$

By naturality of η, we have

$$\mu_{GA'\otimes GB'} \circ T(m^2_{GA',GB'} \circ ((\Lambda_{A'} \circ G(f)) \otimes (\Lambda_{B'} \circ G(g)))) \circ \eta_{GA\otimes GB} \quad = \quad \mu_{GA'\otimes GB'} \circ \eta_{T(GA'\otimes GB')} \circ m^2_{GA',GB'} \circ ((\Lambda_{A'} \circ G(f)) \otimes (\Lambda_{B'} \circ G(g)))$$

and by the unit monad law, we get:

$$\mu_{GA'\otimes GB'}\circ T(m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB} \quad = \quad m^2_{GA',GB'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g)))$$

and therefore (by bifunctoriality of $\otimes$):

$$\mu_{GA'\otimes GB'}\circ T(m^2_{A',B'}\circ((\Lambda_{A'}\circ G(f))\otimes(\Lambda_{B'}\circ G(g))))\circ\eta_{GA\otimes GB} \quad = \quad m^2_{GA',GB'}\circ(\Lambda_{A'}\otimes\Lambda_{B'})\circ(G(f)\otimes G(g))$$

From which it follows (by naturality of g^2) that

$$\mu_{GA'\otimes GB'} \circ T(m^2_{A',B'} \circ ((\Lambda_{A'} \circ G(f)) \otimes (\Lambda_{B'} \circ G(g)))) \circ \eta_{GA\otimes GB} \circ g^2_{A,B} \quad = \quad m^2_{GA',GB'} \circ (\Lambda_{A'} \otimes \Lambda_{B'}) \circ g^2_{TA',TB'} \circ G(f \otimes g)$$

On the other hand, also using the unit monad law we get:

$$\mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'}\circ g^2_{A',B'}) \quad = \quad \mu_{GA'\otimes GB'}\circ T(\eta_{GA'\otimes GB'})\circ T(g^2_{A',B'}) \quad = \quad T(g^2_{A',B'})$$

We are therefore finally left with

$$m^2_{GA',GB'} \circ (\Lambda_{A'} \otimes \Lambda_{B'}) \circ g^2_{TA',TB'} \quad = \quad T(g^2_{A',B'}) \circ \Lambda_{A'\otimes B'} \circ G(m^2_{A',B'})$$

which follows from (15).
Note that

$$G(TA'\otimes TB') \quad \stackrel{g^2_{TA',TB'}}{\longrightarrow} \quad GTA'\otimes GTB' \quad \stackrel{\Lambda_{A'}\otimes\Lambda_{B'}}{\longrightarrow} \quad TGA'\otimes TGB' \quad \stackrel{m^2_{GA',GB'}}{\longrightarrow} \quad T(GA'\otimes GB')$$

and

$$G(TA'\otimes TB') \quad \stackrel{G(m^2_{A',B'})}{\longrightarrow} \quad GT(A'\otimes B') \quad \stackrel{\Lambda_{A'\otimes B'}}{\longrightarrow} \quad TG(A'\otimes B') \quad \stackrel{T(g^2_{A',B'})}{\longrightarrow} \quad T(GA'\otimes GB')$$

Oplax Symmetric Monoidal Coherence of g^2_T and g^0_T. The coherence of g^2_T and g^0_T amount to the following diagrams.

– The associativity diagram:

$$\begin{array}{ccc}
(G_TA \otimes_{\mathbf{Kl}} G_TB) \otimes_{\mathbf{Kl}} G_TC & \xrightarrow{\alpha^{\mathbf{Kl}}_{G_TA,G_TB,G_TC}} & G_TA \otimes_{\mathbf{Kl}} (G_TB \otimes_{\mathbf{Kl}} G_TC) \\
{\scriptstyle g^2_{T,A,B}\otimes_{\mathbf{Kl}}\mathrm{id}^{\mathbf{Kl}}_{G_TC}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}^{\mathbf{Kl}}_{G_TA}\otimes_{\mathbf{Kl}} g^2_{T,B,C}} \\
G_T(A \otimes_{\mathbf{Kl}} B) \otimes_{\mathbf{Kl}} G_TC & & G_TA \otimes_{\mathbf{Kl}} G_T(B \otimes_{\mathbf{Kl}} C) \\
{\scriptstyle g^2_{T,A\otimes_{\mathbf{Kl}}B,C}}\uparrow & & \uparrow{\scriptstyle g^2_{T,A,B\otimes_{\mathbf{Kl}}C}} \\
G_T((A \otimes_{\mathbf{Kl}} B) \otimes_{\mathbf{Kl}} C) & \xrightarrow[G_T(\alpha^{\mathbf{Kl}}_{A,B,C})]{} & G_T(A \otimes_{\mathbf{Kl}} (B \otimes_{\mathbf{Kl}} C))
\end{array} \tag{16}$$

First, recall that $g^2_{T,A,B} = \mathsf{F}_T(g^2_{A,B})$ by definition and that on objects $G_TA = GA$, and also $\mathsf{F}_T(A) = A$ and $A \otimes_{\mathbf{Kl}} B = A \otimes B$. Moreover, $\alpha^{\mathbf{Kl}}_{A,B,C} = \mathsf{F}_T(\alpha_{A,B,C})$ and $\mathrm{id}^{\mathbf{Kl}}_A = \eta_A = \mathsf{F}_T(\mathrm{id}_A)$. Also, since $\eta_{(-)}$ is monoidal, given $\mathbb{C}$-maps f and g we have

$$(\eta_A \circ f) \otimes_{\mathbf{Kl}} (\eta_B \circ g) \quad = \quad m^2_{A,B} \circ ((\eta_A \circ f) \otimes (\eta_B \circ g)) \quad = \quad \eta_{A\otimes B} \circ (f \otimes g) \quad = \quad \mathsf{F}_T(f \otimes g)$$

Finally, thanks to the coherence diagram (14) of distributive laws, for the bottom horizontal map we have

$$\begin{aligned}
G_T(\alpha^{\mathbf{Kl}}_{A,B,C}) \quad &= \quad \Lambda_{A\otimes(B\otimes C)} \circ G(\eta_{A\otimes(B\otimes C)}) \circ G(\alpha_{A,B,C}) \\
&= \quad \eta_{G(A\otimes(B\otimes C))} \circ G(\alpha_{A,B,C}) \quad = \quad \mathsf{F}_T(G(\alpha_{A,B,C}))
\end{aligned}$$

It follows that (16) amounts to the following diagram in $\mathbf{Kl}(T)$:

$$\begin{array}{ccc}
(GA \otimes GB) \otimes GC & \xrightarrow{\mathsf{F}_T(\alpha_{GA,GB,GC})} & GA \otimes (GB \otimes GC) \\
{\scriptstyle \mathsf{F}_T(g^2_{A,B}\otimes \mathrm{id}_{GC})}\uparrow & & \uparrow{\scriptstyle \mathsf{F}_T(\mathrm{id}_{GA}\otimes g^2_{B,C})} \\
G(A \otimes B) \otimes GC & & GA \otimes G(B \otimes C) \\
{\scriptstyle \mathsf{F}_T(g^2_{A\otimes B,C})}\uparrow & & \uparrow{\scriptstyle \mathsf{F}_T(g^2_{A,B\otimes C})} \\
G((A \otimes B) \otimes C) & \xrightarrow[\mathsf{F}_T(G(\alpha_{A,B,C}))]{} & G(A \otimes (B \otimes C))
\end{array}$$

Now we are done since the above diagram is the image under the functor F_T of the associativity coherence diagram of oplax the monoidal functor G.

– The coherence diagrams for units and symmetry are:

$$\begin{array}{ccc}
\mathbf{I} \otimes_{\mathbf{Kl}} G_TA & \xrightarrow{\lambda^{\mathbf{Kl}}_{G_TA}} & G_TA \\
{\scriptstyle g^0_T\otimes_{\mathbf{Kl}}\mathrm{id}_{G_TA}}\downarrow & & \uparrow{\scriptstyle G_T(\lambda^{\mathbf{Kl}}_A)} \\
G_T\mathbf{I} \otimes_{\mathbf{Kl}} G_TA & \xleftarrow[g^2_{T,\mathbf{I},A}]{} & G_T(\mathbf{I} \otimes_{\mathbf{Kl}} A)
\end{array}
\qquad
\begin{array}{ccc}
G_TA \otimes_{\mathbf{Kl}} \mathbf{I} & \xrightarrow{\rho^{\mathbf{Kl}}_{G_TA}} & G_TA \\
{\scriptstyle \mathrm{id}_{G_TA}\otimes_{\mathbf{Kl}} g^0_T}\downarrow & & \uparrow{\scriptstyle G_T(\rho^{\mathbf{Kl}}_A)} \\
G_TA \otimes_{\mathbf{Kl}} G_T\mathbf{I} & \xleftarrow[g^2_{T,A,\mathbf{I}}]{} & G_T(A \otimes_{\mathbf{Kl}} \mathbf{I})
\end{array}$$

$$\begin{array}{ccc} G_TA \otimes_{\mathbf{Kl}} G_TB & \xrightarrow{\gamma^{\mathbf{Kl}}_{G_TA,G_TB}} & G_TB \otimes_{\mathbf{Kl}} G_TA \\ {\scriptstyle g^2_{T,A,B}}\uparrow & & \uparrow{\scriptstyle g^2_{T,B,A}} \\ G_T(A \otimes_{\mathbf{Kl}} B) & \xrightarrow[G_T(\gamma^{\mathbf{Kl}}_{A,B})]{} & G_T(B \otimes_{\mathbf{Kl}} A) \end{array}$$

They are dealt-with similarly. We only detail the case of the unit $\lambda^{\mathbf{Kl}}$. First, as above, we have $g^2_{T,\mathbf{I},A} = \mathsf{F}_T(g^2_{A,B})$ and $g^0_T = \mathsf{F}_T(g^0)$, and on objects: $G_T(A) = A$, $\mathsf{F}_T(A) = A$ and $A \otimes_{\mathbf{Kl}} B = A \otimes B$. Moreover, $\lambda^{\mathbf{Kl}}_A = \mathsf{F}_T(\lambda_A)$ and $\mathrm{id}^{\mathbf{Kl}}_A = \mathsf{F}_T(\mathrm{id}_A)$. Again by monoidality of $\eta_{(-)}$ we have

$$\begin{array}{rcl} g^0_T \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_{G_TA} & = & m^2_{\mathbf{I},A} \circ (\mathsf{F}_T(g^0) \otimes \mathsf{F}_T(\mathrm{id}_{GA})) \quad = \quad m^2_{\mathbf{I},A} \circ ((\eta_{\mathbf{I}} \circ g^0) \otimes (\eta_{GA} \circ \mathrm{id}_{GA})) \\ & = & \eta_{\mathbf{I} \otimes GA} \circ (g^0 \otimes \mathrm{id}_{GA}) \quad = \quad \mathsf{F}_T(g^0 \otimes \mathrm{id}_{GA}) \end{array}$$

Again by the coherence diagram (14) of distributive laws, we have

$$G_T(\lambda^{\mathbf{Kl}}_A) \quad = \quad \Lambda_A \circ G(\eta_A) \circ G(\lambda_A) \quad = \quad \eta_{GA} \circ G(\lambda_A) \quad = \quad \mathsf{F}_T(\lambda_A)$$

Then, as for the associativity coherence law above, we are done since we get the image under the functor F_T of the corresponding unit coherence diagram for the oplax strength of G in $\mathbb{C}$.

The natural map $\epsilon_{T,A}$ is monoidal. The corresponding diagrams are:

$$\begin{array}{ccc} G_TA \otimes_{\mathbf{Kl}} G_TB & \xrightarrow{\epsilon_{T,A} \otimes_{\mathbf{Kl}} \epsilon_{T,B}} & A \otimes_{\mathbf{Kl}} B \\ {\scriptstyle g^2_{T,A,B}}\uparrow & & \| \\ G_T(A \otimes_{\mathbf{Kl}} B) & \xrightarrow[\epsilon_{T,A \otimes_{\mathbf{Kl}} B}]{} & A \otimes_{\mathbf{Kl}} B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & \mathbf{I} \\ & \nearrow^{g^0_T} & \| \\ G_T\mathbf{I} & \xrightarrow[\epsilon_{T,\mathbf{I}}]{} & \mathbf{I} \end{array}$$

Reasonning as above (and in part. using the lax monoidality of $\eta_{(-)}$), these diagrams are quivalent to

$$\begin{array}{ccc} GA \otimes GB & \xrightarrow{\mathsf{F}_T(\epsilon_A \otimes \epsilon_B)} & A \otimes B \\ {\scriptstyle \mathsf{F}_T(g^2_{A,B})}\uparrow & & \| \\ G(A \otimes B) & \xrightarrow[\mathsf{F}_T(\epsilon_{A \otimes B})]{} & A \otimes_{\mathbf{Kl}} B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & \mathbf{I} \\ & \nearrow^{\mathsf{F}_T(g^0)} & \| \\ G\mathbf{I} & \xrightarrow[\mathsf{F}(\epsilon_{\mathbf{I}})]{} & \mathbf{I} \end{array}$$

Now we are done since recalling that F_T is the identity on objects, the above diagrams are the image under F_T of the oplax monoidal coherence digrams of $\epsilon_{(-)}$.

The natural map $\delta_{T,A}$ is monoidal.

$$
\begin{array}{ccc}
G_T A \otimes_{\mathbf{Kl}} G_T B & \xrightarrow{\delta_{T,A} \otimes_{\mathbf{Kl}} \delta_{T,B}} & G_T G_T A \otimes_{\mathbf{Kl}} G_T G_T B \\
{\scriptstyle g^2_{T,A,B}} \uparrow & & \uparrow {\scriptstyle G_T(g^2_{T,A,B}) \circ g^2_{T,G_T A, G_T B}} \\
G_T(A \otimes_{\mathbf{Kl}} B) & \xrightarrow[\delta_{T,A \otimes_{\mathbf{Kl}} B}]{} & G_T G_T (A \otimes_{\mathbf{Kl}} B)
\end{array}
$$

and

$$
\begin{array}{ccccc}
 & & \mathbf{I} & & \\
 & \nearrow^{g^0_T} & & \searrow^{G_T(g^0_T) \circ g^0_T} & \\
G_T \mathbf{I} & & \xrightarrow[\delta_{\mathbf{I}}]{} & & G_T G_T \mathbf{I}
\end{array}
$$

Reasonning as above, using coherence diagram (14) of distributive laws, we have

$$G_T(g^2_{T,A,B}) \quad = \quad \Lambda_{GA \otimes GB} \circ G(\eta_{GA \otimes GB}) \circ g^2_{A,B} \quad = \quad \eta_{G(GA \otimes GB)} \circ G(g^2_{A,B}) \quad = \quad \mathsf{F}_T(g^2_{A,B})$$

and we then conclude as in the case of $\epsilon_{(-)}$ above.

M.3 Monoids and Comonoids

Monoids. Recall from e.g. [27] that a commutative monoid in an SMC $(\mathbb{C}, \otimes, \mathbf{I})$ is a triple $M = (M, u, m)$ where M is an object of $\mathbb{C}$ and u and m are morphisms

$$\mathbf{I} \quad \xrightarrow{u} \quad M \quad \xleftarrow{m} \quad M \otimes M$$

subject to the following coherence diagrams:

$$
\begin{array}{ccccc}
(M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{\mathrm{id}_M \otimes m} & M \otimes M \\
{\scriptstyle m \otimes \mathrm{id}_M} \downarrow & & & & \downarrow {\scriptstyle m} \\
M \otimes M & & \xrightarrow[m]{} & & M
\end{array}
\tag{17}
$$

$$
\begin{array}{ccccc}
\mathbf{I} \otimes M & \xrightarrow{u \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes u} & M \otimes \mathbf{I} \\
 & {\scriptstyle \lambda} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle \rho} & \\
 & & M & &
\end{array}
\tag{18}
$$

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{\gamma} & M \otimes M \\
 {\scriptstyle m} \searrow & & \swarrow {\scriptstyle m} \\
 & M &
\end{array}
\tag{19}
$$

It is well-known (see e.g. [27, Prop. 2]) that we always have $\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}$ in a monoidal category.

Proposition M.9. *If $M = (M, u, m)$ is a monoid object in $\mathbb{C}$, then*

$$\begin{CD} \mathbf{I} \otimes \mathbf{I} @>{u \otimes u}>> M \otimes M \\ @V{\rho_{\mathbf{I}} = \lambda_{\mathbf{I}}}VV @VV{m}V \\ \mathbf{I} @>>{u}> M \end{CD}$$

Proof. By bifunctoriality of $\otimes$, it is equivalent to show

$$\begin{array}{ccccc} \mathbf{I} \otimes \mathbf{I} & \xrightarrow{\mathrm{id}_{\mathbf{I}} \otimes u} & I \otimes M & \xrightarrow{u \otimes \mathrm{id}_M} & M \otimes M \\ \downarrow{\scriptstyle \lambda_{\mathbf{I}}} & & & & \downarrow{\scriptstyle m} \\ \mathbf{I} & & \xrightarrow[u]{\qquad\qquad\qquad} & & M \end{array}$$

But $m \circ (u \otimes \mathrm{id}_M) = \lambda_M$ by the unit law (18), and we are done since by naturality of λ we have

$$\lambda_M \circ (\mathrm{id}_{\mathbf{I}} \otimes u) \quad = \quad u \circ \lambda_{\mathbf{I}}$$

□

The Category Mon($\mathbb{C}$) of Commutative Monoids. The category **Mon**($\mathbb{C}$) of commutative monoids of $\mathbb{C}$ has monoids as objects, and as morphisms from (M, u, m) to (M', u', m'), $\mathbb{C}$-morphisms $f : M \to M'$ making the following two diagrams commute:

$$\begin{CD} M \otimes M @>{f \otimes f}>> M' \otimes M' \\ @V{m}VV @VV{m'}V \\ M @>>{f}> M' \end{CD} \qquad \text{and} \qquad \begin{array}{ccc} & \mathbf{I} & \\ {\scriptstyle u} \swarrow & & \searrow {\scriptstyle u'} \\ M & \xrightarrow[f]{\qquad} & M' \end{array}$$

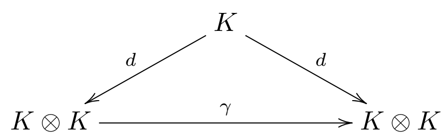
Comonoids. Dually, a commutative monoid in $\mathbb{C}$ is a triple $K = (K, e, d)$ where

$$\mathbf{I} \quad \xleftarrow{e} \quad K \quad \xrightarrow{d} \quad M \otimes M$$

subject to the following coherence diagrams:

$$\begin{array}{ccccc} K & & \xrightarrow{\qquad\qquad d \qquad\qquad} & & K \otimes K \\ \downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle d \otimes \mathrm{id}_K} \\ K \otimes K & \xrightarrow{\mathrm{id}_K \otimes d} & K \otimes (K \otimes K) & \xleftarrow{\alpha} & (K \otimes K) \otimes K \end{array} \tag{20}$$

$$\begin{array}{ccccc} \mathbf{I} \otimes K & \xleftarrow{(e \otimes \mathrm{id}_K)} & K \otimes K & \xrightarrow{(\mathrm{id}_K \otimes e)} & K \otimes \mathbf{I} \\ & {\scriptstyle \lambda} \searrow & \uparrow{\scriptstyle d} & \swarrow {\scriptstyle \rho} & \\ & & K & & \end{array} \tag{21}$$



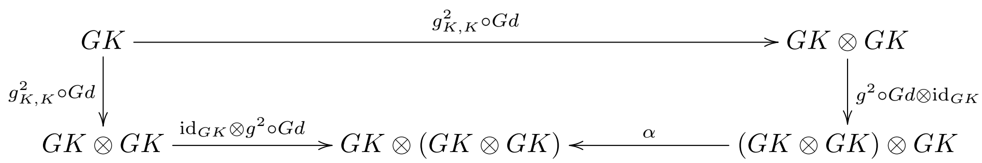
(22)

We record the following simple fact.

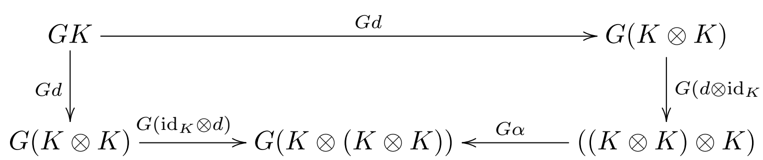
Proposition M.10. *Given symmetric monoidal categories $\mathbb{C}$, $\mathbb{D}$, an oplax symmetric monoidal functor $G : \mathbb{C} \to \mathbb{D}$, and a commutative comonoid (K, e, d) of $\mathbb{C}$, then $(GK, g^0 \circ Ge, g^2_{K,K} \circ Gd)$ is a commutative comonoid in $\mathbb{D}$, where (g^0, g^2) is the oplax strength of G.*

Proof. We check the required diagrams.

Diagram (20) unfolds to:



Note that since (K, e, d) is a comonoid in $\mathbb{C}$, and since G is a functor, we have



By naturality of g^2, we have

$$((g^2 \circ Gd) \otimes \mathrm{id}_{GK}) \circ g^2_{K,K} \quad = \quad (g^2 \otimes \mathrm{id}_{GK}) \circ (Gd \otimes G(\mathrm{id}_K)) \circ g^2_{K,K} \quad = \quad (g^2 \otimes \mathrm{id}_{GK}) \circ g^2_{K \otimes K, K} \circ G(d \otimes \mathrm{id}_K)$$

From which it follows by oplax monoidality of G that

$$\alpha \circ ((g^2 \circ Gd) \otimes \mathrm{id}_{GK}) \circ g^2_{K,K} \circ Gd \quad = \quad (\mathrm{id}_{GK} \otimes g^2) \circ g^2_{K, K \otimes K} \circ G(\alpha) \circ G(d \otimes \mathrm{id}_K) \circ Gd$$

But by functoriality of G, since (K, e, d) is a comonoid in $\mathbb{C}$ we have

$$G(\alpha) \circ G(d \otimes \mathrm{id}_K) \circ Gd \quad = \quad G(\mathrm{id}_K \otimes d) \circ Gd$$

so that

$$\alpha \circ ((g^2 \circ Gd) \otimes \mathrm{id}_{GK}) \circ g^2_{K,K} \circ Gd \quad = \quad (\mathrm{id}_{GK} \otimes g^2) \circ g^2_{K, K \otimes K} \circ G(\mathrm{id}_K \otimes d) \circ Gd$$

The other diagrams are dealt-with similarly. □

The Category Comon($\mathbb{C}$) of Commutative Comonoids. The category **Comon**($\mathbb{C}$) of commutative comonoids of $\mathbb{C}$ has comonoids as objects, and as morphisms from (K, e, d) to (K', e', d'), $\mathbb{C}$-morphisms $f : K \to K'$ making the following two diagrams commute:

$$
\begin{array}{ccc}
K & \xrightarrow{\;f\;} & K' \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\
K \otimes K & \xrightarrow[f \otimes f]{} & K' \otimes K'
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
K & \xrightarrow{\;f\;} & K' \\
{\scriptstyle e}\searrow & & \swarrow{\scriptstyle e'} \\
& \mathbf{I} &
\end{array}
$$

Lifting of Monoids and Comonoids to Kleiseli Categories. We note here the following proposition, to be used in §J (together with Prop. L.1).

Proposition M.11. *Let $\mathbb{C}$ be a symmetric monoidal category.*

(a) Let $T = (T, \mu, \eta)$ be a (lax) symmetric monoidal monad on $\mathbb{C}$.
- *(i) If (M, u, m) is a commutative monoid in $\mathbb{C}$, then $(M, \mathsf{F}_T(u), \mathsf{F}_T(m))$ is a commutative monoid in $\mathbf{Kl}(T)$.*
- *(ii) If (K, e, d) is a commutative comonoid in $\mathbb{C}$, then $(K, \mathsf{F}_T(e), \mathsf{F}_T(d))$ is a commutative comonoid in $\mathbf{Kl}(T)$.*

(b) Let $G = (G, \delta, \epsilon)$ be an oplax symmetric monoidal comonad on $\mathbb{C}$.
- *(i) If (M, u, m) is a commutative monoid in $\mathbb{C}$, then $(M, \mathsf{F}_G(u), \mathsf{F}_G(m))$ is a commutative monoid in $\mathbf{Kl}(G)$.*
- *(ii) If (K, e, d) is a commutative comonoid in $\mathbb{C}$, then $(K, \mathsf{F}_G(e), \mathsf{F}_G(d))$ is a commutative comonoid in $\mathbf{Kl}(G)$.*

We only prove Prop. M.11.(a) since the case M.11.(b) follows by duality.

Proof of Proposition M.11.(ai). Write $(\mathsf{t}^2, \mathsf{t}^0)$ for the (lax) strength of T. Thanks to Prop. M.2, the coherence diagrams of $(M, \mathsf{F}_T(u), \mathsf{F}_T(m))$ amount to the following in $\mathbf{Kl}(T)$.

– Coherence w.r.t. associativity amounts in $\mathbf{Kl}(T)$ to:

$$
\begin{array}{ccccc}
(M \otimes M) \otimes M & \xrightarrow{\mathsf{F}_T(\alpha)} & M \otimes (M \otimes M) & \xrightarrow{\mathrm{id}^{\mathbf{Kl}}_M \otimes_{\mathbf{Kl}} \mathsf{F}_T(m)} & M \otimes M \\
{\scriptstyle \mathsf{F}_T(m) \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_M}\downarrow & & & & \downarrow{\scriptstyle \mathsf{F}_T(m)} \\
M \otimes M & & \xrightarrow[\mathsf{F}_T(m)]{} & & M
\end{array}
$$

Note that

$$\mathsf{F}_T(m) \circ_{\mathbf{Kl}} (\mathsf{F}_T(m) \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_M) \quad = \quad \mu_M \circ T(\eta_{M \otimes M}) \circ T(m) \circ \mathsf{t}^2_{M,M} \circ ((\eta_M \circ m) \otimes (\eta_M))$$

Reasonning similarly as in the proof of Prop. M.7, we have

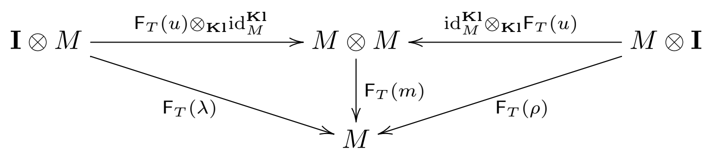
$$
\begin{aligned}
\mathsf{F}_T(m) \circ_{\mathbf{Kl}} (\mathsf{F}_T(m) \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_M) \quad &= \quad T(m) \circ \eta_{M \otimes M} \circ (m \otimes \mathrm{id}_M) \quad = \quad \eta_M \circ m \circ (m \otimes \mathrm{id}_M) \\
&= \quad \mathsf{F}_T(m \circ (m \otimes \mathrm{id}_M))
\end{aligned}
$$

We similarly obtain

$$\mathsf{F}_T(m) \circ_{\mathbf{Kl}} (\mathrm{id}^K l_M \otimes_{\mathbf{Kl}} \mathsf{F}_T(m)) \quad = \quad \mathsf{F}_T(m \circ (\mathrm{id}_M \otimes m))$$

and we are done using the functoriality of F_T and the associativity coherence diagram (17) of monoids.

– Coherence w.r.t. units amounts in $\mathbf{Kl}(T)$ to:

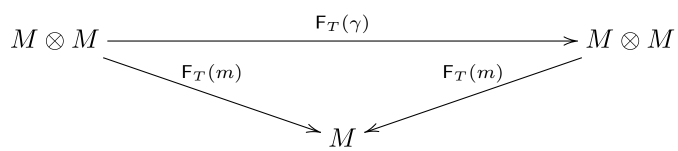


Reasonning as above, we obtain:

$$\begin{array}{lrcl} & \mathsf{F}_T(m) \circ_{\mathbf{Kl}} (\mathsf{F}_T(u) \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_M) & = & \mathsf{F}_T(m \circ (u \otimes \mathrm{id}_M)) \\ \text{and} & \mathsf{F}_T(m) \circ_{\mathbf{Kl}} (\mathrm{id}^{\mathbf{Kl}}_M \otimes_{\mathbf{Kl}} \mathsf{F}_T(u)) & = & \mathsf{F}_T(m \circ (\mathrm{id}_M \otimes u)) \end{array}$$

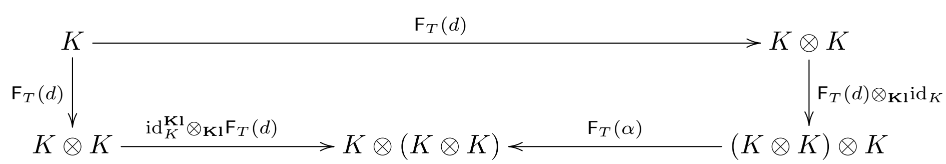
and we are done using the units cohrence diagram (18)

– Coherence w.r.t. symmetry amounts in $\mathbf{Kl}(T)$ to:



and follows directly from diagram (19).

Proof of Proposition M.11.(aii). We proceed similarly as in the case (ai). We only detail the case of coherence w.r.t. associativity, which amounts in $\mathbf{Kl}(T)$ to:



Note that

$$\begin{array}{rcl} (\mathrm{id}^{\mathbf{Kl}}_K \otimes_{\mathbf{Kl}} \mathsf{F}_T(d)) \circ_{\mathbf{Kl}} \mathsf{F}_T(d) & = & \mu_{K\otimes(K\otimes K)} \circ T(\mathrm{id}^{\mathbf{Kl}}_K \otimes_{\mathbf{Kl}} \mathsf{F}_T(d)) \circ \eta_{K\otimes K} \circ d \\ & = & \mu_{K\otimes(K\otimes K)} \circ \eta_{T(K\otimes(K\otimes K))} \circ (\mathrm{id}^{\mathbf{Kl}}_K \otimes_{\mathbf{Kl}} \mathsf{F}_T(d)) \circ \circ d \\ & = & (\mathrm{id}^{\mathbf{Kl}}_K \otimes_{\mathbf{Kl}} \mathsf{F}_T(d)) \circ d \\ & = & \eta_{K\otimes(K\otimes K)} \circ (\mathrm{id}_K \otimes d) \circ d \\ & = & \mathsf{F}_T((\mathrm{id}_K \otimes d) \circ d) \end{array}$$

We similarly obtain

$$(\mathsf{F}_T(d) \otimes_{\mathbf{Kl}} \mathrm{id}^{\mathbf{Kl}}_K) \circ_{\mathbf{Kl}} \mathsf{F}_T(d) \quad = \quad \mathsf{F}_T((d \otimes \mathrm{id}_K) \circ d)$$

and we conclude using the functoriality of F_T and the associativity coherence diagram (20) of comonoids.

The Monad of Monoid Indexing. Following [18, §2.5], a monoid (M, u, m) in a monoidal category $\mathbb{C}$ gives rise to a monad $T = (T, \mu, \eta)$ where $T(-) := (-) \otimes M$,

$$\begin{array}{lllllll} & \eta_A & := & (\mathrm{id}_A \otimes u) \circ \rho_A^{-1} & : & A & \longrightarrow & A \otimes M \\ \text{and} & \mu_A & := & (\mathrm{id}_A \otimes m) \circ \alpha_{A,M,M} & : & (A \otimes M) \otimes M & \longrightarrow & A \otimes M \end{array}$$

It is well-known (see e.g. [18, §2.5] or [27, §6.6]) that (T, μ, η) is a monad. We check here that T is actually a (lax) monoidal monad. The strength of T is

$$m^2_{A,B} \quad : \quad (A \otimes M) \otimes (B \otimes M) \to (A \otimes B) \otimes M \qquad \text{and} \qquad m^0 : \mathbf{I} \to \mathbf{I} \otimes M$$

where $m^2_{A,B}$ is the composite

$$(A \otimes M) \otimes (B \otimes M) \quad \stackrel{\theta_{A,B}}{\longrightarrow} \quad (A \otimes B) \otimes (M \otimes M) \quad \stackrel{\mathrm{id} \otimes m}{\longrightarrow} \quad (A \otimes B) \otimes M$$

where $\theta_{A,B}$ is a natural map made of identities and structure maps of $\mathbb{C}$, and where m^0 is the composite

$$\mathbf{I} \quad \stackrel{\lambda_{\mathbf{I}}^{-1}}{\longrightarrow} \quad \mathbf{I} \otimes \mathbf{I} \quad \stackrel{\mathrm{id}_{\mathbf{I}} \otimes u}{\longrightarrow} \quad \mathbf{I} \otimes M$$

The map $\theta_{A,B}$ is explicitely defined as the following composite:

$$\begin{array}{l} (A{\otimes}M){\otimes}(B{\otimes}M) \quad \stackrel{\alpha}{\longrightarrow} \quad A{\otimes}(M{\otimes}(B{\otimes}M)) \quad \stackrel{\mathrm{id}_A \otimes \gamma}{\longrightarrow} \quad A{\otimes}((B{\otimes}M){\otimes}M) \quad \stackrel{\mathrm{id}_A \otimes \alpha}{\longrightarrow} \\ \qquad\qquad A \otimes (B \otimes (M \otimes M)) \quad \stackrel{\alpha^{-1}}{\longrightarrow} \quad (A \otimes B) \otimes (M \otimes M) \end{array}$$

Note that (T, μ, η) is only a *lax* monad, since the structure maps of monoid objects are in general not isos.

Proposition M.12. *(T, μ, η) is a (lax) symmetric monoidal monad.*

By applying Prop. M.2 to Prop. M.12 we thus get:

Corollary M.13. **Kl**(T) *is symmetric monoidal.*

Proof of Proposition M.12.

$T(-) = (-) \otimes M$ is a (strong) symmetric monoidal functor. The diagrams to check amount to the following:

$$\begin{array}{ccc}
((A \otimes M) \otimes (B \otimes M)) \otimes (C \otimes M) & \xrightarrow{\alpha_{TA,TB,TC}} & (A \otimes M) \otimes ((B \otimes M) \otimes (C \otimes M)) \\
{\scriptstyle ((\mathrm{id}_{A\otimes B} \otimes m)\circ\theta_{A,B})\otimes \mathrm{id}_{C\otimes M}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{A\otimes M} \otimes ((\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{B,C})} \\
((A \otimes B) \otimes M) \otimes (C \otimes M) & & (A \otimes M) \otimes ((B \otimes C) \otimes M) \\
{\scriptstyle (\mathrm{id}_{A\otimes B} \otimes m)\circ\theta_{A\otimes B,C}} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{A,B\otimes C}} \\
((A \otimes B) \otimes C) \otimes M & \xrightarrow[\alpha_{A,B,C}\otimes \mathrm{id}_M]{} & (A \otimes (B \otimes C)) \otimes M
\end{array}$$

which follows from the monoid coherence law (17) of (M, u, m) and the monoidal coherence $\mathbb{C}$, and to

$$\begin{array}{ccc}
\mathbf{I} \otimes (A \otimes M) & \xrightarrow{\lambda_{A\otimes M}} & A \otimes M \\
\downarrow {\scriptstyle ((\mathrm{id}_{\mathbf{I}}\otimes u)\circ\lambda_{\mathbf{I}}^{-1})\otimes \mathrm{id}_{A\otimes M}} & & \uparrow {\scriptstyle \lambda_A \otimes \mathrm{id}_M} \\
(\mathbf{I} \otimes M) \otimes (A \otimes M) & \xrightarrow[(\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{\mathbf{I},A}]{} & (\mathbf{I} \otimes A) \otimes M
\end{array}$$

and

$$\begin{array}{ccc}
(A \otimes M) \otimes \mathbf{I} & \xrightarrow{\rho_{A\otimes M}} & A \otimes M \\
\downarrow {\scriptstyle \mathrm{id}_{A\otimes M}\otimes((\mathrm{id}_{\mathbf{I}}\otimes u)\circ\lambda_{\mathbf{I}}^{-1})} & & \uparrow {\scriptstyle \rho_A \otimes \mathrm{id}_M} \\
(A \otimes M) \otimes (\mathbf{I} \otimes M) & \xrightarrow[(\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{A,\mathbf{I}}]{} & (A \otimes \mathbf{I}) \otimes M
\end{array}$$

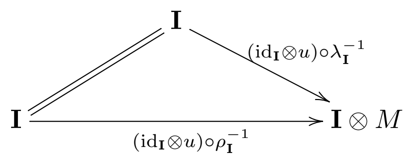
which follow from the monoid coherence laws (18) of (M, u, m) and the monoidal coherence of $\mathbb{C}$ and finally

$$\begin{array}{ccc}
(A \otimes M) \otimes (B \otimes M) & \xrightarrow{\gamma_{TA,TB}} & (B \otimes M) \otimes (A \otimes M) \\
{\scriptstyle (\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{A,B}} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}_{A\otimes B}\otimes m)\circ\theta_{B,A}} \\
(A \otimes B) \otimes M & \xrightarrow[\gamma_{A,B}\otimes \mathrm{id}_M]{} & (B \otimes A) \otimes M
\end{array}$$

which follows from commutative monoid coherence law (19) of (M, u, m) together with the symmetric monoidal coherence of $\mathbb{C}$.

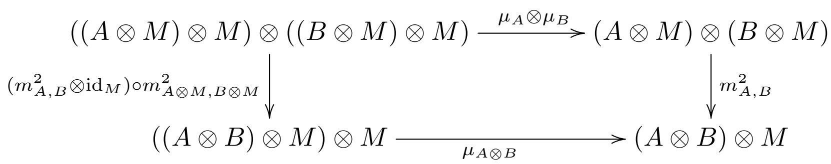
The map $\eta_A : A \to A \otimes M$ is monoidal. We have to check:

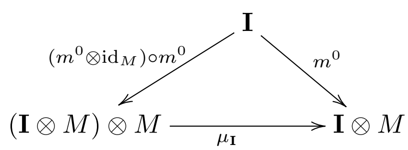
$$\begin{array}{ccc}
A \otimes B & \xrightarrow{((\mathrm{id}_A\otimes u)\circ\rho_A^{-1})\otimes((\mathrm{id}_B\otimes u)\circ\rho_B^{-1})} & (A \otimes M) \otimes (B \otimes M) \\
\| & & \downarrow {\scriptstyle (\mathrm{id}\otimes m)\circ\theta_{A,B}} \\
A \otimes B & \xrightarrow[(\mathrm{id}_{A\otimes B}\otimes u)\circ\rho_{A\otimes B}^{-1}]{} & (A \otimes B) \otimes M
\end{array}$$



The first diagram follows from Prop. M.9. The second one directly follows from the fact that $\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}$ (see e.g. [27, Prop. 2]).

The map $\mu_A : (A \otimes M) \otimes M \to A \otimes M$ *is monoidal.* We check:





for

$$m^2_{A,B} = (\mathrm{id}_{A\otimes B} \otimes m) \circ \theta_{A,B} \qquad \text{and} \qquad m^0 = (\mathrm{id}_{\mathbf{I}} \otimes u) \circ \lambda^{-1}_{\mathbf{I}} \qquad \text{and} \qquad \mu_A = (\mathrm{id}_A \otimes m) \circ \alpha_{A,M,M}$$

The first diagram follows from the monoid coherence laws (17) and (19) together with the symmetric monoidal coherence of $\mathbb{C}$. The second diagram follows from Prop. M.9.

The Comonad of Comonoid Indexing. Dually, a comonoid (K, e, d) in a monoidal category $\mathbb{C}$ gives rise to a comonad $G = (G, \delta, \epsilon)$ where $G(-) := K \otimes (-)$, and

$$\begin{array}{lllllll} & \epsilon_A & := & \lambda_A \circ (e \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & A \\ \text{and} & \delta_A & := & \alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A) & : & K \otimes A & \longrightarrow & K \otimes (K \otimes A) \end{array}$$

Since a comonoid on $\mathbb{C}$ is a monoid on $\mathbb{C}^{\mathrm{op}}$, it is also well-known (again from e.g. [18, §2.5] or [27, §6.8]) that G is a comonad. Dually to §M.3, G is actually *oplax* symmetric monoidal. Its strength is

$$g^2_{A,B} \;:\; K \otimes (A \otimes B) \longrightarrow (K \otimes A) \otimes (K \otimes B) \qquad \text{and} \qquad g^0 \;:\; K \otimes \mathbf{I} \longrightarrow \mathbf{I}$$

where $g^2_{A,B}$ is the composite

$$K \otimes (A \otimes B) \overset{d \otimes \mathrm{id}}{\longrightarrow} (K \otimes K) \otimes (A \otimes B) \overset{\vartheta_{A,B}}{\longrightarrow} (K \otimes A) \otimes (K \otimes B)$$

where $\vartheta_{A,B}$ is a natural map made of identities and structure maps of $\mathbb{C}$, and where g^0 is the composite

$$K \otimes \mathbf{I} \quad \xrightarrow{e\otimes \mathrm{id}_{\mathbf{I}}} \quad \mathbf{I} \otimes \mathbf{I} \quad \xrightarrow{\lambda_{\mathbf{I}}} \quad \mathbf{I}$$

The map $\vartheta_{A,B}$ is explicitely defined as the following composite:

$$\begin{array}{c} (K{\otimes}K){\otimes}(A{\otimes}B) \quad \xrightarrow{\alpha} \quad K{\otimes}(K{\otimes}(A{\otimes}B)) \quad \xrightarrow{\mathrm{id}_A\otimes\alpha^{-1}} \quad K{\otimes}((K{\otimes}A){\otimes}B) \quad \xrightarrow{\gamma} \\ ((K \otimes A) \otimes B) \otimes K \quad \xrightarrow{\alpha} \quad (K \otimes A) \otimes (K \otimes B) \end{array}$$

By duality, from Prop. M.12 we get:

Proposition M.14. *(G, δ, ϵ) is an oplax symmetric monoidal comonad.*

Similarly to Cor. M.13, by applying Prop. M.4 to Prop. M.14 we get:

Corollary M.15. *$\mathbf{Kl}(G)$ is symmetric monoidal.*

The Distributive Law of Comonoid over Monoid Indexing. We now check that there is distributive law Φ of (the comonad of) comonoid indexing over (the monad of) monoid indexing. Moreover, Φ is monoidal in the sense of Prop. M.7.

Proposition M.16. *Consider, in an SMC $(\mathbb{C}, \otimes, \mathbf{I})$, a comonoid (K, e, d) and a monoid (M, u, m), inducing respectivelly the comonad (G, δ, ϵ) with*

$$GA := K{\otimes}A \qquad \epsilon_A := \lambda_A {\circ} (e{\otimes}\mathrm{id}_A) : K{\otimes}A \longrightarrow A \qquad \delta_A := \alpha_{K,K,A} {\circ} (d{\otimes}\mathrm{id}_A) : K{\otimes}A \longrightarrow K{\otimes}(K{\otimes}A)$$

and the monad (T, μ, η) with

$$TA := A{\otimes}M \qquad \eta_A := (\mathrm{id}_A{\otimes}u){\circ}\rho_A^{-1} : A \longrightarrow A{\otimes}M \qquad \mu_A := (\mathrm{id}_A{\otimes}m){\circ}\alpha_{A,M,M} : (A{\otimes}M){\otimes}M \longrightarrow A{\otimes}M$$

Then,

(i) the associativity structure map

$$\Phi_A := \alpha^{-1}_{K,A,M} \quad : \quad GTA = K \otimes (A \otimes M) \quad \Longrightarrow \quad (K \otimes A) \otimes M = TGA$$

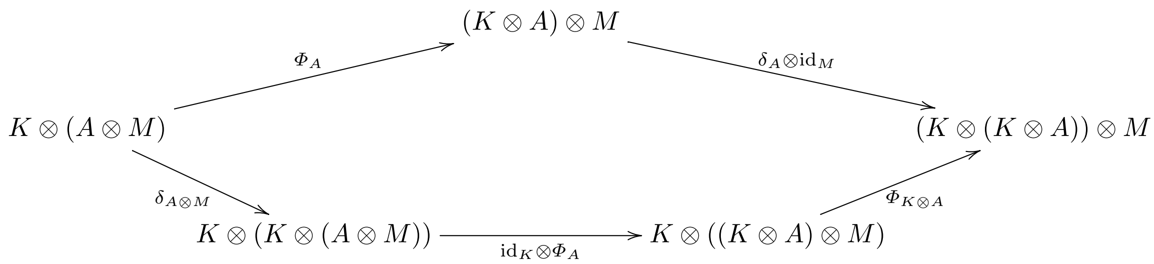
is a distributive law of G over T,

(ii) and it is moreover monoidal (in the sense of Prop. M.7), that is:

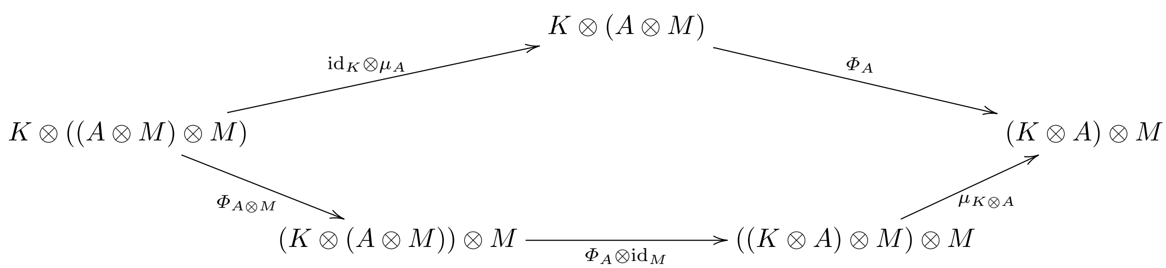
$$\begin{array}{ccc} G(TA \otimes TB) & \xrightarrow{G(m^2_{A,B})} & GT(A \otimes B) \\ \downarrow{\scriptstyle g^2_{TA,TB}} & & \downarrow{\scriptstyle \Phi_{A\otimes B}} \\ GTA \otimes GTB & & TG(A \otimes B) \\ \downarrow{\scriptstyle \Phi_A\otimes\Phi_B} & & \downarrow{\scriptstyle T(g^2_{A,B})} \\ TGA \otimes TGB & \xrightarrow[m^2_{GA,GB}]{} & T(GA \otimes GB) \end{array} \tag{23}$$

where (m^2, m^0) is the (lax) strength of T and (g^2, g^0) is the oplax strength of G.

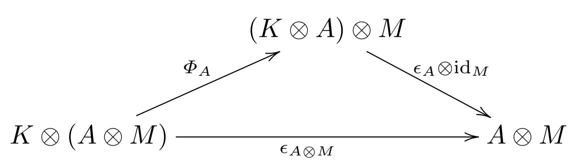
Proof of Proposition M.16.(i). First, note that $\Phi_{(-)}$ is natural by assumption. The diagrams of §M.2 unfold to:



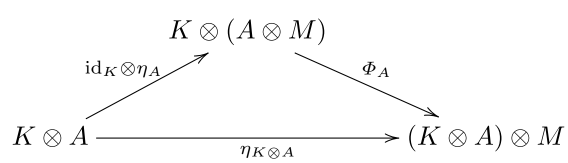
(24)



(25)

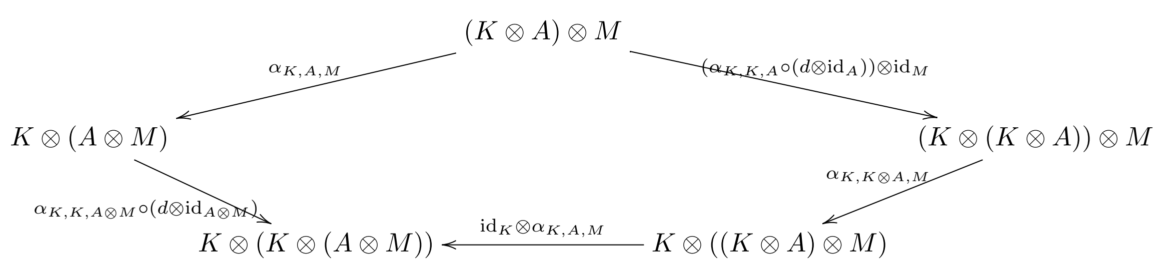


(26)



(27)

– Diagram (24) amounts to



By functoriality of $\otimes$ we have

$$(\alpha_{K,K,A}\circ(d\otimes \mathrm{id}_A))\otimes \mathrm{id}_M \quad = \quad (\alpha_{K,K,A}\circ(d\otimes \mathrm{id}_A))\otimes(\mathrm{id}_M\circ \mathrm{id}_M) \quad = \quad (\alpha_{K,K,A}\otimes \mathrm{id}_M)\circ((d\otimes \mathrm{id}_A)\otimes \mathrm{id}_M)$$

and therefore

$$(\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ ((\alpha_{K,K,A} \circ (d \otimes \mathrm{id}_A)) \otimes \mathrm{id}_M) \quad = \\ (\mathrm{id}_K \otimes \alpha_{K,A,M}) \circ \alpha_{K,K\otimes A,M} \circ (\alpha_{K,K,A} \otimes \mathrm{id}_M) \circ ((d \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

From the pentagon law, it follows that

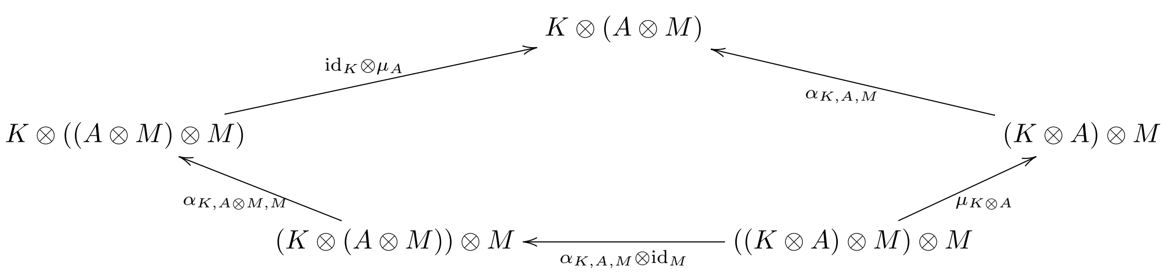
$$(\mathrm{id}_K\otimes\alpha_{K,A,M})\circ\alpha_{K,K\otimes A,M}\circ((\alpha_{K,K,A}\circ(d\otimes \mathrm{id}_A))\otimes \mathrm{id}_M) \quad = \quad \alpha_{K,K,A\otimes M}\circ\alpha_{K\otimes K,A,M}\circ((d\otimes \mathrm{id}_A)\otimes \mathrm{id}_M)$$

and from by naturality of α we get

$$(\mathrm{id}_K\otimes\alpha_{K,A,M})\circ\alpha_{K,K\otimes A,M}\circ((\alpha_{K,K,A}\circ(d\otimes \mathrm{id}_A))\otimes \mathrm{id}_M) \quad = \quad \alpha_{K,K,A\otimes M}\circ(d\otimes(\mathrm{id}_A\otimes \mathrm{id}_M))\circ\alpha_{K,A,M}$$

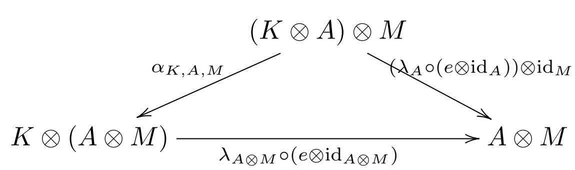
and we are done since $\mathrm{id}_A \otimes \mathrm{id}_M = \mathrm{id}_{A\otimes M}$ by bifunctoriality of $\otimes$.

– Diagram (25), which unfolds to



is dealt-with similarly.

– Diagram (26) amounts to



By bi-functoriality of $\otimes$, we have $\mathrm{id}_{A\otimes M} = \mathrm{id}_A \otimes \mathrm{id}_M$, and by naturality of α it follows that

$$\lambda_{A\otimes M} \circ (e \otimes \mathrm{id}_{A\otimes M}) \circ \alpha_{K,A,M} \quad = \quad \lambda_{A\otimes M} \circ \alpha_{\mathbf{I},A,M} \circ ((e \otimes \mathrm{id}_A) \otimes \mathrm{id}_M)$$

On the other hand, by functoriality of $\otimes$, we have

$$(\lambda_A\circ(e\otimes \mathrm{id}_A))\otimes \mathrm{id}_M \quad = \quad (\lambda_A\circ(e\otimes \mathrm{id}_A))\otimes(\mathrm{id}_M\circ \mathrm{id}_M) \quad = \quad (\lambda_A\otimes \mathrm{id}_M)\circ((e\otimes \mathrm{id}_A)\otimes \mathrm{id}_M)$$

and we are done since $\lambda_{A\otimes M} \circ \alpha_{\mathbf{I},A,M} = \lambda_A \otimes \mathrm{id}_M$ by [27, Prop. 1].

– Diagram (27) unfolds to

$$\begin{array}{ccc} & K \otimes (A \otimes M) & \\ {\scriptstyle \mathrm{id}_K \otimes ((\mathrm{id}_A \otimes u) \circ \rho_A^{-1})} \nearrow & & \nwarrow {\scriptstyle \alpha_{K,A,M}} \\ K \otimes A & \xrightarrow[(\mathrm{id}_{K \otimes A} \otimes u) \circ \rho_{K \otimes A}^{-1}]{} & (K \otimes A) \otimes M \end{array}$$

and is dealt-with similarly, but with [27, Prop. 1] used as follows: Reasoning as for Diagram (26), we are left to show that

$$\alpha_{K,A,\mathbf{I}} \circ \rho_{K \otimes A}^{-1} \quad = \quad \mathrm{id}_K \otimes \rho_A^{-1}$$

which amounts to

$$\rho_{K \otimes A}^{-1} \quad = \quad \alpha_{K,A,\mathbf{I}}^{-1} \circ (\mathrm{id}_K \otimes \rho_A^{-1})$$

and we are done by applying [27, Prop. 1].

Proof of Proposition M.16.(ii). Diagram (23) unfolds to

$$\begin{array}{ccc} K \otimes ((A \otimes M) \otimes (B \otimes M)) & \xrightarrow{\mathrm{id}_K \otimes ((\mathrm{id} \otimes m) \circ \theta_{A,B})} & K \otimes ((A \otimes B) \otimes M) \\ {\scriptstyle \vartheta_{TA,TB} \circ (d \otimes \mathrm{id})} \downarrow & & \downarrow {\scriptstyle \alpha^{-1}} \\ (K \otimes (A \otimes M)) \otimes (K \otimes (B \otimes M)) & & (K \otimes (A \otimes B)) \otimes M \\ {\scriptstyle \alpha^{-1} \otimes \alpha^{-1}} \downarrow & & \downarrow {\scriptstyle ((\vartheta_{A,B} \circ (d \otimes \mathrm{id})) \otimes \mathrm{id}_M} \\ ((K \otimes A) \otimes M) \otimes ((K \otimes B) \otimes M) & \xrightarrow[(\mathrm{id} \otimes m) \circ \theta_{K \otimes A, K \otimes B}]{} & ((K \otimes A) \otimes (K \otimes B)) \otimes M \end{array}$$

But we are done, since modulo symmetric monoidal coherence, the above amounts to

$$\begin{array}{ccc} K \otimes M \otimes M & \xrightarrow{\mathrm{id}_K \otimes m} & K \otimes M \\ {\scriptstyle d \otimes \mathrm{id}_{M \otimes M}} \downarrow & & \downarrow {\scriptstyle d \otimes \mathrm{id}_M} \\ K \otimes K \otimes M \otimes M & \xrightarrow[\mathrm{id}_{K \otimes K} \otimes m]{} & K \otimes K \otimes M \end{array}$$

Table of Contents