Liveness Properties in Geometric Logic for Domain-Theoretic Streams

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We devise a version of Linear Temporal Logic (LTL) on a denotational domain of streams. We investigate this logic in terms of domain theory, (point-free) topology and geometric logic. This yields the first steps toward an extension of the "Domain Theory in Logical Form" paradigm to temporal liveness properties.

We show that the negation-free formulae of LTL induce sober subspaces of streams, but that this is in general not the case in presence of negation. We propose a direct, inductive, translation of negation-free LTL to geometric logic. This translation reflects the approximations used to compute the usual fixpoint representations of LTL modalities.

As a motivating example, we handle a natural input-output specification for the usual filter function on streams.

1. Introduction

We are interested in input-output properties of higher-order programs that handle infinite data, such as streams or non-wellfounded trees. Consider for instance the usual filter function

$$\begin{array}{rcl} \mathsf{filter} & : & (A \to \mathsf{Bool}) \longrightarrow \mathsf{Str}A \longrightarrow \mathsf{Str}A \\ \mathsf{filter}\ p\ (a :: x) & = & \mathsf{if}\ (p\ a)\ \mathsf{then}\ a :: (\mathsf{filter}\ p\ x)\ \mathsf{else}\ (\mathsf{filter}\ p\ x) \end{array}$$

where $\mathsf{Str}A$ stands for the type of streams on A. Assume $p : A \to \mathsf{Bool}$ is a total function that tests for a property P. If x is a stream on A, then $(\mathsf{filter}\ p\ x)$ retains those elements of x which satisfy P. The stream produced by $(\mathsf{filter}\ p\ x)$ is thus only partially defined, unless x has infinitely many elements satisfying P.

Logics like LTL, CTL or the modal μ-calculus are widely used to formulate, on infinite objects, safety and liveness properties (see e.g. [HR07, BS07]). Safety properties state that some "bad" event will not occur, while liveness properties specify that "something good" will happen (see e.g. [BK08]). One typically uses temporal modalities like $\Box$ (*always*) or $\Diamond$ (*eventually*) to write properties of streams and specifications of programs over such data.

A possible specification for filter asserts that $(\mathsf{filter}\ p\ x)$ is a totally defined stream whenever x is a totally defined stream with infinitely many elements satisfying P. We express this with the temporal modalities $\Box$ and $\Diamond$. Let A be finite, and assume given, for each a of type A, a formula Φ_a which holds on $b : A$ exactly when b equals a.[1] Then $\Box \bigvee_a \Phi_a$ selects

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[1] In the setting of [JR21], we would assume $A = \sum_{i=1}^{n} \mathbf{1}$, with Φ_i representing the image of the ith injection.

those streams on A which are totally defined. The formula $\Box\Diamond P$ expresses that a stream has infinitely many elements satisfying P. We can thus state that for all streams $x : \mathsf{Str}A$,

$$x \text{ satisfies } \Box \bigvee_a \Phi_a \text{ and } \Box\Diamond P \quad\Longrightarrow\quad (\mathsf{filter}\ p\ x) \text{ satisfies } \Box \bigvee_a \Phi_a \tag{1}$$

The question we address in this paper is the following. Having in mind that a stream (as opposed to e.g. an integer) is inherently an infinite object, what do we mean exactly by "the stream x satisfies $\Box\Diamond P$"? In our view, the above specification for filter should hold for any stream whatsoever, and not only for those definable in a given programming language.

This leads us to investigate temporal properties on infinite datatypes at the level of denotational semantics. Logics on top of domains are known since quite a long time. Our reference is the paradigm of "Domain Theory in Logical Form" (DTLF) [Abr91] (see also [Zha91]), which allows one to systematically generate a logic from a domain representing a type. These logics are actually obtained by Stone duality, which is at the core of a rich interplay between domain theory, logic and (point-free) topology. This area is presented under various perspectives in a number of sources. We refer to [Abr91, AC98] and (e.g.) [Joh82, Vic89, Vic07, GL13, GvG23]. Some key ideas are put at work in [CZ00].

However, logics on domains given by Stone duality are usually restricted to safety properties. To our knowledge, there is no systematic investigation of liveness properties, such as the ones used in the specification for filter above.

This paper reports on preliminary works, mostly based on an internship of the second author during summer 2023. We devise a version of the logic LTL on a domain of streams $[\![\mathsf{Str}A]\!]$ determined by the recursive type equation $\mathsf{Str}A \cong A \times \mathsf{Str}A$. Each formula Φ of LTL yields a subset $[\![\Phi]\!] \subseteq [\![\mathsf{Str}A]\!]$. We investigate such LTL-definable subsets in terms of domain theory, of (point-free) topology and of a logic called geometric logic.

Our first step is to view domains as topological spaces, so as to benefit from the rich notion of subspace. For instance, (with A finite) the set of ω-words $A^\omega = [\![\Box \bigvee_a \Phi_a]\!]$ turns out to be a discrete sub-poset of $[\![\mathsf{Str}A]\!]$. But as a subspace of $[\![\mathsf{Str}A]\!]$, it becomes equipped with its usual product topology (in the sense of e.g. [PP04]). We observe that LTL formulae without negation induce subspaces of $[\![\mathsf{Str}A]\!]$ which are sober, but that this may fail in presence of negation. The notion of sobriety originates from point-free topology, and has become quite important for the general (point-set) topology of domains (see e.g. [GL13]).

We then turn to geometric logic. The idea is roughly the following. DTLF rests on the fact that finite approximations in a domain can be represented in a propositional logic generated from the topology of the domain. But this is too weak to handle infinitary properties such as those definable with the modalities $\Box$ and $\Diamond$. On the other hand, the sobriety of $[\![\Phi]\!]$ means that we can reason using an abstract notion of approximation induced by the subspace topology. Geometric logic is an infinitary propositional logic which allows for concrete representations of topologies. We provide a direct, inductive, translation of negation-free LTL to a geometric logic based on the domain $[\![\mathsf{Str}A]\!]$. This translation reflects the approximations used to compute the usual fixpoint representations of $\Box, \Diamond$. This shows that for the negation-free fragment, the semantics of LTL can be concretely represented by approximations which live in a natural extension of DTLF for the domain $[\![\mathsf{Str}A]\!]$.

We also check that our translation of negation-free LTL indeed conveys the good approximations to prove that the denotation of filter meets the specification (1) above.

Let us finally mention the scientific context of this work. It is undecidable whether a given higher-order program satisfies a given input-output temporal property written with formulae of the modal μ-calculus [KTU10]. A previous work with the first author provided a refinement type system for proving such properties [JR21]. This type system handles the alternation-free modal μ-calculus on (finitary) polynomial types, which includes LTL. But it is based on guarded recursion and does not allow for non-productive functions such as filter. We ultimately target a similar refinement type systems for a language based on FPC (which extends Plotkin's seminal PCF [Plo77] with recursive types, see e.g. [Pie02]). We think that the present work is a significant step in this direction. On the one hand, DTLF allows for reasoning on denotations using (finitary) type systems [Abr91]. On the other hand, it has been advocated in [KT14] that a form of oracle is needed to handle liveness properties in type systems. And indeed, [JR21] incorporates such oracles in a notion of "iteration term", which in fact makes the system infinitary.[2] We think that our representation of negation-free LTL in geometric logic can lead to an infinitary type system which extends [Abr91], and whose infinitary part can be simulated using iteration terms.

Organization of the paper. The preliminary §2 introduces background on domain theory, and the logic LTL on $[\![\mathsf{Str}A]\!]$. The (point-free) topological approach is presented in §3, and §4 is devoted to geometric logic. Section 5 deals with a deduction system for geometric logic, in connection with the notion of spatiality. The specification of filter is discussed in §6. We conclude in §7. Proofs are available in the Appendices.

2. A Linear Temporal Logic on a Domain of Streams

Let A be a set. A (finite) *word* on A is an element of A^*. A^ω is the set of ω-*words* on A, i.e. the set of all functions $\sigma\colon \mathbb{N} \to A$. We write $u \subseteq v$ when $u \in A^*$ is a prefix of $v \in A^* \cup A^\omega$. The concatenation of $u \in A^*$ with $v \in A^* \cup A^\omega$ is denoted $u \cdot v$ or uv. Given $\sigma \in A^\omega$ and $k \in \mathbb{N}$, we let $\sigma{\upharpoonright}k \in A^\omega$ be the ω-word with $(\sigma{\upharpoonright}k)(n) = \sigma(k+n)$ for all $n \in \mathbb{N}$. For instance, $\sigma{\upharpoonright}0$ is σ, while $\sigma{\upharpoonright}1 = \sigma(1)\cdot\sigma(2)\cdots\sigma(n+1)\cdots$ is σ deprived from its first letter.

2.1. Domains

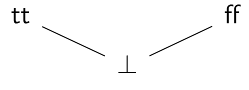
The basic idea of domain theory is to represent a type by partial order $(X, \leq_X)$ thought about as an "information order". The intuition is that $x \leq_X y$ means that y has "more information" than x, or that x is "less defined" than y. Domains are often required to have a least element (representing plain divergence), and are always asked to be stable under certain supremums (so that infinite objects can be thought about as limits of their finite approximations). Our presentation mostly follows [AC98, §1]. See also [Abr91, GL13].

Dcpos and Cpos. Let $(X, \leq)$ be a partial order (or *poset*). An *upper bound* of a subset $S \subseteq X$ is an element $x \in X$ such that $(\forall s \in S)(s \leq x)$. A *least upper bound* (or *supremum*, *sup*) of S is an upper bound ℓ of S such that $\ell \leq x$ for every upper bound x of S. The sup of S is unique whenever it exists, and is usually denoted $\bigvee S$. The notion of *greatest lower bound* (or *infimum*, *inf*) is defined dually. A subset $D \subseteq X$ is *directed* if D is non-empty and for every $x, y \in D$, there is some $z \in D$ such that $x \leq z$ and $y \leq z$.

We say that $(X, \leq)$ is a *dcpo* if every directed $D \subseteq X$ has a sup $\bigvee D \in X$. A *cpo* is a dcpo with a least element (usually denoted $\bot$). Note that each set A is a dcpo for the discrete order (in which x is comparable with y if, and only if, $x = y$). However, such a dcpo A is not a cpo unless A is a singleton.

Example 1 (Flat Domains). Given a set A, the *flat domain* $[\![A]\!]$ is the disjoint union $\{\bot\} + A$ equipped with the partial order $\leq_{[\![A]\!]}$, where $x \leq_{[\![A]\!]} y$ iff $x = y$ or ($x = \bot$ and $y \in A$).

It is easy to see that $([\![A]\!], \leq_{[\![A]\!]})$ is a cpo whose directed subsets have at most one element from A. For instance, the domain $[\![\mathsf{Bool}]\!]$ can be represented by the following Hasse diagram.



[2] Actually, as well as e.g. [NUKT18, SU23], despite a fundamentally different approach (see §7).

Scott-Continuous Functions. Let $X = (X, \leq_X)$ and $Y = (Y, \leq_Y)$ be dcpos. A function $f \colon X \to Y$ is *Scott-continuous* if f is monotone ($x \leq_X x'$ implies $f(x) \leq_Y f(x')$) and if moreover f preserves directed sups, in the sense that for each directed $D \subseteq X$, we have

$$f(\bigvee D) \quad = \quad \bigvee \{f(d) \mid d \in D\}$$

We write **CPO** (resp. **DCPO**) for the category with cpos (resp. dcpos) as objects and with Scott-continuous functions as morphisms. We say that $f \in \mathbf{CPO}[X, Y]$ is strict if $f(\bot_X) = \bot_Y$. A non-strict monotone map between flat domains is necessarily constant.

Given dcpos $X = (X, \leq_X)$ and $Y = (Y, \leq_Y)$, the set of Scott-continuous functions $\mathbf{DCPO}[X, Y]$ is itself a dcpo w.r.t. the *pointwise order*

$$f \leq_{\mathbf{DCPO}[X,Y]} g \quad \text{iff} \quad \forall x \in X,\ f(x) \leq_Y g(x)$$

If Y is actually a cpo, then $\mathbf{DCPO}[X, Y]$ is a cpo whose least element is the constant function $x \in X \mapsto \bot_Y \in Y$, where $\bot_Y$ is the least element of Y.

Example 2 (Streams). Let A be a set. We let $[\![\mathsf{Str}A]\!]$, the cpo of *streams over A*, be $\mathbf{DCPO}[\mathbb{N}, [\![A]\!]]$ with $\mathbb{N}$ discrete. We unfold this important example. Since $\mathbb{N}$ is discrete, $[\![\mathsf{Str}A]\!]$ actually consists of the set $[\![A]\!]^\omega$ equipped with the partial order

$$x \leq_{[\![\mathsf{Str}A]\!]} y \qquad \text{iff} \qquad \forall n \in \mathbb{N},\ x(n) \leq_{[\![A]\!]} y(n)$$

A set $D \subseteq [\![\mathsf{Str}A]\!]$ is directed if, and only if, D is non-empty and each $D(n) = \{x(n) \mid x \in D\}$ has at most one element from A. Then $\bigvee D \in [\![\mathsf{Str}A]\!]$ takes $n \in \mathbb{N}$ to the largest element of $D(n)$. The least element of $[\![\mathsf{Str}A]\!]$ is the stream $\bot^\omega$ of constant value $\bot \in [\![A]\!]$.

Note that $[\![\mathsf{Str}A]\!]$ has "partially defined" elements. Besides the least element $\bot^\omega$, we have e.g. the stream $u \cdot \bot^\omega$ (which agrees with $u \in A^*$ and then is $\bot$ at all sufficiently large positions) or $(a \cdot \bot)^\omega$ (which is a at all even positions, and is $\bot$ everywhere else). The ω-words on A are precisely those streams $x \in [\![\mathsf{Str}A]\!]$ which never take the value $\bot$. Such streams are called *total*. Note that if x is total, then

$$x \quad = \quad \bigvee \{u \cdot \bot^\omega \mid u \in A^* \text{ and } u \subseteq x\}$$

Remark 1. The cpo $[\![\mathsf{Str}A]\!]$ is the usual solution in the category **CPO** of the *domain equation*

$$X \quad \cong \quad [\![A]\!] \times X$$

(where $[\![A]\!] \times X$ is equipped with the pointwise order), see e.g. [AC98, Theorem 7.1.10 and Proposition 7.1.13]. In particular, the constructor $(- :: -)$ of the type $\mathsf{Str}A$ is interpreted as the isomorphism taking $(a, x) \in [\![A]\!] \times [\![\mathsf{Str}A]\!]$ to $a \cdot x \in [\![\mathsf{Str}A]\!]$, with inverse $x \mapsto (x(0), x{\upharpoonright}1)$. Note that $[\![\mathsf{Str}A]\!]$ differs from the usual *Kahn domain* $A^* \cup A^\omega$ (see e.g. [Vic89, Definition 3.7.5 and Example 5.4.4] or [DST19, §7.4], see also [VVK05]).

Remark 2. Each $f \colon X \to_{\mathbf{CPO}} X$ has a *least fixpoint* $\mathsf{Y}(f) \coloneqq \bigvee_{n \in \mathbb{N}} f^n(\bot) \in X$. In particular, filter is interpreted as the Scott-continuous function $[\![\mathsf{filter}]\!]$ taking $p \colon [\![A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Bool}]\!]$ to the least fixpoint of the following function f_p, where X is the cpo $[\![\mathsf{Str}A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Str}A]\!]$.

$$f_p \quad \coloneqq \quad \lambda g.\lambda x.\ \mathsf{if}\ p(x(0))\ \mathsf{then}\ x(0) \cdot g(x{\upharpoonright}1)\ \mathsf{else}\ g(x{\upharpoonright}1) \quad : \quad X \quad \longrightarrow_{\mathbf{CPO}} \quad X$$

Algebraicity. Among the many good properties of $[\![\mathsf{Str}A]\!]$, algebraicity is the crucial one in this work. This property is not used right away, but will be the main assumption of various statements later on.

Let $(X, \leq)$ be a dcpo. We say that $x \in X$ is *finite*[3] if for every directed $D \subseteq X$ such that $x \leq \bigvee D$, there is some $d \in D$ such that $x \leq d$. We say that X is *algebraic* if for every $x \in X$, the set $\{d \in X \mid d \text{ finite and } d \leq x\}$ is directed and has sup x. Each discrete or flat dcpo is algebraic.

[3] Finite elements are called *compact* in [AC98].

Example 3 (Streams). The cpo $[\![\mathsf{Str}A]\!]$ is algebraic, and its finite elements admit a particularly simple description. The *support* of $x \in [\![\mathsf{Str}A]\!]$ is the set supp(x) of "defined letters" of x:

$$\mathrm{supp}(x) \quad := \quad \{n \in \mathbb{N} \mid x(n) \neq \bot\}$$

We say that a stream x has finite support when supp(x) is a finite set. For instance, given a finite word $u \in A^*$ and $n \in \mathbb{N}$, the stream $\bot^n \cdot u \cdot \bot^\omega$ has finite support. On the other hand, total streams, as well as e.g. $(a \cdot \bot)^\omega$, do not have finite support.

For each $x \in [\![\mathsf{Str}A]\!]$, the set $\big\{d \mid d \text{ of finite support and } d \leq_{[\![\mathsf{Str}A]\!]} x\big\}$ is directed and has sup x. Moreover, the finite elements of $[\![\mathsf{Str}A]\!]$ are exactly those of finite support.

2.2. Linear Temporal Logic (LTL)

Syntax and Semantics. Let A be a set. The formulae of $\mathsf{LTL} = \mathsf{LTL}(A)$ are given by

$$\Phi, \Psi \quad ::= \quad a \ \mid\ \mathsf{True} \ \mid\ \mathsf{False} \ \mid\ \Phi \wedge \Psi \ \mid\ \Phi \vee \Psi \ \mid\ \neg\Phi \ \mid\ \bigcirc\Phi \ \mid\ \Phi \,\mathsf{U}\, \Psi \ \mid\ \Phi \,\mathsf{W}\, \Psi$$

where $a \in A$. Hence, besides pure propositional logic, $\mathsf{LTL}(A)$ has *atomic formulae* $a \in A$, and *modalities* $\bigcirc\Phi$ (read "next Φ"), $\Phi \,\mathsf{U}\, \Psi$ (read "Φ until Ψ") and $\Phi \,\mathsf{W}\, \Psi$ (read "Φ weak until Ψ" or "Φ unless Ψ").

The LTL formulae over A are usually interpreted on ω-words over A, see e.g. [BK08, §5]. The interpretation of modalities actually implicitly relies on the bijection $A^\omega \cong A \times A^\omega$. We similarly rely on the isomorphism $[\![\mathsf{Str}A]\!] \cong_{\mathbf{CPO}} [\![A]\!] \times [\![\mathsf{Str}A]\!]$ for interpreting $\mathsf{LTL}(A)$ formulae in $[\![\mathsf{Str}A]\!]$. We define $[\![\Phi]\!] \subseteq [\![\mathsf{Str}A]\!]$ by induction on Φ. The propositional connectives of LTL are interpreted using the usual Boolean algebra structure of the powerset $\mathcal{P}([\![\mathsf{Str}A]\!])$. For $a \in A$, we let $[\![a]\!] := \{x \in [\![\mathsf{Str}A]\!] \mid x(0) = a\}$. The modalities are interpreted as follows.

$$\begin{array}{rcl}
[\![\bigcirc\Phi]\!] & := & \{x \in [\![\mathsf{Str}A]\!] \mid x{\restriction}1 \in [\![\Phi]\!]\} \\
[\![\Phi \,\mathsf{U}\, \Psi]\!] & := & \{x \in [\![\mathsf{Str}A]\!] \mid \exists i \in \mathbb{N},\ x{\restriction}0, \ldots, x{\restriction}(i-1) \in [\![\Phi]\!] \text{ and } x{\restriction}i \in [\![\Psi]\!]\} \\
[\![\Phi \,\mathsf{W}\, \Psi]\!] & := & \{x \in [\![\mathsf{Str}A]\!] \mid \forall i \in \mathbb{N},\ x{\restriction}i \in [\![\Phi]\!]\} \cup [\![\Phi \,\mathsf{U}\, \Psi]\!]
\end{array}$$

We say that $x \in [\![\mathsf{Str}A]\!]$ *satisfies* a formula Φ (notation $x \Vdash \Phi$) when $x \in [\![\Phi]\!]$. It is often convenient to decompose $[\![\bigcirc\Phi]\!]$ as $[\![\bigcirc]\!]([\![\Phi]\!])$, where $[\![\bigcirc]\!] \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ takes S to $\{x \in [\![\mathsf{Str}A]\!] \mid x{\restriction}1 \in S\}$. The modalities U and W may not be easy to grasp. Given LTL formulae Φ and Ψ, we let

$$\begin{array}{rrcll}
\text{"eventually } \Psi\text{"} & \Diamond\Psi & := & \mathsf{True}\,\mathsf{U}\,\Psi & \big([\![\Diamond\Psi]\!] = \{x \in [\![\mathsf{Str}A]\!] \mid \exists i \in \mathbb{N},\ x{\restriction}i \in [\![\Psi]\!]\}\big) \\
\text{"always } \Phi\text{"} & \Box\Phi & := & \Phi\,\mathsf{W}\,\mathsf{False} & \big([\![\Box\Phi]\!] = \{x \in [\![\mathsf{Str}A]\!] \mid \forall i \in \mathbb{N},\ x{\restriction}i \in [\![\Phi]\!]\}\big)
\end{array}$$

Example 4. Consider a stream $x \in [\![\mathsf{Str}A]\!]$.

(1) We have $x \Vdash \bigcirc a$ if, and only if, $x(1) = a$. For instance, $\bot a \bot^\omega \Vdash \bigcirc a$ but $a\bot^\omega \nVdash \bigcirc a$.

(2) We have $x \Vdash \Diamond a$ if, and only if, $x(i) = a$ for some $i \in \mathbb{N}$. For instance, $\bot^n a \bot^\omega \Vdash \Diamond a$ for every $n \in \mathbb{N}$. But $b^\omega \nVdash \Diamond a$ if $b \neq a$.

(3) We have $x \Vdash \Box a$ if, and only if, $x = a^\omega$. If A is finite, then x is total iff $x \Vdash \Box \bigvee_{a \in A} a$.

(4) We have $x \Vdash \Box\Diamond a$ if, and only if, $x(i) = a$ for infinitely many $i \in \mathbb{N}$. E.g. $(\bot a)^\omega \Vdash \Box\Diamond a$.

(5) We have $x \Vdash \Diamond\Box a$ if, and only if, $x(i) = a$ for "ultimately all $i \in \mathbb{N}$". This means that for some $n \in \mathbb{N}$, we have $x(i) = a$ for all $i \geq n$. For instance, $\bot^n a^\omega \Vdash \Diamond\Box a$ for all $n \in \mathbb{N}$. But $(\bot a)^\omega \nVdash \Diamond\Box a$.

Say that Φ and Ψ are (logically) *equivalent*, notation $\Phi \equiv \Psi$, if $[\![\Phi]\!] = [\![\Psi]\!]$. LTL has many redundancies w.r.t. logical equivalence. Besides the usual De Morgan laws, we have e.g.

$$\neg\Box\Phi \ \equiv\ \Diamond\neg\Phi \qquad \neg\Diamond\Phi \ \equiv\ \Box\neg\Phi \qquad \Phi \,\mathsf{W}\, \Psi \ \equiv\ (\Phi \,\mathsf{U}\, \Psi) \vee \Box\Phi$$

Remark 3. The modalities U and W are also "De Morgan" duals, in the following sense. Given Φ and Ψ, it is well-known that $[\![\Phi \mathrel{\mathsf{U}} \Psi]\!]$ and $[\![\Phi \mathrel{\mathsf{W}} \Psi]\!]$ are respectively the least and the greatest fixpoint of the (monotone) map on $\mathcal{P}([\![\mathsf{Str}A]\!])$ taking S to $[\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!](S))$. See e.g. [BK08, Lemmas 5.18 and 5.19]. But $\mathcal{P}([\![\mathsf{Str}A]\!])$ is a complete atomic Boolean algebra, and given a monotone endo-function f on such a Boolean algebra, the least and greatest fixpoints of f are related by $\mathrm{lfp}(f) = \neg\mathrm{gfp}(b \mapsto \neg f(\neg b))$ and $\mathrm{gfp}(f) = \neg\mathrm{lfp}(b \mapsto \neg f(\neg b))$.

Negation-Free LTL. Our main positive results only hold for the negation-free fragment of LTL. An LTL formula is *negation-free* (*n.-f.*) if it contains no negation ($\neg(-)$). Hence, the negation-free formulae of LTL are generated by the above grammar for LTL, but without the production $\neg\Phi$.

Example 5. All formulae of Example 4 are negation-free. Moreover, the negation-free fragment is closed under $\Box(-)$ and $\Diamond(-)$.

Assume A is finite. For any $S \subseteq A$, there is a negation-free formula Ψ_S such that $x \Vdash \Psi_S$ iff $x(0) \in S$. It follows that for any Scott-continuous $p\colon [\![A]\!] \to [\![\mathsf{Bool}]\!]$, there is a negation-free formula Ψ_p such that $x \Vdash \Box\Diamond\Psi_p$ if, and only if, x has infinitely many elements satisfying p.

Most redundancies of LTL mentioned above disappear in the negation-free fragment. This is why we have chosen this set of connectives from the start. In negation-free $\mathsf{LTL}(A)$, all connectives have a De Morgan dual. But negated atomic formulae ($\neg a$ for $a \in A$) are not available. Hence, in contrast with positive normal forms (see e.g. [BK08, Definition 5.20]), negation is *not* definable in negation-free LTL. This positive character is reflected in the following fundamental fact, proved by induction on formulae.

Lemma 1. *If Φ is n.-f. then $[\![\Phi]\!]$ is upward-closed (if $x \in [\![\Phi]\!]$ and $x \leq_{[\![\mathsf{Str}A]\!]} y$ then $y \in [\![\Phi]\!]$).*

Corollary 1. *Let Φ be negation-free. Then $[\![\Phi]\!]$ is closed in $[\![\mathsf{Str}A]\!]$ under directed sups. Moreover, the inclusion $[\![\Phi]\!] \hookrightarrow [\![\mathsf{Str}A]\!]$ is a Scott-continuous order-embedding.*

Hence, $[\![\Phi]\!]$ is a sub-dcpo of $[\![\mathsf{Str}A]\!]$ when Φ is n.-f. But this may not give much information on $[\![\Phi]\!]$. For instance, $A^\omega = [\![\Box \bigvee_{a\in A} a]\!]$ (Example 4(3), A finite) is a discrete dcpo. Building on Lemma 1, we are going to exhibit much more structure on such inclusions $[\![\Phi]\!] \hookrightarrow [\![\mathsf{Str}A]\!]$. But before, we note that Lemma 1 and Corollary 1 may fail in presence of negation.

Example 6. Consider the formula $\neg\Box a$. Note that $a^\omega \not\Vdash \neg\Box a$. But for every finite $d \leq_{[\![\mathsf{Str}A]\!]} a^\omega$, we have $d \Vdash \neg\Box a$. Hence $[\![\neg\Box a]\!]$ is not upward-closed. Moreover, $\{d \text{ finite} \mid d \leq_{[\![\mathsf{Str}A]\!]} a^\omega\}$ is a directed subset of $[\![\neg\Box a]\!]$ which has no sup in $[\![\neg\Box a]\!]$. Hence $[\![\neg\Box a]\!]$ is not a dcpo w.r.t. the restriction of $\leq_{[\![\mathsf{Str}A]\!]}$.

3. The Topological Approach

We shall now look at inclusions $[\![\Phi]\!] \hookrightarrow [\![\mathsf{Str}A]\!]$ from a topological perspective. We recall in §3.1 that the categories (**D**)**CPO** can be embedded in the category **Top** of topological spaces and continuous functions. The highlight is that **Top** has a much richer notion of substructures (called subspaces) than (**D**)**CPO**.

Actually, when looking at (d)cpos as topological spaces, the notion of sobriety from point-free (or "element-free") topology comes to the front. Ample mathematical justifications for the importance of sober spaces in domain theory are gathered in [GL13]. We shall content ourselves with more informal motivations in §3.2. In §3.3, we abstractly prove that $[\![\Phi]\!]$ induces a sober subspace of $[\![\mathsf{Str}A]\!]$ when Φ is negation-free. This will be refined to concrete representations in §4 and §5, using geometric logic.

3.1. Topological Spaces

A *topological space* is a pair $(X, \Omega(X))$ of a set X and a collection $\Omega = \Omega(X)$ of subsets of X, called *open sets*. Ω is called a *topology* on X, and is asked to be stable under arbitrary unions and under finite intersections. In particular, $\emptyset$ and X are open in X (respectively as the empty union and the empty intersection).

A set $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open. Closed sets are stable under finite unions and arbitrary intersections. Hence, any $S \subseteq X$ is contained in a least closed set $\overline{S} \subseteq X$. Each space (X, Ω) is equipped with a *specialization* (pre)order $\leq_\Omega$ on X, defined as

$$x \leq_\Omega y \quad \text{iff} \quad (\forall U \in \Omega)\,(x \in U \implies y \in U)$$

Given $x \in X$, we have $\overline{\{x\}} = \downarrow\! x := \{y \in X \mid y \leq_\Omega x\}$ (see e.g. [GL13, Lemma 4.2.7]). A topology Ω is T_0 when $\leq_\Omega$ is a partial order (see e.g. [GL13, Proposition 4.2.3]).

Given spaces $(X, \Omega(X))$ and $(Y, \Omega(Y))$, a function $f\colon X \to Y$ is *continuous* when its inverse image $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ restricts to a function $\Omega(Y) \to \Omega(X)$, i.e. when $f^{-1}(V) \in \Omega(X)$ for all $V \in \Omega(Y)$. We write **Top** for the category of topological spaces and continuous functions. An *homeomorphism* is an isomorphism in **Top**.

The Scott Topology. The following is well-known. See e.g. [AC98, §1.2] or [GL13, §4].

Let $(X, \leq_X)$ be a dcpo. A subset $U \subseteq X$ is *Scott-open* if U is upward-closed, and if moreover U is inaccessible by directed sups, in the sense that if $\bigvee D \in U$ with $D \subseteq X$ directed, then $D \cap U \neq \emptyset$. This equips X with a T_0 topology, called the *Scott topology*, whose specialization order coincides with $\leq_X$. Note that $C \subseteq X$ is Scott-closed precisely when C is downward-closed and stable under directed sups.

Example 7. When $(X, \leq)$ is algebraic, the sets $\uparrow\! d := \{x \in X \mid d \leq x\}$ with d finite form a sub-basis for the Scott topology. For instance, the Scott-open subsets of $[\![\mathsf{Str}A]\!]$ are arbitrary unions of sets of the form

$$\uparrow\! d \quad = \quad \{x \in [\![\mathsf{Str}A]\!] \mid \forall i \in \mathrm{supp}(d),\ x(i) = d(i)\}$$

with $\mathrm{supp}(d)$ finite. In particular, given $x \in U$ with $U \subseteq [\![\mathsf{Str}A]\!]$ Scott-open, there is a *finite* set $\{i_1, \ldots, i_k\} \subseteq \mathbb{N}$ such that $\{y \mid y(i_1) = x(i_1), \ldots, y(i_k) = x(i_k)\} \subseteq U$.

Beware that $[\![\Phi]\!]$ may not be an open nor a closed subset of $[\![\mathsf{Str}A]\!]$, even when Φ is negation-free. Consider for instance $A^\omega = [\![\Box \bigvee_{a \in A} a]\!]$ (with A non-empty and finite), which contains a^ω but no finite $d \leq_{[\![\mathsf{Str}A]\!]} a^\omega$.

A function $f\colon X \to Y$ between dcpos is Scott-continuous precisely when f is continuous w.r.t. the Scott-topologies on X and Y. It follows that **DCPO** and **CPO** are full subcategories of **Top**. From now on, we shall mostly look at (**D**)**CPO** in this way. Unless stated otherwise, dcpos will always be equipped with their Scott topology.

Subspaces. Our motivation for moving from (**D**)**CPO** to **Top** is that **Top** has a rich notion of subspace. We refer to [BBT20, §1.2]. Given a space (X, Ω) and a subset $P \subseteq X$, the *subspace topology* on P is

$$\Omega{\restriction}P \quad := \quad \{U \cap P \mid U \in \Omega\}$$

The subspace topology on P makes the inclusion function $\iota\colon P \hookrightarrow X$ continuous. It is the "best possible" topology on P in the following sense: given a space $(Y, \Omega(Y))$, a function $f\colon Y \to P$ is continuous if, and only if, the composition $\iota \circ f\colon Y \to X$ is continuous.

$$\begin{tikzcd} & P \arrow[dr, hook, "\iota"] & \\ Y \arrow[ur, "f"] \arrow[rr, "\iota \circ f"'] & & X \end{tikzcd}$$

Example 8. Generalizing Example 4(3) and Lemma 1, A^ω is a discrete sub-dcpo of $[\![\mathsf{Str}A]\!]$.

On the other hand, the subspace topology $\Omega([\![\mathsf{Str}A]\!]){\restriction}A^\omega$ is the usual *product topology* on A^ω (see e.g. [PP04, §III] or [Kec95]). The sets of the form $A^\omega \cap \uparrow\! d$ (with d finite) form a sub-basis for the subspace topology. In fact, its opens are unions of sets of the form $\{\sigma \in A^\omega \mid \sigma(i_1) = a_1, \ldots, \sigma(i_k) = a_k\}$, where $i_1, \ldots, i_k \in \mathbb{N}$ and $a_1, \ldots, a_k \in A$ with $k \geq 0$.

3.2. The Element-Free Setting

The topological setting comes with an intrinsic notion of approximation.

Consider for instance ω-words $\sigma \in A^\omega$ (Example 8). Similarly as with streams in Example 7, given an ω-word σ and an open U, if σ belongs to U, then this fact is witnessed by the knowledge of a finite number of elements of σ.[4] We view the opens U such that $\sigma \in U$ as approximations of σ.

Given a space (X, Ω), we are interested in describing the elements of X by their approximations, represented as suitable sets of opens $\mathcal{F} \subseteq \Omega$. This is the realm of *element-free* (or *point-free*) *topology*. Its central objects, called *frames* (or *locales*), abstract away from the elements of spaces, and only retain the lattice structure of open sets. Besides [GL13], we refer to [Joh82, Joh83, Vic89, PP12, PP21].

Frames. A *complete lattice* is a poset having all sups and all infs. But recall (from e.g. [DP02, Theorem 2.31]) that a poset has all sups if, and only if, it has all infs. Hence, we can see complete lattices indifferently as posets with all sups or as posets with all infs.

A *frame* is a poset L with all sups (and thus all infs), and which satisfies the following *frame distributive law*: for all $S \subseteq L$ and all $a \in L$,

$$a \wedge \bigvee S \quad = \quad \bigvee \{a \wedge s \mid s \in S\}$$

Not every complete lattice is a frame.[5] But every $(\mathcal{P}(X), \subseteq)$ is a frame, and so is the two-elements poset $\mathbf{2} \coloneqq \{0 \leq 1\}$.

Given frames L and K, a *frame morphism* $f \colon L \to K$ is a function which preserves all sups and all *finite* infs. Note that frame morphisms are automatically monotone. We write **Frm** for the category of frames and frame morphisms.

Example 9. Let $(X, \Omega(X))$ be a space. Then $\Omega(X)$ has all sups and they are given by unions. Hence $\Omega(X)$ is a complete lattice. Beware that the inf in $\Omega(X)$ of an arbitrary $S \subseteq \Omega(X)$ is in general not its intersection $\bigcap S$, but the *interior* of $\bigcap S$ (the largest open set contained in $\bigcap S$). However, *finite* infs in $\Omega(X)$ are given by intersections, and $\Omega(X)$ is a frame. In fact, the topologies on a given set X correspond exactly to the sub-frames of $\mathcal{P}(X)$.

Moreover, if $f \colon X \to Y$ is continuous, then its inverse image f^{-1} restricts to a function $\Omega(Y) \to \Omega(X)$. This function is actually a frame morphism $\Omega(f) \in \mathbf{Frm}[\Omega(Y), \Omega(X)]$. In other words, the operation $(X, \Omega(X)) \mapsto \Omega(X)$ extends to a functor Ω from the category **Top** to $\mathbf{Frm}^{\mathrm{op}}$, the opposite of **Frm**. The category $\mathbf{Frm}^{\mathrm{op}}$ is the category of *locales*.

The Space of Points. We see a frame L as a collection of formal approximations. Suitable subsets of L describe "converging" sets of formal approximations, and constitute the elements of a space, the space of points of L. The idea is as follows. Given a space X and $x \in X$, let

$$\mathcal{F}_x \quad \coloneqq \quad \{U \in \Omega(X) \mid x \in U\}$$

Note the following properties of $\mathcal{F}_x$ w.r.t. the frame structure of $\Omega(X)$. First, $\mathcal{F}_x$ is stable under finite intersections ($x \in X$, and $x \in U \cap V$ iff $x \in U$ and $x \in V$). Second, given $S \subseteq \Omega(X)$ with $\bigcup S \in \mathcal{F}_x$, we have $U \in \mathcal{F}_x$ for some $U \in S$ (if $x \in \bigcup S$ then $x \in U$ for some $U \in S$). Hence, the characteristic function of $\mathcal{F}_x \subseteq \Omega(X)$ is a frame morphism $\Omega(X) \to \mathbf{2}$.

A *point* of a frame L is an element of $\mathrm{pt}(L) \coloneqq \mathbf{Frm}[L, \mathbf{2}]$. We shall always identify a point $\mathcal{F} \in \mathbf{Frm}[L, \mathbf{2}]$ with the set $\{a \in L \mid \mathcal{F}(a) = 1\}$. Given $a \in L$, let

$$\mathsf{ext}(a) \quad \coloneqq \quad \{\mathcal{F} \in \mathrm{pt}(L) \mid a \in \mathcal{F}\}$$

The function $\mathsf{ext} \colon L \to \mathcal{P}(\mathrm{pt}(L))$ is a frame morphism (see e.g. [Joh82, Lemma II.1.6]). In particular, its image is a sub-frame of $\mathcal{P}(\mathrm{pt}(L))$, and is thus a topology $\Omega(\mathrm{pt}(L))$ on $\mathrm{pt}(L)$. The *space of points* of L is $(\mathrm{pt}(L), \Omega(\mathrm{pt}(L)))$.

[4] Dually, the knowledge of the whole ω-word σ may be needed to testify that $\sigma \notin U$.

[5] Consider e.g. a finite (and thus complete) non-distributive lattice, see [DP02, Example 4.6(6)].

The operation $L \mapsto \mathrm{pt}(L)$ extends to a functor $\mathrm{pt}\colon \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$ which is right adjoint to Ω (see e.g. [Joh82, Theorem II.1.4]). The action of $\mathrm{pt}\colon \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$ on a frame morphism $f\colon L \to K$ is the continuous function $\mathrm{pt}(K) \to \mathrm{pt}(L)$ which takes $\mathcal{F} \in \mathbf{Frm}[K, \mathbf{2}]$ to $\mathcal{F} \circ f \in \mathbf{Frm}[L, \mathbf{2}]$. The unit at $X \in \mathbf{Top}$ of the adjunction $\Omega \dashv \mathrm{pt}$ is the continuous function $\eta_X\colon X \to \mathrm{pt}(\Omega(X))$ taking x to $\mathcal{F}_x$ (see e.g. [Joh82, §II.1.6]).

Sober Spaces. The function $\eta_X\colon X \to \mathrm{pt}(\Omega(X))$ continuously maps the space X to its space of "converging formal approximations" $\mathrm{pt}(\Omega(X))$. But $\mathrm{pt}(\Omega(X))$ may not correctly represent X. A space X is *sober* if η_X is a bijection (in which case η_X is automatically an homeomorphism, see [Joh82, §II.1.6]).

Given a frame L, the space of points $\mathrm{pt}(L)$ is always sober ([Joh82, Lemma II.1.7]). Hence (by functoriality of Ω and pt), if X is homeomorphic to $\mathrm{pt}(L)$, then X is sober as well. It follows from [Joh82, Lemma II.1.6(ii)] that A^ω is sober for its product topology.[6] But not every dcpo is sober ([Joh82, II.1.9]). For the following, see e.g. [Joh82, Theorem VII.2.6].

Proposition 1. *Algebraic dcpos are sober.*

Remark 4. In fact, a sober space is always T_0, and is moreover a dcpo w.r.t. its specialization order ([Joh82, Lemmas II.1.6(i) and II.1.9]). This provides a functor $\mathbf{Frm}^{\mathrm{op}} \to \mathbf{DCPO}$ which is actually right adjoint to the composite $\mathbf{DCPO} \hookrightarrow \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$, yielding the *Scott adjunction* of [DL22].

In particular, a sober dcpo is completely determined by the specialization order of its space of points. On the other hand, beware that the composite $\mathbf{Frm}^{\mathrm{op}} \to \mathbf{DCPO} \to \mathbf{Frm}^{\mathrm{op}}$ may loose a lot of structure (e.g. it takes the product topology on A^ω to the discrete (Scott) topology).

3.3. Sobriety of Subspaces

Proposition 1 and Remark 4 imply that for an algebraic dcpo X, the topological notion of approximation coincides with the domain-theoretic one. But we are interested in subspaces of the algebraic cpo $[\![\mathsf{Str}A]\!]$. Discussing the sobriety of such subspaces involves going further into the point-free setting. While the main results of this §3.3 are important for this paper, the technical developments are used again only in §5.

Let (X, Ω) be a space, and consider some $P \subseteq X$. The subspace inclusion $\iota\colon (P, \Omega{\restriction}P) \hookrightarrow (X, \Omega)$ induces the surjective frame morphism $\iota^\star := \Omega(\iota)\colon \Omega \twoheadrightarrow \Omega{\restriction}P$ which takes $U \in \Omega$ to $(U \cap P) \in \Omega{\restriction}P$. The following is a handy reformulation of sobriety for $(P, \Omega{\restriction}P)$.

Lemma 2. *Assume that (X, Ω) is sober. Then the following are equivalent.*

(i) $(P, \Omega{\restriction}P)$ is sober.

(ii) For each $x \in X$, we have $x \in P$ if, and only if, $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$ for some $\mathcal{G} \in \mathrm{pt}(\Omega{\restriction}P)$.

$$\begin{tikzcd} \Omega \arrow[rr, two heads, "\iota^\star"] \arrow[dr, "\mathcal{F}_x"'] & & \Omega{\restriction}P \arrow[dl, dashed, "\mathcal{G}"] \\ & \mathbf{2} & \end{tikzcd}$$

Let L be a frame. A *quotient frame* of L is an isomorphism-class of surjective frame morphisms $L \twoheadrightarrow K$. We are going to discuss an abstract but mathematically powerful representation of the quotient frame $\Omega \twoheadrightarrow \Omega{\restriction}P$. We use tools from [Joh82, §II.2] and [PP12, §VI.1] on the dual notion of *sub-locale*.

Everything starts from Galois connections and related adjointness properties, for which we refer to [DP02, 7.23–7.34]. Fix a frame morphism $f\colon L \to K$. Since $f\colon L \to K$ preserves all sups, it has an *upper adjoint* $f_\star\colon K \to L$. This means that for all $a \in L$ and all $b \in K$,

$$f(a) \leq_K b \qquad \text{if, and only if,} \qquad a \leq_L f_\star(b)$$

[6] Actually, [Joh82, Lemma II.1.6(ii)] states that each T_2 space is sober.

The pair $(f, f_\star)$ thus forms a Galois connection, and $f_\star$ is (uniquely) determined by

$$f_\star(b) \quad = \quad \bigvee_L \{a \mid f(a) \leq_K b\}$$

The function $f_\star$ is in general not a frame morphism, but it always preserves all infs.

The composition $j := f_\star \circ f : L \to L$ is a *nucleus* in the sense of [Joh82, §II2.2]: we have (i) $j(a \wedge a') = j(a) \wedge j(a')$, (ii) $a \leq j(a)$ and (iii), $j(j(a)) \leq j(a)$. Nuclei are monotone and idempotent. If $j : L \to L$ is a nucleus, then the set $L_j := \{a \in L \mid j(a) = a\}$ of *j-fixpoints* is a frame and $j : L \to L_j$ is a frame morphism ([Joh82, Lemma II.2.2]). Note that the finite infs in L_j are those of L. But the sup of $S \subseteq L_j$ in L_j is $j(\bigvee_L S)$.

Consider now a subspace inclusion $\iota : (P, \Omega\restriction P) \hookrightarrow (X, \Omega)$. Following [PP12, §VI.1.1], we write $\widetilde{P}$ for the frame of j-fixpoints, where $j := \iota_\star \circ \iota^\star$ and $\iota_\star$ is the upper adjoint of $\iota^\star$.

$$\Omega \xrightarrow{\ \iota^\star\ } \!\!\!\!\rightarrow \Omega\restriction P \xrightarrow{\ \iota_\star\ } \Omega$$

We rely on the following description of the nucleus j (from which [PP12, §VI.1.1] gives an explicit representation of j-fixpoints that we shall not use directly).

Remark 5. Given an open $U \in \Omega$ of X, we have

$$j(U) \quad = \quad \bigcup \{V \in \Omega \mid V \cap P = U \cap P\}$$

The proof of [Joh82, Theorem II.2.3] gives Lemma 3 below. Recall that order-isomorphisms preserve all existing sups and infs ([DP02, Lemma 2.27(ii)]).

Lemma 3. *The function $\iota_\star : \Omega\restriction P \to \Omega$ co-restricts to a frame isomorphism $\iota_\star : \Omega\restriction P \to \widetilde{P}$.*

We use $j = \iota_\star \iota^\star : \Omega \twoheadrightarrow \widetilde{P}$ to represent the quotient frame induced by the subspace inclusion $\iota : (P, \Omega\restriction P) \hookrightarrow (X, \Omega)$. The frame $\widetilde{P}$ turns out to be a good tool for studying the sobriety of $(P, \Omega\restriction P)$. Following [PP12, VI.1.3], given $x \in X$ we let $\widetilde{x} := X \setminus \overline{\{x\}} = X \setminus \downarrow x$. We shall now see that it is useful to characterize when $\widetilde{x} \in \widetilde{P}$ (i.e. when $j(\widetilde{x}) = \widetilde{x}$).

Remark 6. Given $x \in X$ and $U \in \Omega$, we have $U \subseteq \widetilde{x}$ if, and only if, $x \notin U$.

Lemma 4. *Let $x \in X$ and $\widetilde{\mathcal{F}}_x := \{U \in \widetilde{P} \mid x \in U\}$. Then $\widetilde{\mathcal{F}}_x \in \mathrm{pt}(\widetilde{P})$ if and only if $\widetilde{x} \in \widetilde{P}$.*

Proposition 2. *Let $P \subseteq X$ with (X, Ω) sober. Then the following are equivalent.*

(i) $(P, \Omega\restriction P)$ is sober.

(ii) For each $x \in X$, we have $x \in P$ if, and only if, $\widetilde{x} \in \widetilde{P}$.

In condition (ii) above, we actually always have $\widetilde{x} \in \widetilde{P}$ when $x \in P$ (see [PP12, VI.1.3.1]).

Proposition 2 will yield a general sufficient condition for the sobriety of $(P, \Omega\restriction P)$ (Theorem 1 below), from which we will obtain the case of negation-free LTL (Corollary 3).

One further step into the point-free setting gives us sharper results. A space (X, Ω) is T_D when for each $x \in X$, there is some open $U \in \Omega$ such that $x \in U$ and $(U \setminus \{x\}) \in \Omega$. See [PP12, §I.2]. It is shown in [PP12, Proposition VI.1.3.1] that if X is a (possibly not sober) T_D space, then condition (ii) of Proposition 2 holds for *any* $P \subseteq X$. It follows that if X is sober *and* T_D, then each $P \subseteq X$ induces a sober subspace.

Consider now the case of a sober space (X, Ω) which is not T_D. Hence, there is some $x \in X$ such that for all open U with $x \in U$, the set $U \setminus \{x\}$ is not open.

Lemma 5. *Let $x \in X$ as above and set $P := X \setminus \{x\}$. Then $(P, \Omega\restriction P)$ is not sober.*

It follows from [PP12, §I.2.1] that A^ω is T_D for the product topology.[7] But $[\![\mathsf{Str}A]\!]$ is not T_D, unless $A = \emptyset$. Consider $a^\omega \in [\![\mathsf{Str}A]\!]$. Then any Scott-open U containing a^ω contains also some finite $d \leq_{[\![\mathsf{Str}A]\!]} a^\omega$. Hence $U \setminus \{a^\omega\}$ is not upward-closed and thus not Scott-open.

[7] Actually, each T_1 space is T_D ([PP12, §I.2.1]).

Corollary 2. *$[\![\neg\Box a]\!] = [\![\mathsf{Str}A]\!] \setminus \{a^\omega\}$ is not a sober subspace of $[\![\mathsf{Str}A]\!]$.*

Let (X, Ω) be sober, and let $P \subseteq X$ be upward-closed for $\leq_\Omega$. Assume $x \notin P$. Then $P \setminus {\downarrow}x = P$, and thus $P \cap \widetilde{x} = P$. Hence $j(\widetilde{x}) = \bigcup\{V \in \Omega \mid V \cap P = P\}$, so $j(\widetilde{x}) = X$ and $\widetilde{x} \notin \widetilde{P}$. It follows that $x \in P$ precisely when $\widetilde{x} \in \widetilde{P}$, and Proposition 2 gives the following.

Theorem 1. *If (X, Ω) is sober and $P \subseteq X$ is upward-closed for $\leq_\Omega$, then $(P, \Omega\restriction P)$ is sober.*

Corollary 3. *If (X, Ω) is a sober dcpo and if $P \subseteq X$ is upward-closed, then $(P, \Omega\restriction P)$ is sober. In particular, if Φ is negation-free, then $[\![\Phi]\!]$ is a sober subspace of $[\![\mathsf{Str}A]\!]$.*

The importance we give to the negation-free fragment of LTL ultimately rests on Corollaries 2 and 3. But the frame $\widetilde{P}$ seems too abstract to be used concretely.

4. Geometric Logic

Geometric logic is an infinitary propositional logic which describes frames. Very roughly, the idea is that if a theory $\mathtt{Th}$ in geometric logic represents a frame L, then the models of $\mathtt{Th}$ can be organized in a space which is homeomorphic to $\mathrm{pt}(L)$, the space of points of L.

We will represent the (sober) space $[\![\mathsf{Str}A]\!]$ by a theory $\mathtt{T}[\![\mathsf{Str}A]\!]$ in geometric logic. Further, for each negation-free formula Φ of LTL, we shall (inductively) devise a theory $\mathtt{T}[\![\Phi]\!]$ such that $\mathtt{T}[\![\mathsf{Str}A]\!] \cup \mathtt{T}[\![\Phi]\!]$ represents the (sober) subspace induced by $[\![\Phi]\!] \hookrightarrow [\![\mathsf{Str}A]\!]$. This will provide a concrete presentation of the corresponding quotient frame.

Our approach to geometric logic here is not the usual one, as presented in e.g. [Vic07] (see also [Joh02, §D]). The relations between the two approaches are discussed in §5.

4.1. Geometric Theories

Formulae and Valuations. Let At be a set of *atomic propositions*. The *conjunctive* and the *geometric* formulae over At are respectively defined as

$$\gamma, \gamma' \in \mathtt{Conj}(At) \quad ::= \quad p \ \mid\ \mathtt{true} \ \mid\ \gamma \wedge \gamma' \qquad \text{and} \qquad \varphi, \psi, \theta \in \mathtt{Geom}(At) \quad ::= \quad \bigvee S$$

where $p \in At$ and $S \subseteq \mathtt{Conj}(At)$. A *valuation* of At is a function $\nu \colon At \to \mathbf{2}$. Given $\chi \in \mathtt{Conj}(At) \cup \mathtt{Geom}(At)$, the *satisfaction relation* $\nu \models \chi$ is defined by

$$\begin{array}{llllll} \nu \models \mathtt{true} & & & \nu \models \gamma \wedge \gamma' & \text{iff} & \nu \models \gamma \text{ and } \nu \models \gamma' \\ \nu \models p & \text{iff} & \nu(p) = 1 & \nu \models \bigvee S & \text{iff} & \text{there exists } \gamma \in S \text{ such that } \nu \models \gamma \end{array}$$

We let $\mathtt{false}$ be the geometric formula $\bigvee \emptyset$. We may write γ for the geometric formula $\bigvee\{\gamma\}$. Given conjunctive formulae $(\gamma_i \mid i \in I)$, we write $\bigvee_{i \in I} \gamma_i$ for the geometric formula $\bigvee\{\gamma_i \mid i \in I\}$. Note that $\bigvee_{i \in I} \gamma_i = \bigvee_{j \in J} \gamma'_j$ if there is a bijection $f \colon I \to J$ with $\gamma_i = \gamma'_{f(i)}$.

Remark 7. There is no primitive notion of conjunction or disjunction on geometric formulae, but they can be defined. Given $(\varphi_i \mid i \in I)$ with $\varphi_i = \bigvee\{\gamma_{i,j} \mid j \in J_i\}$, we define $\bigvee_{i \in I} \varphi_i$ to be the geometric formula $\bigvee\{\gamma_{i,j} \mid i \in I \text{ and } j \in J_i\}$. We then have $\nu \models \bigvee_{i \in I} \varphi_i$ iff $\nu \models \varphi_i$ for some $i \in I$.

Similarly, given $\varphi = \bigvee_{i \in I} \gamma_i$ and $\psi = \bigvee_{j \in J} \gamma'_j$, we define $\varphi \wedge \psi := \bigvee\{\gamma_i \wedge \gamma'_j \mid (i, j) \in I \times J\}$. Then $\nu \models \varphi \wedge \psi$ iff $\nu \models \varphi$ and $\nu \models \psi$.

Sequents and Theories. A *sequent* over At is a pair $\psi \vdash \varphi$ of geometric formulae $\varphi, \psi \in \mathtt{Geom}(At)$. A valuation ν of At is a *model* of $\psi \vdash \varphi$ if $\nu \models \psi$ implies $\nu \models \varphi$. Note that ν is a model of the sequent $\varphi \vdash \mathtt{false}$ if, and only if, $\nu \not\models \varphi$.

A *geometric theory* over At is a set $\mathtt{T}$ of sequents over At. A valuation ν of At is a *model* of $\mathtt{T}$ if ν is a model of all the sequents of $\mathtt{T}$. We write $\mathbf{Mod}(\mathtt{T})$ for the set of models of $\mathtt{T}$.

An *antecedent-free sequent* has the form $\mathtt{true} \vdash \varphi$, and is denoted $\vdash \varphi$ (or even φ) for short. An *antecedent-free theory* consists of antecedent-free sequents only.

Algebraic Dcpos. Algebraic dcpos have a natural representation by geometric theories. This relies on the following well-known facts, for which we refer to [AC98, §1.1].

An *ideal* on a poset $(P, \leq)$ is a subset $J \subseteq P$ which is downward-closed and directed. The set $\mathrm{Idl}(P)$ of ideals on P is an algebraic dcpo for inclusion. The operation $P \mapsto \mathrm{Idl}(P)$ is left adjoint to the forgetful functor from **DCPO** to the category of posets and monotone maps.

Let X be a dcpo, and let $\mathrm{Fin}(X)$ be its sub-poset of finite elements. Given an ideal $J \in \mathrm{Idl}(\mathrm{Fin}(X))$, since J is directed we have $\bigvee J \in X$. For the following, see e.g. [AC98, Proof of Proposition 1.1.21(2)].

Lemma 6. *Let X be an algebraic dcpo. The function $J \in \mathrm{Idl}(\mathrm{Fin}(X)) \mapsto \bigvee J \in X$ is an order-isomorphism. Its inverse $X \to \mathrm{Idl}(\mathrm{Fin}(X))$ takes x to $\{d \in \mathrm{Fin}(X) \mid d \leq x\}$.*

We represent an algebraic dcpo $X = (X, \leq_X)$ by a geometric theory $\mathtt{T}(X)$ over $At = \mathrm{Fin}(X)$. The theory $\mathtt{T}(X)$ consists of $\vdash \bigvee \mathrm{Fin}(X)$ together with all the sequents

$$d \vdash d' \quad (\text{if } d' \leq_X d) \qquad\qquad d \wedge d' \vdash \bigvee \{d'' \in \mathrm{Fin}(X) \mid d \leq_X d'' \text{ and } d' \leq_X d''\}$$

where $d, d' \in \mathrm{Fin}(X)$. Note that $J \subseteq \mathrm{Fin}(X)$ is an ideal if, and only if, its characteristic function $\mathrm{Fin}(X) \to \mathbf{2}$ is a model of $\mathtt{T}(X)$. Combined with Lemma 6, this yields Proposition 3 below. If $x \in X$, let $\nu(x) \colon At \to \mathbf{2}$ be the characteristic function of $\{d \in \mathrm{Fin}(X) \mid d \leq_X x\}$.

Proposition 3. *The map $x \mapsto \nu(x)$ is a bijection $X \to \mathbf{Mod}(\mathtt{T}(X))$.*

When X is actually an algebraic cpo, let $\mathtt{T}_\perp(X)$ be the theory obtained from $\mathtt{T}(X)$ by replacing the antecedent-free sequent $\vdash \bigvee \mathrm{Fin}(X)$ with $\vdash \perp_X$ (where $\perp_X \in \mathrm{Fin}(X)$ is the least element of X). The theories $\mathtt{T}(X)$ and $\mathtt{T}_\perp(X)$ have exactly the same models. Hence Proposition 3 also holds with $\mathtt{T}_\perp(X)$ in place of $\mathtt{T}(X)$.

Example 10 (Streams). We further simplify the theory representing the cpo $[\![\mathsf{Str}A]\!]$.

Consider $x, y \in [\![\mathsf{Str}A]\!]$ such that $\uparrow x \cap \uparrow y \neq \emptyset$ (i.e. such that $x, y \leq_{[\![\mathsf{Str}A]\!]} z$ for some $z \in [\![\mathsf{Str}A]\!]$). Then, since $[\![A]\!]$ is a flat cpo, we have $x(n) = y(n)$ for all $n \in \mathrm{supp}(x) \cap \mathrm{supp}(y)$. It follows that the set $\{x, y\}$ has a sup $x \vee_{[\![\mathsf{Str}A]\!]} y$ in $[\![\mathsf{Str}A]\!]$. Note that $x \vee_{[\![\mathsf{Str}A]\!]} y$ is finite whenever so are x and y.

We can thus represent $[\![\mathsf{Str}A]\!]$ with the following theory $\mathtt{T}[\![\mathsf{Str}A]\!]$, where $d, d' \in \mathrm{Fin}([\![\mathsf{Str}A]\!])$.

$$\begin{array}{rcll@{\qquad}rcll} d & \vdash & d' & (\text{if } d' \leq_{[\![\mathsf{Str}A]\!]} d) & d \wedge d' & \vdash & \mathtt{false} & (\text{if } \uparrow d \cap \uparrow d' = \emptyset) \\ & \vdash & \perp^\omega & & d \wedge d' & \vdash & d \vee_{[\![\mathsf{Str}A]\!]} d' & (\text{if } \uparrow d \cap \uparrow d' \neq \emptyset) \end{array}$$

Remark 8. Note that the theory $\mathtt{T}([\![\mathsf{Str}A]\!])$ of Example 10 only involves finite geometric formulae. Actually, this amounts to the fact that $[\![\mathsf{Str}A]\!]$ is *spectral* in its Scott topology (see [GvG23, Corollary 7.48 and Definition 6.2]).[8] Roughly, being a spectral space means that the topology can be generated from a distributive lattice (as opposed to a frame). See e.g. [GvG23, Proposition 3.26 and Theorem 6.1] for details.

Spectral cpos include those known as "SFP" domains (see e.g. [Abr91, §2.2]). SFP domains are also called "bifinite" domains (see e.g. [AC98, Definition 5.2.2 and Theorem 5.2.7]). They are stable under most common domain operations and have solutions for recursive type equations (see e.g. [Abr91, §2.2]).

In the paradigm of "Domain Theory in Logical Form", spectral domains are particularly important because the logic of the underlying distributive lattice can be incorporated into finitary type systems [Abr91] (see also [AC98, §10.5] for a simple instance).[9]

[8] Note that $[\![\mathsf{Str}A]\!]$ is always compact, since any Scott-open containing $\perp^\omega$ contains the whole of $[\![\mathsf{Str}A]\!]$. This contrasts with the product topology on A^ω, which is compact iff A is finite (see e.g. [PP04, §III.3.5]).

[9] In the terminology of §5, this is because frames induced by distributive lattices are always spatial (see e.g. [Joh82, Theorem II.3.4]).

4.2. The Sober Space of Models

Given a geometric theory $\mathtt{T}$ over At, we will now equip $\mathbf{Mod}(\mathtt{T})$ with a sober topology induced by a quotient of $\mathtt{Geom}(At)$. In particular, this will extend the bijection of Proposition 3 to an homeomorphism between an algebraic dcpo X and the space of models $\mathbf{Mod}(\mathtt{T}(X))$.

In view of Example 10, we may have $\mathbf{Mod}(\mathtt{T}) = \mathbf{Mod}(\mathtt{U})$ for different theories $\mathtt{T}$ and $\mathtt{U}$ over At. The topology on $M = \mathbf{Mod}(\mathtt{T})$ depends on M (and At), but not on the theory $\mathtt{T}$ such that $M = \mathbf{Mod}(\mathtt{T})$. Fix a set of atomic propositions At and let M be a set of the form $\mathbf{Mod}(\mathtt{T})$ for some theory $\mathtt{T}$ over At. Define

$$\mathsf{mod}_M \colon \mathtt{Geom}(At) \to \mathcal{P}(M), \quad \varphi \mapsto \{\nu \in M \mid \nu \models \varphi\}$$

Let $\Omega(M) \subseteq \mathcal{P}(M)$ be the image of mod_M. We have $\mathsf{mod}_M(\mathtt{true}) = M$, and (via Remark 7),

$$\mathsf{mod}_M(\varphi \wedge \psi) = \mathsf{mod}_M(\varphi) \cap \mathsf{mod}_M(\psi) \qquad \text{and} \qquad \mathsf{mod}_M\left(\bigvee_{i\in I} \varphi_i\right) = \bigcup_{i\in I} \mathsf{mod}_M(\varphi_i)$$

It follows that $(\Omega(M), \subseteq)$ is stable under the sups and the finite infs of $(\mathcal{P}(M), \subseteq)$. Hence $(\Omega(M), \subseteq)$ is a sub-frame of $(\mathcal{P}(M), \subseteq)$, and $(M, \Omega(M))$ is a topological space.

Given a theory $\mathtt{T}$, we write $\mathbf{Mod}(\mathtt{T})$ for $(\mathbf{Mod}(\mathtt{T}), \Omega(\mathbf{Mod}(\mathtt{T})))$, the *space of models of* $\mathtt{T}$. Since the space $\mathbf{Mod}(\mathtt{T})$ only depends on the models of $\mathtt{T}$, the following directly applies to the theory $\mathtt{T}[\![\mathsf{Str}A]\!]$ of Example 10.

Proposition 4. *Let X be an algebraic dcpo. The bijection $x \mapsto \nu(x)$ of Proposition 3 extends to an homeomorphism from X to $\mathbf{Mod}(\mathtt{T}(X))$.*

Spaces of models are always sober. To this end, given M as above, we quotient $\mathtt{Geom}(At)$ under the preorder $\preceq_M$ with $\varphi \preceq_M \psi$ iff $\mathsf{mod}_M(\varphi) \subseteq \mathsf{mod}_M(\psi)$. The relation $\sim_M$ of *M-equivalence* on $\mathtt{Geom}(At)$ is defined as $\varphi \sim_M \psi$ iff $\varphi \preceq_M \psi$ and $\psi \preceq_M \varphi$ (i.e. $\mathsf{mod}_M(\varphi) = \mathsf{mod}_M(\psi)$). We let $[\varphi]_M$ be the $\sim_M$-class of φ, and $\mathtt{Geom}(At)/M$ be the set of $\sim_M$-classes of geometric formulae. We write $\leq_M$ for the partial order on $\mathtt{Geom}(At)/M$ induced by the preorder $\preceq_M$ (see e.g. [GL13, §2.3.1]).

The function mod_M yields an order-isomorphism $(\mathtt{Geom}(At)/M, \leq_M) \to (\Omega(M), \subseteq)$. Since order-isomorphisms preserve all existing sups and infs ([DP02, Lemma 2.27(ii)]), we obtain

Lemma 7. *$(\mathtt{Geom}(At)/M, \leq_M)$ is a frame with greatest element $[\mathtt{true}]_M$, and*

$$\begin{aligned}
[\textstyle\bigvee_{i\in I} \gamma_i]_M \wedge [\bigvee_{j\in J} \gamma'_j]_M &= [\textstyle\bigvee \{\gamma_i \wedge \gamma'_j \mid i \in I \text{ and } j \in J\}]_M \\
\textstyle\bigvee_{i\in I} [\bigvee_{j\in J_i} \gamma_{i,j}]_M &= [\textstyle\bigvee \{\gamma_{i,j} \mid i \in I \text{ and } j \in J_i\}]_M
\end{aligned}$$

Theorem 2. *Let $\mathtt{T}$ be a geometric theory over At. The function taking $\nu \in \mathbf{Mod}(\mathtt{T})$ to $\{[\varphi]_{\mathbf{Mod}(\mathtt{T})} \mid \nu \models \varphi\}$ is an homeomorphism from $\mathbf{Mod}(\mathtt{T})$ to $\mathrm{pt}(\mathtt{Geom}(At)/\mathbf{Mod}(\mathtt{T}))$.*

Corollary 4. *Let $\mathtt{T}$ be a geometric theory. The space $\mathbf{Mod}(\mathtt{T})$ is sober.*

Subspaces. Consider a (sober) space (X, Ω). Assume that X is represented by a geometric theory $\mathtt{T}$ over At, in the sense that X is homeomorphic to the space $\mathbf{Mod}(\mathtt{T})$. Given a subset $P \subseteq X$, there might be a theory $\mathtt{U}$ over At such that the bijection $X \cong \mathbf{Mod}(\mathtt{T})$ restricts to a bijection $P \cong \mathbf{Mod}(\mathtt{T} \cup \mathtt{U})$. In this case, Proposition 5 below implies that the subspace $(P, \Omega{\restriction}P)$ is homeomorphic to the space $\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})$, so that $(P, \Omega{\restriction}P)$ is sober and $\Omega{\restriction}P$ is isomorphic to $\mathtt{Geom}(At)/\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})$. In such situations, we write $\mathbf{Mod}_{\mathtt{T}}(\mathtt{U})$ for the space $\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})$.

Proposition 5. *Given geometric theories $\mathtt{T}$ and $\mathtt{U}$ on At, the space $\mathbf{Mod}_{\mathtt{T}}(\mathtt{U})$ is* equal *to the subspace induced by the inclusion $\mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \subseteq \mathbf{Mod}(\mathtt{T})$.*

Example 11. Let X be an algebraic dcpo, and let $P \coloneqq \uparrow y$ for some fixed $y \in X$. Given $x \in X$, we have $x \in P$ precisely when $\nu(x)$ is a model of $\mathtt{U}(y) \coloneqq \{\vdash d \mid d \leq_X y\}$. Hence, the subspace induced by $P \subseteq X$ is homeomorphic to $\mathbf{Mod}_{\mathtt{T}(X)}(\mathtt{U}(y))$.

Assume now $X = [\![\mathsf{Str}A]\!]$, and let $a \in A$. Since the subspace induced by the LTL formula $\neg\Box a$ is not sober (Corollary 2), it follows from Proposition 5 and Corollary 4 that there is *no* geometric theory $\mathtt{T}$ over $\mathrm{Fin}([\![\mathsf{Str}A]\!])$ such that for all $x \in [\![\mathsf{Str}A]\!]$, we have $x \in [\![\neg\Box a]\!]$ iff $\nu(x) \in \mathbf{Mod}(\mathtt{T})$.

4.3. Operations on Theories

We shall now see that for each negation-free formula Φ of LTL, there is a geometric theory $\mathtt{T}[\![\Phi]\!]$ such that for all streams x, we have $x \in [\![\Phi]\!]$ iff $\nu(x) \in \mathbf{Mod}(\mathtt{T}[\![\Phi]\!])$ (where $\nu(x)$ is as in Propositions 3 and 4). This may not be possible if Φ contains negations (Example 11).

To this end, we devise operations on theories which represent unions and intersections of sets of models. Given theories $(\mathtt{T}_i \mid i \in I)$ over At, we let $\curlywedge_{i\in I} \mathtt{T}_i \coloneqq \bigcup_{i\in I} \mathtt{T}_i$. Then we have

$$\mathbf{Mod}\left(\curlywedge_{i\in I} \mathtt{T}_i\right) \quad = \quad \bigcap_{i\in I} \mathbf{Mod}(\mathtt{T}_i)$$

Intersections of sets of models can thus be represented by unions of theories. It is more difficult to devise an operation on theories for *unions* of sets of models. A solution is provided by the following crucial construction. Let $(\mathtt{T}_i \mid i \in I)$ be theories, all over At, with $\mathtt{T}_i = \{\psi_{i,j} \vdash \varphi_{i,j} \mid j \in J_i\}$.

(1) If I is finite, we let $\curlyvee_{i\in I} \mathtt{T}_i \coloneqq \left\{\bigwedge_{i\in I} \psi_{i,f(i)} \vdash \bigvee_{i\in I} \varphi_{i,f(i)} \mid f \in \prod_{i\in I} J_i\right\}$.

(2) If I is infinite, and all $\mathtt{T}_i$'s are antecedent-free, $\curlyvee_{i\in I} \mathtt{T}_i \coloneqq \left\{\vdash \bigvee_{i\in I} \varphi_{i,f(i)} \mid f \in \prod_{i\in I} J_i\right\}$.

Note that if $(\mathtt{T}_i \mid i \in I)$ consists of countably many countable (antecedent-free) theories, then $\curlywedge_{i\in I} \mathtt{T}_i$ is always countable while $\curlyvee_{i\in I} \mathtt{T}_i$ may be uncountable.

Proposition 6. *In both cases above, we have (using the Axiom of Choice when I is infinite)*

$$\mathbf{Mod}\left(\curlyvee_{i\in I} \mathtt{T}_i\right) \quad = \quad \bigcup_{i\in I} \mathbf{Mod}(\mathtt{T}_i)$$

Example 12. Let X be an algebraic dcpo, and let $P \subseteq X$ be upward-closed. Then $P = \bigcup_{y\in P} \uparrow y$. Hence, given $x \in X$, we have $x \in P$ exactly when $\nu(x)$ is a model of $\curlyvee_{y\in P} \mathtt{U}(y)$, where $\mathtt{U}(y)$ is as in Example 11.

4.3.1. Translation of Negation-Free LTL Formulae

Recall from Lemma 1 that if Φ is a negation-free formula of $\mathsf{LTL}(A)$, then $[\![\Phi]\!]$ is upward-closed in $[\![\mathsf{Str}A]\!]$. Example 12 thus provides a geometric theory over $\mathrm{Fin}([\![\mathsf{Str}A]\!])$ for $[\![\Phi]\!]$.[10] But we shall get more information by explicitly defining a geometric theory $\mathtt{T}[\![\Phi]\!]$ by induction on Φ. Actually, it is even better to work with a stratified presentation of negation-free LTL.

Our stratification of negation-free LTL formulae is based on the following expected fact. Recall from §2.2 the map $[\![\bigcirc]\!]$ taking $S \in \mathcal{P}([\![\mathsf{Str}A]\!])$ to $\{x \mid x{\restriction}1 \in S\} \in \mathcal{P}([\![\mathsf{Str}A]\!])$.

Lemma 8. *Fix set A.*

(1) The function $[\![\bigcirc]\!] \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ preserves all unions and all intersections.

(2) Given LTL formulae Φ, Ψ, let $H_{\Phi,\Psi}$ take $S \in \mathcal{P}([\![\mathsf{Str}A]\!])$ to $[\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!](S))$. Then we have $[\![\Phi \mathbin{\mathsf{U}} \Psi]\!] = \bigcup_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{False}]\!])$ and $[\![\Phi \mathbin{\mathsf{W}} \Psi]\!] = \bigcap_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{True}]\!])$.

Figure 1 presents a stratified grammar for negation-free $\mathsf{LTL}(A)$. We let $G = G(A)$ be the set of all formulae Φ_1, Ψ_1 from the second layer in Figure 1. $G_\delta = G_\delta(A)$ consists of formulae Φ_2, Ψ_2 from the third layer. The negation-free $\mathsf{LTL}(A)$ formulae are the those from the last layer.

[10] In fact, we shall see in Remark 13 (§5) that when A is countable, if $P \subseteq [\![\mathsf{Str}A]\!]$ induces a sober subspace, then this subspace is homeomorphic to $\mathbf{Mod}_{\mathtt{T}[\![\mathsf{Str}A]\!]}(\mathtt{U})$ for some (abstractly given) theory $\mathtt{U}$.

$$
\begin{array}{rcl}
\Phi_0, \Psi_0 & ::= & \mathsf{False} \mid \mathsf{True} \mid a \\
 & \mid & \Phi_0 \vee \Psi_0 \mid \Phi_0 \wedge \Psi_0 \mid \bigcirc\Phi_0 \\
 & & \\
G(A) \ni \Phi_1, \Psi_1 & ::= & \Phi_0 \\
 & \mid & \Phi_1 \vee \Psi_1 \mid \Phi_1 \wedge \Psi_1 \mid \bigcirc\Phi_1 \mid \Phi_1 \mathbin{\mathsf{U}} \Psi_1 \\
 & & \\
G_\delta(A) \ni \Phi_2, \Psi_2 & ::= & \Phi_1 \in G(A) \\
 & \mid & \Phi_2 \vee \Psi_2 \mid \Phi_2 \wedge \Psi_2 \mid \bigcirc\Phi_2 \mid \Phi_2 \mathbin{\mathsf{W}} \Psi_2 \\
 & & \\
\text{n.-f. } \mathsf{LTL}(A) \ni \Phi_3, \Psi_3 & ::= & \Phi_2 \in G_\delta(A) \\
 & \mid & \Phi_3 \vee \Psi_3 \mid \Phi_3 \wedge \Psi_3 \mid \bigcirc\Phi_3 \mid \Phi_3 \mathbin{\mathsf{U}} \Psi_3 \mid \Phi_3 \mathbin{\mathsf{W}} \Psi_3
\end{array}
$$

Figure 1. Stratified grammar for negation-free $\mathsf{LTL}(A)$, where $a \in A$.

Example 13. Recall from §2.2 that $\Diamond\Psi = (\mathsf{True} \mathbin{\mathsf{U}} \Psi)$ and $\Box\Phi = (\Phi \mathbin{\mathsf{W}} \mathsf{False})$. Hence, G is closed under $\Diamond(-)$ and G_δ is closed under $\Box(-)$. But G_δ is (crucially) *not* closed under $\Diamond(-)$. In particular, we have $\bigcirc a, \Diamond a \in G$ and $\Box a, \Box\Diamond a \in G_\delta$. On the other hand, the negation-free formula $\Diamond\Box a$ is not a G_δ formula.

When looking at $\Diamond\Psi$ and $\Box\Phi$ via Lemma 8(2), it is convenient to simplify the functions $H_{\mathsf{True},\Psi}$ and $H_{\Phi,\mathsf{False}}$ to respectively $[\![\Psi]\!] \cup [\![\bigcirc]\!](-)$ and $[\![\Phi]\!] \cap [\![\bigcirc]\!](-)$. This amounts to restate Lemma 8(2) as $[\![\Diamond\Psi]\!] = \bigcup_{m\in\mathbb{N}}[\![\Psi]\!] \cup [\![\bigcirc\Psi]\!] \cup \cdots \cup [\![\bigcirc^m\Psi]\!]$, and similarly for $[\![\Box\Phi]\!]$.

Remark 9. The interpretations of formulae from G or G_δ have the expected topological complexity. Namely, if $\Phi_1 \in G$, then $[\![\Phi_1]\!]$ is Scott-open in $[\![\mathsf{Str}A]\!]$. If $\Phi_2 \in G_\delta$, then $[\![\Phi_2]\!]$ is a countable intersection of Scott-opens (i.e. a G_δ subset of $[\![\mathsf{Str}A]\!]$).

This stratification of negation-free LTL allows for a stratified translation to geometric theories. In fact, each LTL formula $\Phi_1 \in G$ can be translated to a single geometric formula $\mathtt{F}[\![\Phi_1]\!]$, with $\mathtt{F}[\![\Phi_0]\!]$ finite when Φ_0 is from the first layer. Formulae Φ from the last two layers will be translated to antecedent-free theories $\mathtt{T}[\![\Phi]\!]$, with $\mathtt{T}[\![\Phi_2]\!]$ countable when $\Phi_2 \in G_\delta$.

Fix a set A and let $At := \mathrm{Fin}([\![\mathsf{Str}A]\!])$. We devise operations on geometric formulae and theories which mimic the action of $[\![\bigcirc]\!](-)$ on $\mathcal{P}([\![\mathsf{Str}A]\!])$. We begin with geometric formulae. The idea is that given $x \in [\![\mathsf{Str}A]\!]$ and $d \in \mathrm{Fin}([\![\mathsf{Str}A]\!])$, we have $d \leq_{[\![\mathsf{Str}A]\!]} x{\upharpoonright}1$ exactly when $(\bot \cdot d) \leq_{[\![\mathsf{Str}A]\!]} x$. The geometric formula $\bigcirc\varphi$ is then defined by propagating the stream operation $d \mapsto \bot \cdot d$ in φ. We set $\bigcirc d := \bot \cdot d$ and

$$
\bigcirc\mathtt{true} := \mathtt{true} \qquad \bigcirc(\gamma \wedge \gamma') := (\bigcirc\gamma) \wedge (\bigcirc\gamma') \qquad \bigcirc \textstyle\bigvee_{i\in I}\gamma_i := \bigvee_{i\in I}\bigcirc\gamma_i
$$

Given a theory $\mathtt{Th}$ over At, we let $\bigcirc\mathtt{Th} := \{\bigcirc\psi \vdash \bigcirc\varphi \mid (\psi \vdash \varphi) \in \mathtt{Th}\}$. Note that $\bigcirc\mathtt{Th}$ is antecedent-free whenever so is $\mathtt{Th}$. Recall the map $x \mapsto \nu(x)$ of Propositions 3 and 4.

Lemma 9. *Let $x \in [\![\mathsf{Str}A]\!]$.*

(1) We have $\nu(x) \models \bigcirc\varphi$ if, and only if, $\nu(x{\upharpoonright}1) \models \varphi$.

(2) We have $\nu(x) \in \mathbf{Mod}(\bigcirc\mathtt{Th})$ if, and only if, $\nu(x{\upharpoonright}1) \in \mathbf{Mod}(\mathtt{Th})$.

We now define a geometric formula $\mathtt{F}[\![\Phi_1]\!]$ over At by induction on $\Phi_1 \in G$:

$$
\begin{array}{lcllcl}
\mathtt{F}[\![a]\!] & := & a \cdot \bot^\omega & & & \\
\mathtt{F}[\![\mathsf{True}]\!] & := & \mathtt{true} & \mathtt{F}[\![\Phi_1 \wedge \Psi_1]\!] & := & \mathtt{F}[\![\Phi_1]\!] \wedge \mathtt{F}[\![\Psi_1]\!] \\
\mathtt{F}[\![\mathsf{False}]\!] & := & \mathtt{false} & \mathtt{F}[\![\Phi_1 \vee \Psi_1]\!] & := & \mathtt{F}[\![\Phi_1]\!] \vee \mathtt{F}[\![\Psi_1]\!] \\
\mathtt{F}[\![\bigcirc\Phi_1]\!] & := & \bigcirc\mathtt{F}[\![\Phi_1]\!] & \mathtt{F}[\![\Phi_1 \mathbin{\mathsf{U}} \Psi_1]\!] & := & \bigvee_{n\in\mathbb{N}} \mathtt{H}^n_{\mathtt{F}[\![\Phi_1]\!],\mathtt{F}[\![\Psi_1]\!]}(\mathtt{false})
\end{array}
$$

where $\mathtt{H}_{\varphi,\psi}(\theta) := \psi \vee (\varphi \wedge \bigcirc(\theta))$. (We silently included the case of Φ_0 from the first layer.)

Lemma 10. *Let $\Phi_1 \in G$. Given $x \in [\![\mathsf{Str}A]\!]$, we have $x \in [\![\Phi_1]\!]$ if, and only if, $\nu(x) \models \mathtt{F}[\![\Phi_1]\!]$.*

Finally, the antecedent-free theory $\mathtt{T}[\![\Phi_3]\!]$ is defined by induction on Φ_3 as follows:

$$\begin{array}{lcllcl} \mathtt{T}[\![\Phi_1]\!] & \coloneqq & \{\vdash \mathtt{F}[\![\Phi_1]\!]\} & \mathtt{T}[\![\bigcirc\Phi_3]\!] & \coloneqq & \bigcirc\mathtt{T}[\![\Phi_3]\!] \\ \mathtt{T}[\![\Phi_3 \wedge \Psi_3]\!] & \coloneqq & \mathtt{T}([\![\Phi_3]\!]) \curlywedge \mathtt{T}([\![\Psi_3]\!]) & \mathtt{T}[\![\Phi_3 \mathbin{\mathsf{W}} \Psi_3]\!] & \coloneqq & \bigcurlywedge_{n\in\mathbb{N}} \mathtt{TH}^n_{\mathtt{T}[\![\Phi_3]\!],\mathtt{T}[\![\Psi_3]\!]}(\{\vdash \mathtt{true}\}) \\ \mathtt{T}[\![\Phi_3 \vee \Psi_3]\!] & \coloneqq & \mathtt{T}([\![\Phi_3]\!]) \curlyvee \mathtt{T}([\![\Psi_3]\!]) & \mathtt{T}[\![\Phi_3 \mathbin{\mathsf{U}} \Psi_3]\!] & \coloneqq & \bigcurlyvee_{n\in\mathbb{N}} \mathtt{TH}^n_{\mathtt{T}[\![\Phi_3]\!],\mathtt{T}[\![\Psi_3]\!]}(\{\vdash \mathtt{false}\}) \end{array}$$

where $\mathtt{TH}_{\mathtt{T},\mathtt{U}}(\mathtt{V}) \coloneqq \mathtt{U} \curlyvee (\mathtt{T} \curlywedge \bigcirc\mathtt{V})$. (We silently included the case of $\Phi_2 \in G_\delta$.)

Theorem 3. *Let Φ be negation-free. For $x \in [\![\mathsf{Str}A]\!]$, we have $x \in [\![\Phi]\!]$ if, and only if, $\nu(x) \in \mathbf{Mod}(\mathtt{T}[\![\Phi]\!])$.*

Remark 10. A direct inspection reveals that $\mathtt{F}[\![\Phi_0]\!]$ is indeed a finite geometric formula when Φ_0 is from the first layer. Similarly, the geometric theory $\mathtt{T}[\![\Phi_2]\!]$ contains only countably-many sequents when $\Phi_2 \in G_\delta$.

Remark 11. Recall that LTL formulae Φ, Ψ are *equivalent*, notation $\Phi \equiv \Psi$, when $[\![\Phi]\!] = [\![\Psi]\!]$. The following standard equivalences are obtained similarly as in [BK08, §5.1.4].

$$\begin{array}{lcllcllcl} \bigcirc\mathsf{False} & \equiv & \mathsf{False} & \bigcirc(\Phi \vee \Psi) & \equiv & \bigcirc\Phi \vee \bigcirc\Psi & \bigcirc(\Phi \mathbin{\mathsf{U}} \Psi) & \equiv & (\bigcirc\Phi) \mathbin{\mathsf{U}} (\bigcirc\Psi) \\ \bigcirc\mathsf{True} & \equiv & \mathsf{True} & \bigcirc(\Phi \wedge \Psi) & \equiv & \bigcirc\Phi \wedge \bigcirc\Psi & \bigcirc(\Phi \mathbin{\mathsf{W}} \Psi) & \equiv & (\bigcirc\Phi) \mathbin{\mathsf{W}} (\bigcirc\Psi) \end{array}$$

Hence, up to equivalence, we can push the $\bigcirc$'s to atoms $a \in A$. In particular, we may assume that $\bigcirc$ occurs only in first layer's formulae of the form $\bigcirc^n a$.

Example 14. Let Φ_0 be an LTL formula from the first layer in Figure 1. Up to equivalence (Remark 11), we can assume that Φ_0 is in disjunctive normal form, and actually that Φ_0 is a disjunction of conjunctions of formulae of the form $\bigcirc^n a$. Then $\mathtt{F}[\![\Phi_0]\!]$ is simply a disjunction of conjunctions of atomic propositions of the form $(\bot^n \cdot a \cdot \bot^\omega) \in \mathrm{Fin}([\![\mathsf{Str}A]\!])$.

Consider the formula $\Phi_1 \coloneqq \Diamond\Phi_0 \in G$. Recall that $\Diamond\Phi_0 = (\mathsf{True} \mathbin{\mathsf{U}} \Phi_0)$, and note that $\mathtt{H}_{\mathtt{true},\varphi}(\theta) = \varphi \vee (\mathtt{true} \wedge \bigcirc\theta)$. Up to the replacement of $\mathtt{true} \wedge \bigcirc\theta$ by $\bigcirc\theta$, we get that $\mathtt{F}[\![\Phi_1]\!]$ is the geometric formula $\bigvee_{m\in\mathbb{N}} (\mathtt{F}[\![\Phi_0]\!] \vee \bigcirc\mathtt{F}[\![\Phi_0]\!] \vee \cdots \vee \bigcirc^m\mathtt{F}[\![\Phi_0]\!])$. This mirrors the formulation of Lemma 8(2) in Example 13.

We turn to the formula $\Phi_2 \coloneqq \Box\Diamond\Phi_0 \in G_\delta$. We have $\Box\Phi_1 = (\Phi_1 \mathbin{\mathsf{W}} \mathsf{False})$ and we simplify $\mathtt{H}_{\varphi,\mathtt{false}}(\theta)$ to $\varphi \wedge \bigcirc\theta$. The theory $\mathtt{T}[\![\Phi_2]\!]$ then consists of all the sequents

$$\vdash \quad \bigwedge_{n\le N} \bigvee_{m\in\mathbb{N}} \bigcirc^n\mathtt{F}[\![\Phi_0]\!] \vee \bigcirc^{n+1}\mathtt{F}[\![\Phi_0]\!] \vee \cdots \vee \bigcirc^{n+m}\mathtt{F}[\![\Phi_0]\!]$$

where N ranges over $\mathbb{N}$. This mirrors the fact that $[\![\Box\Diamond\Phi_0]\!]$ is the set of those streams $x \in [\![\mathsf{Str}A]\!]$ such that for each $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$ with $x{\restriction}(n+m) \in [\![\Phi_0]\!]$.

Example 15. Continuing Example 14, we now consider the case of $\Phi_3 \coloneqq \Diamond\Box a$ with $a \in A$. For this more involved example, we allow ourselves some simplifications that we deliberately avoided in Example 14. Namely, for $\mathtt{T}[\![\Box\Phi]\!]$ and $\mathtt{T}[\![\Diamond\Psi]\!]$ we take respectively $\bigcurlywedge_{n\in\mathbb{N}} \bigcirc^n\mathtt{T}[\![\Phi]\!]$ and $\bigcurlyvee_{m\in\mathbb{N}} \bigcirc^m\mathtt{T}[\![\Psi]\!]$. Then we have

$$\begin{array}{rcl} \mathtt{T}[\![\Box a]\!] & = & \{\bigcirc^n\mathtt{F}[\![a]\!] \mid n \in \mathbb{N}\} \\ \mathtt{T}[\![\Diamond\Box a]\!] & = & \left\{\bigvee_{m\in\mathbb{N}} \bigcirc^{m+f(m)}\mathtt{F}[\![a]\!] \mid f\colon \mathbb{N} \to \mathbb{N}\right\} \end{array}$$

The uncountable theory $\mathtt{T}[\![\Diamond\Box a]\!]$ relies on the (classical) choice principle behind Proposition 6. It expresses that given a stream $x \in [\![\mathsf{Str}A]\!]$, we have $x \notin [\![\Diamond\Box a]\!]$ if, and only if, there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $x(m + f(m)) \neq a$ for all $m \in \mathbb{N}$. In particular, if $x \notin [\![\Diamond\Box a]\!]$, then the function $g\colon m \mapsto m + f(m)$ finds arbitrary large $n = g(m)$ such that $x(n) \neq a$.

Consider a negation-free $\Phi \in \mathsf{LTL}(A)$. We see the subset $[\![\Phi]\!] \subseteq [\![\mathsf{Str}A]\!]$ as a subspace rather than as a sub-dcpo (cf. Example 8). This subspace $[\![\Phi]\!] = ([\![\Phi]\!], \Omega([\![\mathsf{Str}A]\!]){\restriction}[\![\Phi]\!])$ is always sober (Corollary 3). With geometric logic, we gained a description of the subspace $[\![\Phi]\!]$ as the space of models of the theory $\mathtt{T}[\![\Phi]\!]$. Namely, the subspace $[\![\Phi]\!]$ is homeomorphic

$$(\textsc{Th})\ \frac{}{\varphi \vdash_{\mathtt{T}} \psi}\ (\text{if } (\varphi \vdash \psi) \in \mathtt{T}) \qquad (\textsc{Ax})\ \frac{}{\varphi \vdash_{\mathtt{T}} \varphi} \qquad (\textsc{Cut})\ \frac{\varphi \vdash_{\mathtt{T}} \theta \qquad \theta \vdash_{\mathtt{T}} \psi}{\varphi \vdash_{\mathtt{T}} \psi}$$

$$(\wedge\text{-L}_1)\ \frac{}{\varphi \wedge \psi \vdash_{\mathtt{T}} \varphi} \qquad (\wedge\text{-L}_2)\ \frac{}{\varphi \wedge \psi \vdash_{\mathtt{T}} \psi} \qquad (\wedge\text{-R})\ \frac{\theta \vdash_{\mathtt{T}} \varphi \qquad \theta \vdash_{\mathtt{T}} \psi}{\theta \vdash_{\mathtt{T}} \varphi \wedge \psi}$$

$$(\mathtt{true}\text{-R})\ \frac{}{\varphi \vdash_{\mathtt{T}} \mathtt{true}} \qquad (\bigvee\text{-L})\ \frac{\text{for all } i \in I,\ \varphi_i \vdash_{\mathtt{T}} \psi}{\bigvee_{i \in I} \varphi_i \vdash_{\mathtt{T}} \psi} \qquad (\bigvee\text{-R})\ \frac{}{\varphi_i \vdash_{\mathtt{T}} \bigvee_{i \in I} \varphi_i}$$

Figure 2. Deduction rules for geometric logic over a theory $\mathtt{T}$.

to $\mathbf{Mod}_{[\![\mathsf{Str}A]\!]}(\mathtt{T}[\![\Phi]\!])$ (Theorem 3 and Proposition 5), while the frame $\Omega([\![\mathsf{Str}A]\!])\restriction[\![\Phi]\!]$ is isomorphic to $\mathtt{Geom}(At)/\mathbf{Mod}_{[\![\mathsf{Str}A]\!]}(\mathtt{T}[\![\Phi]\!])$ (Theorem 2). Note that in the latter, two geometric formulae are equivalent exactly when they have the same $\mathtt{T}[\![\Phi]\!]$-models. Hence geometric formulae generate the frame $\Omega([\![\mathsf{Str}A]\!])\restriction[\![\Phi]\!]$. Moreover, we have seen in Examples 14 and 15 concrete cases in which the theory $\mathtt{T}[\![\Phi]\!]$ explicitly represents approximations of $[\![\Phi]\!]$.

However, a limitation of this approach is that the frame $\mathtt{Geom}(At)/\mathbf{Mod}_{[\![\mathsf{Str}A]\!]}(\mathtt{T}[\![\Phi]\!])$ is defined by purely semantic means. We discuss this in 5 below, which gives a complete deduction system for $\mathtt{T}[\![\Phi]\!]$ in the G_δ case. We now comment on potential extensions to LTL with negation.

Remark 12. Say that an LTL formula Φ is an F *formula* if Φ is the negation of a G formula. The F_σ formulae are the negations of the G_δ ones. For instance, $\neg a$ (with $a \in A$) is a simple non-trivial F formula, while $\neg\Box a \equiv \Diamond\neg a$ is an F_σ formula. It follows from Remark 9 that F formulae induce Scott-*closed* subsets of $[\![\mathsf{Str}A]\!]$, and that the F_σ ones induce countable unions of Scott-closed sets (i.e. F_σ sets).

Now, if $\Phi = \neg\Phi_1$ with $\Phi_1 \in G$, then the subspace $[\![\Phi]\!]$ is represented by the geometric theory $\{\mathtt{F}[\![\Phi_1]\!] \vdash \mathtt{false}\}$. Hence Theorem 3 extends to F formulae (so that Proposition 5 and Theorem 2 can be applied in this case). But beware that this does not hold in general for F_σ formulae, since the subspace $[\![\neg\Box a]\!] = [\![\Diamond\neg a]\!]$ is not representable in geometric logic (in the sense of Example 11). In particular, there is no geometric theory $\mathtt{T}$ such that $\mathbf{Mod}(\mathtt{T}) = \bigcup_{m\in\mathbb{N}} \mathbf{Mod}\{\bigcirc^m \mathtt{F}[\![a]\!] \vdash \mathtt{false}\}$, and Proposition 6 does not extend to infinitely many arbitrary theories.

5. Free Frames and Spatiality

Our approach to geometric logic in §4 focuses on *spaces* of *models*. However, the literature rather considers geometric logic as a formal way to present frames by *generators and relations*. A customary tool for this is the notion of *congruence preorder* (see [Vic89, §4 and §6.1–2] and [Hec15, §3]). It is folklore that congruence preorders can be presented using an (infinitary) deduction system for geometric logic (in the spirit of e.g. [Vic07, §2.2] and [Joh02, D1.1.7(m) and §D1.3]).

Let $\mathtt{T}$ be a geometric theory over At. The *deduction relation* $\vdash_{\mathtt{T}}$ on $\mathtt{Geom}(At)$ is defined by the rules in Figure 2, using the constructs of Remark 7. The relation $\vdash_{\mathtt{T}}$ is a preorder. Similarly as in §4.2, we quotient $\mathtt{Geom}(At)$ under the relation $\sim_{\mathtt{T}}$ of $\mathtt{T}$-*equivalence* defined as $\varphi \sim_{\mathtt{T}} \psi$ iff $\varphi \vdash_{\mathtt{T}} \psi$ and $\psi \vdash_{\mathtt{T}} \varphi$. We write $[\varphi]_{\mathtt{T}}$ for the $\sim_{\mathtt{T}}$-class of φ and $\mathtt{Geom}(At)/\mathtt{T}$ for the set of $\sim_{\mathtt{T}}$-classes. We let $\leq_{\mathtt{T}}$ be the partial order on $\mathtt{Geom}(At)/\mathtt{T}$ induced by $\vdash_{\mathtt{T}}$. In contrast with [Joh02, §D1.3 and D1.4.14] (see also [Vic07, §2.2]), we do not need to enforce frame distributivity in Figure 2, since it is hardwired in the constructs of Remark 7.

Lemma 11. $(\mathtt{Geom}(At)/\mathtt{T}, \leq_{\mathtt{T}})$ *is a frame.*

Spatiality. How does $\texttt{Geom}(At)/\texttt{T}$ relate to the frame $\texttt{Geom}(At)/\mathbf{Mod}(\texttt{T})$ of §4.2? To answer this question, we compare the preorders $\vdash_{\texttt{T}}$ and $\preceq_{\mathbf{Mod}(\texttt{T})}$. First, an easy induction on derivations proves the following *soundness* property:

$$\psi \vdash_{\texttt{T}} \varphi \quad \Longrightarrow \quad \psi \preceq_{\mathbf{Mod}(\texttt{T})} \varphi$$

The converse implication is a form of *completeness*: if every model of $\texttt{T}$ is a model of the sequent $\psi \vdash \varphi$, then $\psi \vdash_{\texttt{T}} \varphi$ is derivable. This may fail in geometric logic, by lack of spatiality. A frame L is said to be *spatial* when given $a \not\leq b$ in L, there is a point $\mathcal{F} \in \mathrm{pt}(L)$ such that $a \in \mathcal{F}$ and $b \notin \mathcal{F}$. A frame is spatial precisely when it is isomorphic to $\Omega(X)$ for some space X, but not every frame is spatial ([Joh82, §II.1.5]).

Assume now that $\texttt{Geom}(At)/\texttt{T}$ is spatial, and let $\psi \nvdash_{\texttt{T}} \varphi$. Hence $[\psi]_{\texttt{T}} \not\leq_{\texttt{T}} [\varphi]_{\texttt{T}}$ and there is a point $\mathcal{F}$ of $\texttt{Geom}(At)/\texttt{T}$ such that $[\psi]_{\texttt{T}} \in \mathcal{F}$ and $[\varphi]_{\texttt{T}} \notin \mathcal{F}$. Let $\nu\colon At \to \mathbf{2}$ take $p \in At$ to 1 iff $[p]_{\texttt{T}} \in \mathcal{F}$.

Lemma 12. *Let $\mathcal{F}$ and ν as above. Then for every $\theta \in \texttt{Geom}(At)$, we have $\nu \models \theta$ if, and only if, $[\theta]_{\texttt{T}} \in \mathcal{F}$. In particular, ν is a model of $\texttt{T}$ with $\nu \models \psi$ and $\nu \not\models \varphi$.*

Hence, when $\texttt{Geom}(At)/\texttt{T}$ is spatial, the preorders $\vdash_{\texttt{T}}$ and $\preceq_{\mathbf{Mod}(\texttt{T})}$ coincide, and $\texttt{Geom}(At)/\texttt{T}$ is thus isomorphic to $\texttt{Geom}(At)/\mathbf{Mod}(\texttt{T})$. Since $\texttt{Geom}(At)/\mathbf{Mod}(\texttt{T})$ is always isomorphic to the frame of opens of the space $\mathbf{Mod}(\texttt{T})$ (Theorem 2), we have the following.

Theorem 4. *Let $\texttt{T}$ be a geometric theory over At. Then the following are equivalent.*

(i) The frame $\texttt{Geom}(At)/\texttt{T}$ is spatial.

(ii) Deduction in $\vdash_{\texttt{T}}$ is complete ($\psi \vdash_{\texttt{T}} \varphi$ if, and only if, $\psi \preceq_{\mathbf{Mod}(\texttt{T})} \varphi$).

(iii) The frame $\texttt{Geom}(At)/\texttt{T}$ is isomorphic to $\texttt{Geom}(At)/\mathbf{Mod}(\texttt{T})$.

Free Frames and Congruence Preorders. In view of Theorem 4, it is interesting to know when a frame $\texttt{Geom}(At)/\texttt{T}$ is spatial. We discuss this using the following notions. Write $\texttt{E}$ for the empty theory.

Let $\mathbf{U}\colon \mathbf{Frm} \to \mathbf{Set}$ be the forgetful functor. A *free frame* on At is the data of a frame $L(At)$ and of a function $\eta_{At}\colon At \to \mathbf{U}L(At)$ such that for each frame L and each function $f\colon At \to \mathbf{U}L$, there is a unique frame morphism $f^\star\colon L(At) \to L$ making the following diagram to commute.

$$\begin{tikzcd} At \arrow[rr, "f"] \arrow[dr, "\eta_{At}"'] & & \mathbf{U}L \\ & \mathbf{U}L(At) \arrow[ur, dashed, "\mathbf{U}f^\star"'] & \end{tikzcd}$$

Proposition 7. $(\texttt{Geom}(At)/\texttt{E}, \leq_{\texttt{E}})$ *(with the function $p \in At \mapsto [p]_{\texttt{T}}$) is a free frame on At.*

The operation $At \mapsto \texttt{Geom}(At)/\texttt{E}$ thus yields a left adjoint to $\mathbf{U}\colon \mathbf{Frm} \to \mathbf{Set}$ (see e.g. [ML98, Theorem IV.1.2]). Actually, the set of Scott-opens of the cpo $\mathcal{P}(At)$ is also a free frame on At. See [Joh82, Lemma VII.4.9] (see also [Hec15, Theorem 3.1] and [Joh02, C1.1.4 and C4.1.6]). Since free frames are unique up to isomorphism, it follows that the frame $\texttt{Geom}(At)/\texttt{E}$ is spatial.

It is customary to present a frame by quotienting a free frame under a congruence preorder. We refer to [Hec15, §3.4]. A *congruence preorder* on a frame $(L, \leq)$ is a preorder $\preceq$ on L such that $\leq \subseteq \preceq$ and such that for each arbitrary (resp. finite) $S \subseteq L$, we have $\bigvee S \preceq b$ whenever $a \preceq b$ for all $a \in S$ (resp. $b \preceq \bigwedge S$ whenever $b \preceq a$ for all $a \in S$). Given a congruence preorder $\preceq$ on L, let $a \sim b$ iff $a \preceq b$ and $b \preceq a$.

Each binary relation R on L is contained in a least congruence preorder $\overline{R}$ on L.

Proposition 8. *Given geometric theories $\texttt{T}, \texttt{U}$ over At, let $R := \{([\varphi]_{\texttt{T}}, [\psi]_{\texttt{T}}) \mid (\varphi \vdash \psi) \in \texttt{T} \cup \texttt{U}\}$. Then $\texttt{Geom}(At)/(\texttt{T} \cup \texttt{U})$ is isomorphic to the quotient of $\texttt{Geom}(At)/\texttt{T}$ by $\sim_{\overline{R}}$.*

In particular, $\mathtt{Geom}(At)/\mathtt{T}$ is isomorphic to the quotient of the free frame $\mathtt{Geom}(At)/\mathtt{E}$ under $\sim_{\overline{R}}$, where $R = \{([\varphi]_\mathtt{E}, [\psi]_\mathtt{E}) \mid (\varphi \vdash \psi) \in \mathtt{T}\}$.

Example 16 ([Joh02, D1.1.7(m) and D1.4.14]). Each frame $(L, \leq_L)$ is isomorphic to the frame $\mathtt{Geom}(At)/\mathtt{T}(L)$, where $\mathtt{T}(L)$ is the following theory over $At := \{\underline{a} \mid a \in L\}$.

$$\begin{array}{rclcrcl} \underline{a} & \vdash & \underline{b} & (\text{if } a \leq_L b) & & \vdash & \underline{\top_L} \\ \underline{\bigvee_L S} & \vdash & \bigvee \{\underline{a} \mid a \in S\} & (\text{if } S \subseteq L) & \underline{a} \wedge \underline{b} & \vdash & \underline{a \wedge_L b} \end{array}$$

Hence, any frame is a quotient of a free frame by a geometric theory. It then follows from [Joh82, §II.1.5] that $\mathtt{Geom}(At)/\mathtt{T}$ may not be spatial. On the positive side, we have

Theorem 5 ([Hec15, Corollary 3.15])**.** *Let R be a* countable *binary relation on a free frame $L(At)$. Then the quotient of $L(At)$ under $\sim_{\overline{R}}$ is a spatial frame.*

Corollary 5. *Let $\mathtt{T}$ be a geometric theory over At. If $\mathtt{T}$ is countable, then the frame $\mathtt{Geom}(At)/\mathtt{T}$ is spatial. In particular, if Φ is a G_δ formula of $\mathsf{LTL}(A)$, then $\mathtt{Geom}(At)/\mathtt{T}[\![\Phi]\!]$ is spatial, where $At = \mathrm{Fin}([\![\mathsf{Str}A]\!])$.*

Hence, when Φ is a G_δ formula of LTL, deduction in $\vdash_{\mathtt{T}[\![\Phi]\!]}$ completely axiomatizes $\mathbf{Mod}(\mathtt{T}[\![\Psi]\!])$, and the frame $\mathtt{Geom}(At)/\mathtt{T}[\![\Phi]\!]$ is isomorphic to $\mathtt{Geom}(At)/\mathbf{Mod}(\mathtt{T}[\![\Phi]\!])$.

We must warn the reader on the following subtle points, which actually motivate the explicit constructions of §4.3.1.

Remark 13. Consider a space (X, Ω). It follows from Example 16 that there is a theory $\mathtt{T}_\Omega$ over $At_\Omega := \{\underline{U} \mid U \in \Omega\}$ such that Ω is isomorphic to $\mathtt{Geom}(At_\Omega)/\mathtt{T}_\Omega$ (so that $\mathtt{Geom}(At_\Omega)/\mathtt{T}_\Omega$ is spatial). But beware that when X is an algebraic dcpo, the theory $\mathtt{T} := \mathtt{T}(X)$ of §4.1 is over the set of atomic propositions $At := \mathrm{Fin}(X)$, which differs from At_Ω. In particular, the theories $\mathtt{T}$ and $\mathtt{T}_\Omega$ differ, and there is *a priori* no reason for Ω to be isomorphic to $\mathtt{Geom}(At)/\mathtt{T}$.

An algebraic dcpo X is said to be *ω-algebraic* when $\mathrm{Fin}(X)$ is (at most) countable. For instance, $[\![\mathsf{Str}A]\!]$ is ω-algebraic precisely when A is (at most) countable.

Assume X is ω-algebraic. Set $At := \mathrm{Fin}(X)$ and $\mathtt{T} := \mathtt{T}(X)$ as above. The theory $\mathtt{T}$ is countable, and Corollary 5 applies. Hence, the frame $\mathtt{Geom}(At)/\mathtt{T}$ is spatial and thus isomorphic to the frame Ω (Theorem 4, Theorem 2 and Proposition 4).

Consider now a subset $P \subseteq X$. Recall from §3.3 that the quotient frame $\Omega \twoheadrightarrow \Omega{\restriction}P$ can be represented as the frame of j-fixpoints for a nucleus j on Ω. It is then a consequence of Proposition 8 and [Hec15, §3.4] that the frame $\Omega{\restriction}P$ is isomorphic to $\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})$, where

$$\mathtt{U} \quad := \quad \{\varphi \vdash \psi \mid [\varphi]_\mathtt{T} \leq_\mathtt{T} j([\psi]_\mathtt{T})\}$$

In particular, the frame $\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})$ spatial and thus isomorphic to $\mathtt{Geom}(At)/\mathbf{Mod}_\mathtt{T}(\mathtt{U})$. But beware that this does *not* imply that the *space* $(P, \Omega{\restriction}P)$ is represented by the space of models $\mathbf{Mod}_\mathtt{T}(\mathtt{U})$, unless $(P, \Omega{\restriction}P)$ is sober, since in this case we have

$$(P, \Omega{\restriction}P) \;\cong\; \mathrm{pt}(\Omega{\restriction}P) \;\cong\; \mathrm{pt}(\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})) \;\cong\; \mathrm{pt}(\mathtt{Geom}(At)/\mathbf{Mod}_\mathtt{T}(\mathtt{U})) \;\cong\; \mathbf{Mod}_\mathtt{T}(\mathtt{U})$$

In the case of streams $[\![\mathsf{Str}A]\!]$ (with A countable), it follows that for *any* $P \subseteq [\![\mathsf{Str}A]\!]$ there is a geometric theory $\mathtt{U}$ on At which represents the *frame* $\Omega{\restriction}P$. For instance, with $P := [\![\neg\Box a]\!]$, the isomorphism $\Omega{\restriction}P \cong \mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})$ lifts to homeomorphisms

$$\mathrm{pt}(\Omega{\restriction}P) \;\cong\; \mathrm{pt}(\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})) \;\cong\; \mathrm{pt}(\mathtt{Geom}(At)/\mathbf{Mod}_\mathtt{T}(\mathtt{U})) \;\cong\; \mathbf{Mod}_\mathtt{T}(\mathtt{U})$$

But we have $(P, \Omega{\restriction}P) \not\cong \mathbf{Mod}_\mathtt{T}(\mathtt{U})$ since $(P, \Omega{\restriction}P) \not\cong \mathrm{pt}(\Omega{\restriction}P)$ as $(P, \Omega{\restriction}P)$ is not sober (Corollary 2).

6. A Specification for the Denotation of Filter

The core of this paper consists of the results presented above concerning LTL on streams. However, the long term goal of this work is to reason on input-output (negation-free) LTL properties of functions. We now briefly sketch how our results can help to handle our motivating example, namely the filter function on streams. This is mostly preliminary; we leave as future work the elaboration of a general solution.

We work with the function $[\![\mathsf{filter}]\!]$ of Remark 2. Fix a finite set A and a Scott-continuous function $p\colon [\![A]\!] \to [\![\mathsf{Bool}]\!]$. Assume that for all $a \in A$, we have $p(a) \neq \bot_{[\![\mathsf{Bool}]\!]}$. Let $\Psi = \Psi_p$ as in Example 5 and set $\Phi := \bigvee_{a \in A} a$. The specification (1) for filter leads to the following specification for $[\![\mathsf{filter}]\!]$:

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total}, \qquad x \Vdash \Box\Diamond\Psi \quad \Longrightarrow \quad ([\![\mathsf{filter}]\!]\ p\ x) \Vdash \Box\Phi \tag{2}$$

where we refrained from writing $x \Vdash \Box\Phi$ for the assumption that x is total.

We use the notations of Remark 2. In particular, $[\![\mathsf{filter}]\!]p$ is the least fixpoint of the Scott-continuous function $f_p\colon X \to X$, where $X := \mathbf{CPO}[[\![\mathsf{Str}A]\!], [\![\mathsf{Str}A]\!]]$. In symbols, we have $[\![\mathsf{filter}]\!]p = \mathsf{Y}(f_p) = \bigvee_{n\in\mathbb{N}} f_p^n(\bot_X)$.

The standard method to reason on such fixpoints is the rule of *fixpoint induction* (see e.g. [AC98, §6.2]). This rule asserts that given a subset S of a cpo X, and a morphism $f\colon X \to_{\mathbf{CPO}} X$, we have $\mathsf{Y}(f) \in S$ provided (i) $\bot_X \in S$, (ii) S is stable under sups of ω-chains, and (iii) $f(x) \in S$ whenever $x \in S$. In our case, the subset of interest is $S := \{f \mid x \text{ total and } x \Vdash \Box\Diamond\Psi \ \Rightarrow\ f(x) \Vdash \Box\Phi\}$. But fixpoint induction cannot be applied since $\bot_X \notin S$ (as $\bot_X$ takes any $x \in [\![\mathsf{Str}A]\!]$ to $\bot^\omega \not\Vdash \Box\Phi$).

We can proceed as follows, with the help of §4.3.1. Given $k, n \in \mathbb{N}$ with $k \leq n$, let

$$\begin{array}{rcl} \psi_{n,k} & := & \bigvee \left\{ \bigwedge_{1 \leq j \leq k} \bigcirc^{i_j} \mathtt{F}[\![\Psi]\!] \mid 0 \leq i_1 < \cdots < i_k < n \right\} \\ \varphi_k & := & \bigwedge_{m<k} \bigcirc^m \mathtt{F}[\![\Phi]\!] \end{array}$$

Note that $\nu(x) \in \mathbf{Mod}\,(\mathtt{T}[\![\Box\Diamond\Psi]\!])$ if, and only if, $(\forall k \in \mathbb{N})(\exists n \geq k)(\nu(x) \models \psi_{n,k})$. It follows that condition (2) can be obtained from the following.

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total},\ \forall k \in \mathbb{N},\ \forall n \geq k, \quad \nu(x) \models \psi_{n,k} \quad \Longrightarrow \quad \nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k \tag{3}$$

Condition (3) is a consequence of Lemma 13 below. The main inductive argument is encapsulated in item (1).

Lemma 13. *Write g_n for $f_p^n(\bot_X)\colon [\![\mathsf{Str}A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Str}A]\!]$. Let $x \in [\![\mathsf{Str}A]\!]$ be a total stream.*

(1) Assume $k \leq n$. If $\nu(x) \models \psi_{n,k}$, then $\nu(g_n(x)) \models \varphi_k$.

(2) Let $n, k \in \mathbb{N}$. If $\nu(g_n(x)) \models \varphi_k$, then $\nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k$.

7. Conclusion

In this paper, we conducted a semantic study of a logic LTL on a domain of streams $[\![\mathsf{Str}A]\!]$. We showed that the negation-free formulae of LTL induce sober subspaces of $[\![\mathsf{Str}A]\!]$, and that this may fail in presence of negation. We proposed an inductive translation of negation-free LTL to geometric logic. This translation reflects the semantics of LTL, and we use it to prove that the denotation of filter satisfies the specification (1).

Further Works. First, the logic LTL on $[\![\mathsf{Str}A]\!]$ deserves further studies, in particular regarding decidability and possible axiomatizations.

We think an important next step would be to propose a refinement type system in the spirit of [JR21], but for an extension of PCF with streams. More precisely, the system of [JR21] crucially relies on controlled unfoldings of (formula level) fixpoints. We think that our translation to geometric logic could provide a domain-theoretic analogue for that, yielding a system grounded on DTLF (in the form of [Abr91, §4.3] or e.g. [AC98, §10.5]). This may rely on a deduction system for either LTL or geometric logic.In any case, we expect to need an analogue of the iteration terms of [JR21], which actually could simulate (enough of) the infinitary aspects of geometric logic. Also, an important task in this direction would be to formulate sufficiently general reasoning principles for program-level fixpoints.

Further, we expect to handle alternation-free modal μ-properties[11] on (finitary) polynomial types, thus targeting a system which as a whole would be based on FPC. But polynomial types involve sums, and sums are not universal in **CPO**, in contrast with **DCPO** and with the category $\mathbf{CPO}_\perp$ of *strict* functions. We think of working with Call-By-Push-Value (CBPV) [Lev03, Lev22] for the usual adjunction between **DCPO** and $\mathbf{CPO}_\perp$. On the long run, it would be nice if this basis could extend to enriched models of CBPV, so as to handle further computational effects. Print and global store are particularly relevant, as an important trend in proving temporal properties considers programs generating streams of events. Major works in this line include [SSVH08, HC14, HL17, NUKT18, KT14, UST18, NUKT18, SU23]. In contrast with ours, these approaches are based on trace semantics of syntactic expressions rather than denotational domains.[12]

In a different direction, we think our approach based on geometric logic could extend to linear types [HJK00], for instance targeting systems like [NW03, Win04], and relying on the categorical study of [BF06].

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[11] This corresponds to "alternation depth 1" in [BW18, §2.2]. See also [BS07, §7] and [SV10].

[12] See e.g. [NUKT18, Theorem 4.1 (and Figure 6)] or [SU23, Theorem 1 (and Definition 20 from the full version)].

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A. Proofs of §2 (A Linear Temporal Logic on a Domain of Streams)

A.1. Proofs of §2.1 (Domains)

A.1.1. Proof of Remark 1

We first discuss Remark 1. Consider the cpo $[\![\mathsf{Str}A]\!]$ equipped with the isomorphism

$$[\![A]\!] \times [\![\mathsf{Str}A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Str}A]\!], \quad (a, x) \mapsto a \cdot x$$

(where $(-) \times (-)$ is equipped with the pointwise order), with inverse $x \mapsto (x(0), x{\restriction}1)$. It is thus clear that $[\![\mathsf{Str}A]\!]$ is *a* solution in **CPO** of the recursive domain equation

$$X \quad \cong \quad [\![A]\!] \times X$$

But in Remark 1, we claimed that $[\![\mathsf{Str}A]\!]$ is actually *the solution of* $X \cong [\![A]\!] \times X$ *in the usual sense.* By this, we mean the following. In presence of general recursive types, one has to solve equations

$$F(X, X) \quad \cong \quad X \tag{4}$$

where

$$F \quad : \quad \mathbf{CPO}^{\mathrm{op}} \times \mathbf{CPO} \quad \longrightarrow \quad \mathbf{CPO}$$

is a functor of mixed variance. The usual solution, as presented in e.g. [AC98, §7.1], is to replace **CPO** with the category $\mathbf{CPO}^{\mathrm{ip}}$ of *injection-projection pairs* ([AC98, Definition 7.1.8]), and to replace F with a *covariant* functor $F^{\mathrm{ip}} : \mathbf{CPO}^{\mathrm{ip}} \times \mathbf{CPO}^{\mathrm{ip}} \to \mathbf{CPO}^{\mathrm{ip}}$. Then instead of (4) one solves the following

$$F^{\mathrm{ip}}(X, X) \quad \cong \quad X \tag{5}$$

in $\mathbf{CPO}^{\mathrm{ip}}$. In turn, equation (5) is usually solved using colimits of ω-chains ([AC98, Proposition 7.1.3]). But by [AC98, Theorem 7.1.10], colimits of ω-chains in $\mathbf{CPO}^{\mathrm{ip}}$ can actually be computed from limits of ω^{op}-chains in **CPO**.

We return to the case of streams, and consider the functor

$$F \quad = \quad [\![A]\!] \times (-) \quad : \quad \mathbf{CPO} \quad \longrightarrow \quad \mathbf{CPO}$$

Then by [AC98, Definition 7.1.17 and Proof of Theorem 7.1.10], we have to consider the limit of the ω^{op}-chain

$$\mathbf{1} \xleftarrow{\;!\;} F(\mathbf{1}) \xleftarrow{F(!)} F^2(\mathbf{1}) \dashleftarrow F^n(\mathbf{1}) \xleftarrow{F^n(!)} F^{n+1}(\mathbf{1}) \dashleftarrow$$

where $\mathbf{1}$ is the terminal cpo $\{\bot\}$. Now, by [AC98, Proposition 7.1.13], the limit of the above ω^{op}-chain is given by

$$\left\{\alpha \in \textstyle\prod_{n \in \mathbb{N}} F^n(\mathbf{1}) \mid \alpha(n) = F^n(!)(\alpha(n+1))\right\}$$

It is easy to see that $[\![\mathsf{Str}A]\!]$ is isomorphic to this limit. The key is to define for each $n \in \mathbb{N}$ an isomorphism $\iota_n : [\![A]\!]^n \to F^n(\mathbf{1})$ with $\iota_0 = \mathrm{id}_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}$ and

$$\begin{array}{lrcl} \iota_{n+1} \quad : & [\![A]\!]^{n+1} & \longrightarrow & F^{n+1}(\mathbf{1}) = [\![A]\!] \times F^n(\mathbf{1}) \\ & (a_1, a_2, \dots, a_{n+1}) & \longmapsto & (a_1, \iota_n(a_2, \dots, a_{n+1})) \end{array}$$

and to observe that the following commutes

$$\begin{tikzcd} {[\![A]\!]^{n+1}} \arrow[r, "\iota_{n+1}"] \arrow[d, "(a_1,\dots,a_n,a_{n+1}) \mapsto (a_1,\dots,a_n)"'] & F^{n+1}(\mathbf{1}) \arrow[d, "F^n(!)"] \\ {[\![A]\!]^n} \arrow[r, "\iota_n"] & F^n(\mathbf{1}) \end{tikzcd}$$

A.1.2. Proof of Example 3

Let A be a set. We show that the cpo $[\![\mathsf{Str}A]\!]$ is algebraic, and that its finite elements are precisely those of finite support.

Lemma 14. *Given* $x \in [\![\mathsf{Str}A]\!]$*, the set* $\{d \mid d \textit{ of finite support and } d \leq_{[\![\mathsf{Str}A]\!]} x\}$ *is directed.*

Proof. First, the set is non-empty since e.g. $x(0) \cdot \bot^\omega$ has finite (possibly empty) support and is $\leq_{[\![\mathsf{Str}A]\!]} x$.

Let $d, d' \leq_{[\![\mathsf{Str}A]\!]} x$, with d and d' of finite support. Since $[\![A]\!]$ is flat, for all $n \in \mathrm{supp}(d) \cap \mathrm{supp}(d')$, we have $d(n) = d'(n)$. It follows that for each $n \in \mathbb{N}$, the set $\{d(n), d'(n)\}$ has a greatest element, so that $d(n) \bigvee_{[\![A]\!]} d'(n)$ exists in $[\![A]\!]$. We can thus define

$$d'' \quad : \quad n \in \mathbb{N} \quad \longmapsto \quad d(n) \bigvee\nolimits_{[\![A]\!]} d'(n)$$

Then d'' has finite support. Moreover, we have $d'' \leq_{[\![\mathsf{Str}A]\!]} x$ and $d, d' \leq_{[\![\mathsf{Str}A]\!]} d''$. ☐

Lemma 15. *Given* $x \in [\![\mathsf{Str}A]\!]$*, we have*

$$x \quad = \bigvee \left\{d \mid d \textit{ of finite support and } d \leq_{[\![\mathsf{Str}A]\!]} x\right\}$$

Proof. It is clear that x is an upper bound of $\{d \mid d \text{ of finite support and } d \leq_{[\![\mathsf{Str}A]\!]} x\}$.

Let now $y \in [\![\mathsf{Str}A]\!]$ such that $d \leq_{[\![\mathsf{Str}A]\!]} y$ for all d of finite support such that $d \leq_{[\![\mathsf{Str}A]\!]} x$. We show that $x \leq_{[\![\mathsf{Str}A]\!]} y$. Let $n \in \mathbb{N}$ and $d \coloneqq \bot^n \cdot x(n) \cdot \bot^\omega$. Then d is of finite support and $d \leq_{[\![\mathsf{Str}A]\!]} x$. Hence $d \leq_{[\![\mathsf{Str}A]\!]} y$ and $x(n) \leq_{[\![A]\!]} y(n)$. It follows that for all $n \in \mathbb{N}$ we have $x(n) \leq_{[\![A]\!]} y(n)$, and by definition we get $x \leq_{[\![\mathsf{Str}A]\!]} y$. ☐

Lemma 16. $x \in [\![\mathsf{Str}A]\!]$ *is finite if, and only if,* x *has finite support.*

Proof. Assume first that x is finite. Since

$$x \quad \leq \quad \bigvee \left\{d \mid d \text{ of finite support and } d \leq_{[\![\mathsf{Str}A]\!]} x\right\}$$

where the set in the r.-h.s. is directed, there is some d of finite support such that $d \leq_{[\![\mathsf{Str}A]\!]} x$ and $x \leq_{[\![\mathsf{Str}A]\!]} d$. Hence $x = d$ has finite support.

Conversely, assume d has finite support, and let $D \subseteq [\![\mathsf{Str}A]\!]$ be directed with $d \leq_{[\![\mathsf{Str}A]\!]} \bigvee D$. Let $n \in \mathrm{supp}(d)$. The set $D(n) = \{x(n) \mid x \in D\}$ is directed, and since $[\![A]\!]$ is flat, this set a greatest element. Hence $d(n) \leq_{[\![A]\!]} x_n(n)$ for some $x_n \in D$. Since $\mathrm{supp}(d)$ is finite and (again) since D is directed, we obtain that $d \leq_{[\![\mathsf{Str}A]\!]} x$ for some $x \in D$. ☐

A.2. Proofs of §2.2 (Linear Temporal Logic (LTL))

Lemma 17 (Remark 3)**.** *Given a complete lattice L and a (monotone) function $f \colon L \to L$, we write* $\mathrm{lfp}(f)$ *and* $\mathrm{gfp}(f)$ *for the least and the greatest fixpoint of f, respectively.*

(1) Given LTL *formulae* Φ, Ψ*, let* $H_{\Phi,\Psi} \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ *take S to* $[\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!](S))$*. Then*

$$[\![\Phi \mathbin{\mathsf{U}} \Psi]\!] \; = \; \mathrm{lfp}(H_{\Phi,\Psi}) \qquad \textit{and} \qquad [\![\Phi \mathbin{\mathsf{W}} \Psi]\!] \; = \; \mathrm{gfp}(H_{\Phi,\Psi})$$

(2) Given a complete atomic Boolean algebra and a monotone $f \colon B \to B$, we have

$$\mathrm{lfp}(f) \; = \; \neg\mathrm{gfp}(b \mapsto \neg f(\neg b)) \qquad \textit{and} \qquad \mathrm{gfp}(f) \; = \; \neg\mathrm{lfp}(b \mapsto \neg f(\neg b))$$

Proof.

(1) The proof mimics that of [BK08, Lemmas 5.18 and 5.19]. First note that we have

$$\Phi \mathsf{W} \Psi \quad\equiv\quad (\Phi \mathsf{U} \Psi) \vee \Box\Phi$$

and $[\![\Phi \mathsf{U} \Psi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$.

We show that $[\![\Phi \mathsf{U} \Psi]\!]$ and $[\![\Phi \mathsf{W} \Psi]\!]$ are indeed fixpoints of $H_{\Phi,\Psi}$, that is

$$\begin{array}{rcl} [\![\Phi \mathsf{U} \Psi]\!] & = & [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{U} \Psi]\!]) \\ [\![\Phi \mathsf{W} \Psi]\!] & = & [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{W} \Psi]\!]) \end{array}$$

The case of $[\![\Phi \mathsf{U} \Psi]\!]$ is a trivial unfolding of the definition: given $x \in [\![\mathsf{Str}A]\!]$, we have

$$\begin{array}{rll} x \in [\![\Phi \mathsf{U} \Psi]\!] & \text{iff} & \text{there is } i \geq 0 \text{ with } x\restriction 0, \dots, x\restriction(i-1) \in [\![\Phi]\!] \text{ and } x\restriction i \in [\![\Psi]\!] \\ & \text{iff} & x \in [\![\Psi]\!] \text{ or } (\text{there is } i \geq 1 \text{ with } x\restriction 0, \dots, x\restriction(i-1) \in [\![\Phi]\!] \text{ and } x\restriction i \in [\![\Psi]\!]) \\ & \text{iff} & x \in [\![\Psi]\!] \text{ or } (x \in [\![\Phi]\!] \text{ and there is } i \geq 1 \text{ with } x\restriction 1, \dots, x\restriction(i-1) \in [\![\Phi]\!] \text{ and } x\restriction i \in [\![\Psi]\!]) \\ & \text{iff} & x \in [\![\Psi]\!] \text{ or } (x \in [\![\Phi]\!] \text{ and } x\restriction 1 \in [\![\Phi \mathsf{U} \Psi]\!]) \\ & \text{iff} & x \in [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{U} \Psi]\!]) \end{array}$$

We now turn to $[\![\Phi \mathsf{W} \Psi]\!]$. Since $[\![\Phi \mathsf{U} \Psi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$, and since $[\![\bigcirc]\!]$ is monotone (for inclusion), we have

$$[\![\Phi \mathsf{U} \Psi]\!] \quad\subseteq\quad [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{W} \Psi]\!])$$

Let $x \in [\![\Phi \mathsf{W} \Psi]\!] \setminus [\![\Phi \mathsf{U} \Psi]\!]$, i.e. $x \in [\![\Box\Phi]\!]$. Since $[\![\Box\Phi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$, we get

$$x \quad\in\quad [\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Box\Phi]\!] \quad\subseteq\quad [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{W} \Psi]\!])$$

It follows that

$$[\![\Phi \mathsf{W} \Psi]\!] \quad\subseteq\quad [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{W} \Psi]\!])$$

For the converse inclusion, if $x \in [\![\Psi]\!]$, then we have $x \in [\![\Phi \mathsf{U} \Psi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$. Assume now $x \in [\![\Phi]\!] \cap [\![\bigcirc]\!][\![\Phi \mathsf{W} \Psi]\!]$. If $x \in [\![\bigcirc]\!][\![\Phi \mathsf{U} \Psi]\!]$, then we obtain $x \in [\![\Phi \mathsf{U} \Psi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$. Otherwise, we have $x \in [\![\bigcirc]\!][\![\Box\Phi]\!]$, and since $x \in [\![\Phi]\!]$, this gives $x \in [\![\Box\Phi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$.

Hence $[\![\Phi \mathsf{U} \Psi]\!]$ and $[\![\Phi \mathsf{W} \Psi]\!]$ are indeed fixpoints of $H_{\Phi,\Psi}$.

We show that $[\![\Phi \mathsf{U} \Psi]\!]$ is the *least* fixpoint of $H_{\Phi,\Psi}$. Let $P \in \mathcal{P}([\![\mathsf{Str}A]\!])$ such that $P = H_{\Phi,\Psi}(P)$. We have to show that $[\![\Phi \mathsf{U} \Psi]\!] \subseteq P$. But if $x \in [\![\Phi \mathsf{U} \Psi]\!]$, there is some $i \geq 0$ such that $x\restriction 0, \dots, x\restriction(i-1) \in [\![\Phi]\!]$ and $x\restriction i \in [\![\Psi]\!]$. Since P is a fixpoint of $H_{\Phi,\Psi}$, we get $x\restriction i \in P$. Again since P is a fixpoint of $H_{\Phi,\Psi}$, we obtain $x\restriction(i-1), \dots, x\restriction 0 \in P$. Hence $x = x\restriction 0 \in P$ and we are done.

It remains to show that $[\![\Phi \mathsf{W} \Psi]\!]$ is the greatest fixpoint of $H_{\Phi,\Psi}$. Let $Q \in \mathcal{P}([\![\mathsf{Str}A]\!])$ such that

$$Q \quad=\quad [\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!]Q)$$

We have to show that $Q \subseteq [\![\Phi \mathsf{W} \Psi]\!]$. Let $x \in Q$. If $x \in [\![\Box\Phi]\!]$ then we are done. Otherwise, there is a *least* $i \geq 0$ such that $x\restriction i \notin [\![\Phi]\!]$. Hence $x\restriction 0, \dots x\restriction(i-1) \in [\![\Phi]\!]$. Since $x \in Q = H_{\Phi,\Psi}(Q)$, it follows that $x\restriction 1, \dots x\restriction i \in Q$. Hence $x\restriction i \in [\![\Psi]\!]$ since $x\restriction i \notin [\![\Phi]\!]$. It follows that $x \in [\![\Phi \mathsf{U} \Psi]\!] \subseteq [\![\Phi \mathsf{W} \Psi]\!]$.

(2) Let $f \colon B \to B$ be monotone with B a complete atomic Boolean algebra. By the Knaster-Tarski Fixpoint Theorem (see e.g. [DP02, 2.35]), we have

$$\begin{array}{rcl} \mathrm{lfp}(f) & = & \bigvee\{a \in B \mid f(a) \leq a\} \\ \mathrm{gfp}(f) & = & \bigwedge\{a \in B \mid a \leq f(a)\} \end{array}$$

Let g be the monotone function $B \to B$ which takes a to $\neg f(\neg a)$. We show

$$\mathrm{gfp}(f) \quad = \quad \neg \mathrm{lfp}(g)$$

The other equation is then obtained by duality. Since B is complete and atomic, it follows from e.g. [DP02, Theorem 10.24] that $\neg \bigvee S = \bigwedge \{\neg s \mid s \in S\}$ for every $S \subseteq B$. We then compute

$$\begin{array}{rcl}
\neg \mathrm{lfp}(g) & = & \neg \bigvee \{a \mid g(a) \leq a\} \\
& = & \bigwedge \{\neg a \mid g(a) \leq a\} \\
& = & \bigwedge \{\neg a \mid \neg f(\neg a) \leq a\} \\
& = & \bigwedge \{\neg a \mid \neg a \leq f(\neg a)\} \\
& = & \bigwedge \{b \mid b \leq f(b)\} \\
& = & \mathrm{gfp}(f)
\end{array}$$

□

Lemma 18 (Lemma 1)**.** *If Φ is negation-free then $[\![\Phi]\!]$ is upward-closed in $[\![\mathsf{Str}A]\!]$ (if $x \in [\![\Phi]\!]$ and $x \leq_{[\![\mathsf{Str}A]\!]} y$ then $y \in [\![\Phi]\!]$).*

Proof. We reason by induction Φ. Since upward-closed sets are stable under (arbitrary) unions and intersections, we just have to consider the cases of atomic formulae $a \in A$ and of modalities.

Concerning atomic formulae, if $x \in [\![a]\!]$, then $x(0) = a$. Since $[\![A]\!]$ is a flat cpo, we have $y(0) = a$ for all $y \geq_{[\![\mathsf{Str}A]\!]} x$. Hence $[\![a]\!]$ is upward-closed.

The cases of modalities $\bigcirc\Phi$, $(\Phi \mathbin{\mathsf{U}} \Psi)$ and $(\Phi \mathbin{\mathsf{W}} \Psi)$ follow from the induction hypothesis and the fact that $x \leq_{[\![\mathsf{Str}A]\!]} y$ implies $x{\upharpoonright}i \leq_{[\![\mathsf{Str}A]\!]} y{\upharpoonright}i$ for all $i \in \mathbb{N}$. □

B. Proofs of §3 (The Topological Approach)

B.1. Proofs of §3.3 (Sobriety of Subspaces)

Let (X, Ω) be a topological space, and fix $P \subseteq X$. Write $\iota \colon (P, \Omega{\upharpoonright}P) \hookrightarrow (X, \Omega)$ for the subspace inclusion, and let $\iota^\star = \Omega(\iota) \colon \Omega \twoheadrightarrow \Omega{\upharpoonright}P$ be the induced surjective frame morphism.

The following simple observation is used repeatedly below.

Lemma 19. *Given $x \in P$, we have $\mathcal{F}_x = \mathcal{F}_x^{\Omega\upharpoonright P} \circ \iota^\star$ where $\mathcal{F}_x^{\Omega\upharpoonright P} := \{(U \cap P) \in \Omega{\upharpoonright}P \mid x \in U \cap P\}$.*

Proof. Given $U \in \Omega$, if $U \in \mathcal{F}$ then $x \in U$, so that $x \in U \cap P$ and $\iota^\star(U) = (U \cap P) \in \mathcal{F}_x^{\Omega\upharpoonright P}$. Conversely, if $\iota^\star(U) \in \mathcal{F}_x^{\Omega\upharpoonright P}$, then in particular $x \in U$ so that $U \in \mathcal{F}_x$. □

Lemma 20 (Lemma 2)**.** *Assume that (X, Ω) is sober. Then the following are equivalent.*

(i) $(P, \Omega{\upharpoonright}P)$ is sober.

(ii) For each $x \in X$, we have $x \in P$ if, and only if, there is some $\mathcal{G} \in \mathrm{pt}(\Omega{\upharpoonright}P)$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$.

$$\begin{tikzcd}
\Omega \arrow[rr, two heads, "\iota^\star"] \arrow[dr, "\mathcal{F}_x"'] & & \Omega{\upharpoonright}P \arrow[dl, dashed, "\mathcal{G}"] \\
& \mathbf{2} &
\end{tikzcd}$$

Proof. Let (X, Ω) be sober, and fix some $P \subseteq X$. In view of Lemma 19, we just have to show that the following are equivalent.

(i) The space $(P, \Omega{\upharpoonright}P)$ is sober.

(ii) For each $x \in X$, we have $x \in P$ whenever there is some $\mathcal{G} \in \mathrm{pt}(\Omega{\upharpoonright}P)$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$.

We discuss each implication separately.

(i) $\Rightarrow$ (ii). Assume that $(P, \Omega{\restriction}P)$ is sober, and let $x \in X$. Let $\mathcal{G} \in \mathrm{pt}(\Omega{\restriction}P)$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$. We have to show that $x \in P$. Since $(P, \Omega{\restriction}P)$ is sober, we have $\mathcal{G} = \mathcal{F}_y^{\Omega{\restriction}P}$ for some unique $y \in P$. But we have seen above that $\mathcal{F}_y = \mathcal{F}_y^{\Omega{\restriction}P} \circ \iota^\star$. Hence $\mathcal{F}_x = \mathcal{F}_y$ and $x = y$ since X is sober.

(ii) $\Rightarrow$ (i). Assume condition (ii) and let $\mathcal{G} \in \mathrm{pt}(\Omega{\restriction}P)$. We have to show that $\mathcal{G} = \mathcal{F}_x^{\Omega{\restriction}P}$ for a unique $x \in P$.

Since X is sober, there is some $x \in X$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$. Condition (ii) implies that $x \in P$, so that $\mathcal{F}_x = \mathcal{F}_x^{\Omega{\restriction}P} \circ \iota^\star$. But since $\iota^\star$ is a surjective frame morphism, it is in particular an epimorphism in **Frm**. Hence $\mathcal{G} \circ \iota^\star = \mathcal{F}_x^{\Omega{\restriction}P} \circ \iota^\star$ implies $\mathcal{G} = \mathcal{F}_x^{\Omega{\restriction}P}$.

Moreover, if $\mathcal{G} = \mathcal{F}_y^{\Omega{\restriction}P}$ for some $y \in P$, then since $\mathcal{F}_y = \mathcal{F}_y^{\Omega{\restriction}P} \circ \iota^\star$, we have $\mathcal{F}_x = \mathcal{F}_y$. Hence $y = x$ since X is sober. ☐

Write $j \colon \Omega \to \Omega$ for the nucleus induced by the surjective frame morphism $\iota^\star \colon \Omega \twoheadrightarrow \Omega{\restriction}P$, and write $\widetilde{P}$ for the frame of j-fixpoints ($\widetilde{P} = \{U \in \Omega \mid j(U) = U\}$).

Lemma 21 (Remark 5)**.** *Given an open $U \in \Omega$ of X, we have*

$$j(U) \quad = \quad \bigcup \{V \in \Omega \mid V \cap P = U \cap P\}$$

Proof. Fix $U \in \Omega$. By definition, we have

$$\begin{array}{rcl} j(U) & = & \iota_\star(\iota^\star(U)) \\ & = & \bigcup \{V \in \Omega \mid \iota^\star(V) \subseteq \iota^\star(U)\} \\ & = & \bigcup \{V \in \Omega \mid V \cap P \subseteq U \cap P\} \end{array}$$

Hence, if $V \in \Omega$ is such that $V \cap P = U \cap P$, then in particular $V \cap P \subseteq U \cap P$ and thus $V \subseteq j(U)$. It follows that $\bigcup \{V \in \Omega \mid V \cap P = U \cap P\} \subseteq j(U)$.

It remains to show that $j(U) \subseteq \bigcup \{V \in \Omega \mid V \cap P = U \cap P\}$. Since $j(U) \in \Omega$, we are done if we show that $j(U) \cap P = U \cap P$. But we have $j(U) \cap P = \bigcup \{V \cap P \mid V \cap P \subseteq U \cap P\}$ so that $j(U) \cap P \subseteq U \cap P$. Since $U \subseteq j(U)$, we obtain $j(U) \cap P = U \cap P$, as required. ☐

Recall that given $x \in \Omega$, we set $\widetilde{x} = X \setminus \overline{\{x\}} = X \setminus \downarrow x$, where $y \in \downarrow x$ iff $y \leq_\Omega x$, with $\leq_\Omega$ the specialization (pre)order of (X, Ω).

Lemma 22 (Remark 6)**.** *Given $x \in X$ and $U \in \Omega$, we have $U \subseteq \widetilde{x}$ if, and only if, $x \notin U$.*

Proof. If $U \subseteq \widetilde{x}$, then we obviously have $x \notin U$ since $x \notin \widetilde{x}$. Assume conversely that $U \not\subseteq \widetilde{x}$. Hence there is some $y \in U$ such that $y \notin \widetilde{x}$. This implies $y \leq_\Omega x$ with $y \in U$, so that $x \in U$. ☐

Lemma 23 (Lemma 4)**.** *Given $x \in X$, let $\widetilde{\mathcal{F}}_x \coloneqq \{U \in \widetilde{P} \mid x \in U\}$. Then $\widetilde{\mathcal{F}}_x \in \mathrm{pt}(\widetilde{P})$ if, and only if, $\widetilde{x} \in \widetilde{P}$.*

Proof. Assume first that $\widetilde{x} \in \widetilde{P}$. It is clear that $\widetilde{\mathcal{F}}_x$ is upward-closed and stable under finite intersections. Let $S \subseteq \widetilde{P}$ such that for all $U \in S$, we have $U \subseteq \widetilde{x}$. Then $\bigcup S \subseteq \widetilde{x}$ and $j(\bigcup S) \subseteq j(\widetilde{x}) = \widetilde{x}$ (since $\widetilde{x} \in \widetilde{P}$). Hence, if $x \in j(\bigcup S)$, we have $j(\bigcup S) \not\subseteq \widetilde{x}$, and there is some $U \in S$ such that $U \not\subseteq \widetilde{x}$, i.e. $x \in U$.

Assume conversely that $\widetilde{\mathcal{F}}_x \in \mathrm{pt}(\widetilde{P})$. Let $S \coloneqq \{V \in \widetilde{P} \mid V \subseteq \widetilde{x}\}$. Then for all $V \in S$, we have $x \notin V$ and thus $V \notin \widetilde{\mathcal{F}}_x$. Hence $j(\bigcup S) \notin \widetilde{\mathcal{F}}_x$, so that $x \notin j(\bigcup S)$ and $j(\bigcup S) \subseteq \widetilde{x}$. But this implies $j(\widetilde{x}) \subseteq \widetilde{x}$ since $\bigcup S = \widetilde{x}$. ☐

It is well-known that $x \in P$ implies $\widetilde{x} \in \widetilde{P}$ ([PP12, Remark VI.1.3.1]).

Proposition 9 (Proposition 2)**.** *Assume that (X, Ω) is sober. Then the following are equivalent.*

(i) $(P, \Omega \restriction P)$ *is sober.*

(ii) For each $x \in X$*, we have* $x \in P$ *if, and only if,* $\widetilde{x} \in \widetilde{P}$*.*

Proof. We prove each implication separately.

(ii) $\Rightarrow$ **(i).** We show that $(P, \Omega \restriction P)$ is sober whenever $\widetilde{x} \in \widetilde{P}$ implies $x \in P$ for all $x \in X$. By Lemma 19 and Lemma 20 (i.e. Lemma 2), it is sufficient to show that given $x \in X$, we have $x \in P$ whenever there is some $\mathcal{G} \in \mathrm{pt}(\Omega \restriction P)$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$.

Assume that for all $x \in X$, we have $x \in P$ whenever $\widetilde{x} \in \widetilde{P}$. Let $x \in X$ such that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$ for some $\mathcal{G} \in \mathrm{pt}(\Omega \restriction P)$. We have to show $x \in P$. Assume toward a contradiction that $x \notin P$. By our assumption, this implies that $\widetilde{x} \notin \widetilde{P}$. Hence there is some $y \in j(\widetilde{x}) \setminus \widetilde{x}$. In turn, there is some $V \in \Omega$ such that $y \in V$ and $V \cap P = \widetilde{x} \cap P$. Since $y \notin \widetilde{x} = X \setminus \downarrow x$, we have $y \leq_\Omega x$, and since $y \in V$, we get $x \in V$. It follows that $V \in \mathcal{F}_x$, so that $\iota^\star(V) = (V \cap P) \in \mathcal{G}$. But since $V \cap P = \widetilde{x} \cap P = \iota^\star(\widetilde{x})$, we thus get $\widetilde{x} \in \mathcal{F}_x$, a contradiction since $x \notin \widetilde{x}$.

(i) $\Rightarrow$ **(ii).** Assume that $(P, \Omega \restriction P)$ is sober, and let $x \in X$ such that $\widetilde{x} \in \widetilde{P}$. We have to show that $x \in P$. We apply Lemma 20 (i.e. Lemma 2). Actually, we are going to construct a point $\widetilde{\mathcal{F}} \in \mathrm{pt}(\widetilde{P})$ such that $\mathcal{F}_x = \widetilde{\mathcal{F}} \circ j$. Since $j = \iota_\star \circ \iota^\star$ (where $\iota_\star$ is the upper adjoint of $\iota^\star$), Lemma 3 then gives the result.

Let $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_x$ be the set of all $U \in \widetilde{P}$ such that $x \in U$. Since $\widetilde{x} \in \widetilde{P}$, we have $\widetilde{\mathcal{F}}_x \in \mathrm{pt}(\widetilde{P})$ by Lemma 23 (i.e. Lemma 4). In order to obtain $x \in P$, it thus remains to show that

$$\mathcal{F}_x \quad = \quad \widetilde{\mathcal{F}}_x \circ j$$

But given $U \in \mathcal{F}_x$, we have $x \in U \subseteq j(U)$, so that $j(U) \in \widetilde{\mathcal{F}}_x$. Conversely, let $U \in \Omega$ such that $U \notin \mathcal{F}_x$. We have $x \notin U$ and thus $U \subseteq \widetilde{x}$. But then $j(U) \subseteq j(\widetilde{x}) = \widetilde{x}$. Hence $x \notin j(U)$ and $j(U) \notin \widetilde{\mathcal{F}}_x$. $\square$

Lemma 24 (Lemma 5)**.** *Assume that* (X, Ω) *is sober and let* $x \in X$ *such that for all* $U \in \Omega$ *with* $x \in U$*, we have* $U \setminus \{x\} \notin \Omega$*. Set* $P \coloneqq X \setminus \{x\}$*. Then* $(P, \Omega \restriction P)$ *is not sober.*

Proof. Let $\iota \colon (P, \Omega \restriction P) \hookrightarrow (X, \Omega)$ be the subspace inclusion. Since $x \notin P$, by By Lemma 19 and Lemma 20 (i.e. Lemma 2), it is sufficient to show that $\mathcal{F}_x = \mathcal{G} \circ \iota^\star$ for some $\mathcal{G} \in \mathrm{pt}(\Omega \restriction P)$. We appeal to Lemma 3, and instead provide a $\widetilde{\mathcal{F}} \in \mathrm{pt}(\Omega \restriction P)$ such that $\mathcal{F}_x = \widetilde{\mathcal{F}} \circ j$ where $j \colon \Omega \to \Omega$ is the nucleus induced by ι. Let $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_x$ be the set of all $U \in \widetilde{P}$ such that $x \in U$.

Note that for each $U \in \Omega$, we have

$$j(U) \quad = \quad \bigcup \{ V \in \Omega \mid V \setminus \{x\} = U \setminus \{x\} \}$$

We first show that $\widetilde{x} \in \widetilde{P}$. This amounts to showing that $j(\widetilde{x}) \subseteq \widetilde{x}$, i.e. that $x \notin j(\widetilde{x})$. Assume toward a contradiction that $x \in j(\widetilde{x})$. Hence there is some $V \in \Omega$ such that $x \in V$ and $V \setminus \{x\} = (X \setminus \overline{\{x\}}) \setminus \{x\}$. But $(X \setminus \overline{\{x\}}) \setminus \{x\} = X \setminus \overline{\{x\}}$ is open, while $V \setminus \{x\}$ is not, a contradiction.

We thus get $\widetilde{\mathcal{F}}_x \in \mathrm{pt}(\widetilde{P})$ by Lemma 23 (i.e. Lemma 4). We are left with proving

$$\mathcal{F}_x \quad = \quad \widetilde{\mathcal{F}}_x \circ j$$

Given $U \in \mathcal{F}_x$, we have $x \in U \subseteq j(U)$ and thus $j(U) \in \widetilde{\mathcal{F}}_x$. Conversely, let $U \in \Omega$ such that $x \in j(U)$. Hence there is some $V \in \Omega$ such that $x \in V$ and $V \setminus \{x\} = U \setminus \{x\}$. Since $(V \setminus \{x\}) \notin \Omega$, we have $(U \setminus \{x\}) \notin \Omega$. Hence $x \in U$ and $U \in \mathcal{F}_x$. $\square$

C. Proofs of §4 (Geometric Logic)

C.1. Proofs of §4.1 (Geometric Theories)

Lemma 25 (Remark 7)**.**

(1) Given $(\varphi_i \mid i \in I)$ with $\varphi_i = \bigvee \{\gamma_{i,j} \mid j \in J_i\}$, let $\bigvee_{i\in I} \varphi_i \coloneqq \bigvee \{\gamma_{i,j} \mid i \in I \text{ and } j \in J_i\}$. Then

$$\nu \models \bigvee_{i\in I} \varphi_i \quad \text{iff} \quad \text{there exists } i \in I \text{ such that } \nu \models \varphi_i$$

(2) Given $\varphi = \bigvee_{i\in I} \gamma_i$ and $\psi = \bigvee_{j\in J} \gamma'_j$, let $\varphi \wedge \psi \coloneqq \bigvee_{(i,j)\in I\times J} \gamma_i \wedge \gamma'_j$. Then

$$\nu \models \varphi \wedge \psi \quad \text{iff} \quad \nu \models \varphi \text{ and } \nu \models \psi$$

Proof.

(1) Since

$$\begin{array}{rll} \nu \models \bigvee_{i\in I} \varphi_i & \text{iff} & \text{there are } i \in I \text{ and } j \in J_i \text{ such that } \nu \models \gamma_{i,j} \\ & \text{iff} & \text{there is } i \in I \text{ such that } \nu \models \varphi_i \end{array}$$

(2) Since

$$\begin{array}{rll} \nu \models \varphi \wedge \psi & \text{iff} & \text{there is } (i,j) \in I \times J \text{ such that } \nu \models \gamma_i \wedge \gamma'_j \\ & \text{iff} & \text{there is } (i,j) \in I \times J \text{ such that } \nu \models \gamma_i \text{ and } \nu \models \gamma'_i \\ & \text{iff} & (\text{there is } i \in I \text{ such that } \nu \models \gamma_i) \text{ and } (\text{there is } j \in J \text{ such that } \nu \models \gamma'_j) \\ & \text{iff} & \nu \models \varphi \text{ and } \nu \models \psi \end{array}$$

□

C.2. Proofs of §4.2 (The Sober Space of Models)

Let $X = (X, \leq_X)$ be an algebraic dcpo. Recall the geometric theory $\mathtt{T}(X)$ over $At = \mathrm{Fin}(X)$, namely

$$d \vdash d' \quad (\text{if } d' \leq_X d) \qquad \vdash \bigvee \mathrm{Fin}(X) \qquad d \wedge d' \vdash \bigvee \{d'' \in \mathrm{Fin}(X) \mid d \leq_X d'' \text{ and } d' \leq_X d''\}$$

where $d, d' \in \mathrm{Fin}(X)$.

Recall also that given $x \in X$, $\nu(x)\colon At \to \mathbf{2}$ is the characteristic function of $\{d \in \mathrm{Fin}(X) \mid d \leq_X x\}$.

Proposition 10 (Proposition 4)**.** *Let X be an algebraic dcpo. The bijection $x \mapsto \nu(x)$ of Proposition 3 extends to an homeomorphism from X to $\mathbf{Mod}(\mathtt{T}(X))$.*

Proof. By [AC98, Proposition 1.21(1)], the finite elements of $\mathrm{Idl}(\mathrm{Fin}(X))$ are the principal ideals (i.e. those of the form $\downarrow_{\mathrm{Fin}(X)} d = \{d' \in \mathrm{Fin}(X) \mid d' \leq_X d\}$). Hence, the order-isomorphism $X \cong \mathrm{Idl}(\mathrm{Fin}(X))$ of Lemma 6 takes a basic Scott-open $\uparrow d \in \Omega(X)$ to the basic Scott-open $\Uparrow(\downarrow_{\mathrm{Fin}(X)} d) \in \Omega(\mathrm{Idl}(\mathrm{Fin}(X)))$, where

$$\Uparrow \left(\downarrow_{\mathrm{Fin}(X)} d\right) \quad = \quad \left\{J \in \mathrm{Idl}(\mathrm{Fin}(X)) \mid \downarrow_{\mathrm{Fin}(X)} d \subseteq J\right\}$$

Now, writing $\nu\colon At \to \mathbf{2}$ for the characteristic function of $J \in \mathrm{Idl}(\mathrm{Fin}(X))$, we have $\downarrow_{\mathrm{Fin}(X)} d \subseteq J$ if, and only if, $\nu \models d$. In other words, under Lemma 6, the basic opens of X correspond exactly to the $\mathsf{mod}_{\mathbf{Mod}(\mathtt{T}(X))}(d)$, for d an atomic proposition over $At = \mathrm{Fin}(X)$. This directly extends to Scott-opens $U \in \Omega(X)$ on the one hand, and opens $\mathsf{mod}_{\mathbf{Mod}(\mathtt{T}(X))}(\varphi) \in \Omega(\mathbf{Mod}(\mathtt{T}(X)))$ on the other. □

C.2.1. Proof of Theorem 2

We shall now prove Theorem 2.

Theorem 6 (Theorem 2)**.** *Let* $\mathtt{T}$ *be a geometric theory over* At*. The function taking* $\nu \in \mathbf{Mod}(\mathtt{T})$ *to* $\{[\varphi]_{\mathbf{Mod}(\mathtt{T})} \mid \nu \models \varphi\}$ *is an homeomorphism from* $\mathbf{Mod}(\mathtt{T})$ *to* $\mathrm{pt}(\mathtt{Geom}(At)/\mathbf{Mod}(\mathtt{T}))$.

Fix a geometric theory $\mathtt{T}$ over At. For notational simplicity, we let $M := \mathbf{Mod}(\mathtt{T})$ and $L := \mathtt{Geom}(At)/M$. But beware that the proof of Theorem 6 (i.e. Theorem 2) relies on the theory $\mathtt{T}$.

Define

$$\begin{array}{rcrcl} f & : & (At \to \mathbf{2}) & \longrightarrow & \mathcal{P}(L) \\ & & \nu & \longmapsto & \{[\varphi]_M \mid \nu \models \varphi\} \end{array}$$

Note that f is injective since $f(\nu) = f(\nu')$ implies that for all $p \in At$ we have $\nu \models p$ if, and only if, $\nu' \models p$.

Lemma 26. *Given* $\nu \colon At \to \mathbf{2}$*, we have* $\nu \in M$ *if, and only if,* $f(\nu) \in \mathrm{pt}(L)$*.*

Proof. Assume first that $\nu \in M$. We show that $f(\nu) \in \mathrm{pt}(L)$. First, if $[\varphi]_M \leq_M [\psi]_M$ and $[\varphi]_M \in f(\nu)$, then $\nu \models \varphi$. Hence $\nu \models \psi$ since $\mathsf{mod}_M(\varphi) \subseteq \mathsf{mod}_M(\psi)$ and $\nu \in M$. It follows that $[\varphi]_M \in f(\nu)$. Second, assume $[\varphi]_M, [\varphi]_M \in f(\nu)$. Then $\nu \models \varphi$ and $\nu \models \psi$. It follows that $\nu \models \varphi \wedge \psi$, and by Lemma 7 we get $[\varphi]_M \wedge [\psi]_M \in f(\nu)$. Finally, assume $\bigvee_{i \in I}[\varphi_i]_M \in f(\nu)$. By Lemma 7 we get that $\nu \models \bigvee_{i \in I} \varphi_i$. Hence for some $i \in I$ we have $\nu \models \varphi_i$, and thus $[\varphi_i]_M \in f(\nu)$.

Conversely, assume that $f(\nu) \in \mathrm{pt}(L)$. We show that $\nu \in M$. Let $\varphi \vdash \psi$ be a sequent of $\mathtt{T}$. Since $\varphi \preceq_M \psi$, we have $[\psi]_M \in f(\nu)$ whenever $[\varphi]_M \in f(\nu)$. Hence, we have $\nu \models \psi$ whenever $\nu \models \varphi$. ☐

Given $\mathcal{F} \in \mathrm{pt}(L)$, let $\nu_{\mathcal{F}}$ be the valuation which takes $p \in At$ to 1 iff $[p]_M \in \mathcal{F}$.

Lemma 27. *For each* $\varphi \in \mathtt{Geom}(At)$*, we have* $[\varphi]_M \in \mathcal{F}$ *if, and only if,* $\nu_{\mathcal{F}} \models \varphi$*.*

Proof. We first show by induction on $\gamma \in \mathtt{Conj}(At)$ that $\nu_{\mathcal{F}} \models \gamma$ exactly when $[\bigvee\{\gamma\}]_M \in \mathcal{F}$ (thus making explicit that $\bigvee\{\gamma\}$ is the conjunctive formula $\gamma \in \mathtt{Conj}(At)$ seen as a geometric formula).

Case of $p \in At$. Since by definition of $\nu_{\mathcal{F}}$, we have $\nu_{\mathcal{F}} \models p$ if, and only if, $[p]_M = [\bigvee\{p\}]_M \in \mathcal{F}$.

Case of $\mathtt{true}$. On the one hand, we have $\nu_{\mathcal{F}} \models \mathtt{true}$. On the other hand, we have $[\bigvee\{\mathtt{true}\}]_M \in \mathcal{F}$ by Lemma 7.

Case of $\gamma_1 \wedge \gamma_2$. We have $\nu_{\mathcal{F}} \models \gamma_1 \wedge \gamma_2$ iff $\nu_{\mathcal{F}} \models \gamma_1$ and $\nu_{\mathcal{F}} \models \gamma_2$. By induction hypothesis, for $i = 1, 2$ we have $\nu_{\mathcal{F}} \models \gamma_i$ iff $[\bigvee\{\gamma_i\}]_M \in \mathcal{F}$. On the other hand, by Lemma 7 we have

$$[\textstyle\bigvee\{\gamma\}]_M \wedge [\bigvee\{\gamma'\}]_M \quad = \quad [\bigvee\{\gamma \wedge \gamma'\}]_M$$

Hence $[\bigvee\{\gamma \wedge \gamma'\}]_M \in \mathcal{F}$ if, and only if, $[\bigvee\{\gamma\}]_M, [\bigvee\{\gamma'\}]_M \in \mathcal{F}$. It follows that $\nu_{\mathcal{F}} \models \gamma_1 \wedge \gamma_2$ iff $[\bigvee\{\gamma \wedge \gamma'\}]_M \in \mathcal{F}$.

We now consider the case of $\varphi = \bigvee_{i \in I} \gamma_i$. Note that for *every* ν of At, we have

$$\begin{array}{rll} \nu \models \varphi & \text{iff} & \text{there is } i \in I \text{ such that } \nu \models \gamma_i \\ & \text{iff} & \text{there is } i \in I \text{ such that } \nu \models \bigvee\{\gamma_i\} \\ & \text{iff} & \nu \models \bigvee_{i \in I} \bigvee\{\gamma_i\} \end{array}$$

where $\bigvee_{i \in I} \bigvee\{\gamma_i\}$ is the operation on *geometric* formulae of Lemma 25 (i.e. Remark 7). Hence by Lemma 7 we have

$$[\varphi]_M \quad = \quad \textstyle\bigvee_{i \in I}[\bigvee\{\gamma_i\}]_M$$

We finally conclude with the following, which relies on the above inductive property on conjunctive formulae:

$$\begin{array}{lll} \nu_{\mathcal{F}} \models \varphi & \text{iff} & \text{there is } i \in I \text{ such that } \nu_{\mathcal{F}} \models \gamma_i \\ & \text{iff} & \text{there is } i \in I \text{ such that } [\bigvee\{\gamma_i\}]_M \in \mathcal{F} \\ & \text{iff} & [\varphi]_M \in \mathcal{F} \end{array}$$

□

Hence, given $\mathcal{F} \in \mathrm{pt}(L)$ we have

$$\begin{array}{rcl} f(\nu_{\mathcal{F}}) & = & \{[\varphi]_M \mid \nu_{\mathcal{F}} \models \varphi\} \\ & = & \{[\varphi]_M \mid [\varphi]_M \in \mathcal{F}\} \\ & = & \mathcal{F} \end{array}$$

It follows that we have a bijection

$$\begin{array}{rcccl} f & : & M & \longrightarrow & \mathrm{pt}(L) \\ & & \nu & \longmapsto & \{[\varphi]_M \mid \nu \models \varphi\} \end{array}$$

We can now prove Theorem 6 (i.e. Theorem 2).

Proof Theorem 6. It remains to prove that $f\colon M \to \mathrm{pt}(\mathtt{Geom}(At)/M)$ is an homeomorphism.

Write g for the frame isomorphism $(\mathtt{Geom}(At)/M, \leq_M) \to (\Omega(M), \subseteq)$ induced by $\mathsf{mod}_M\colon \mathtt{Geom}(At) \to \Omega(M)$. Note that

$$g([\varphi]_M) \quad = \quad \mathsf{mod}_M(\varphi) \quad = \quad \{\nu \in M \mid \nu \models \varphi\}$$

Recall from §3.2 the unit at $(M, \Omega(M))$ of the adjunction $\Omega \dashv \mathrm{pt}$, namely

$$\begin{array}{rcccl} \eta_M & : & M & \longrightarrow & \mathrm{pt}(\Omega(M)) \\ & & \nu & \longmapsto & \{U \in \Omega(M) \mid \nu \in U\} \end{array}$$

Since g is an isomorphism, we have

$$\begin{array}{rcl} \eta_M(\nu) & = & \{g([\varphi]_M) \mid \nu \in g([\varphi]_M)\} \\ & = & \{g([\varphi]_M) \mid \nu \models \varphi\} \end{array}$$

The underlying function of η_M thus factors as the composite $\mathrm{pt}(g) \circ f$, where $\mathrm{pt}(g)$ stands for the underlying bijection of the homeomorphism $\mathrm{pt}(g)\colon \mathrm{pt}(\Omega(M)) \to \mathrm{pt}(\mathtt{Geom}(At)/M)$ (recall the contravariant action of $\mathrm{pt}\colon \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$). It follows that the underlying function of η_M is a bijection as a composition of two bijections. But by [Joh82, §II.1.6], η_M is then automatically an homeomorphism. It follows that $f = \mathrm{pt}(g)^{-1} \circ \eta_M$ is an homeomorphism. □

C.2.2. Proof of Proposition 5

We now turn to Proposition 5.

Proposition 11 (Proposition 5)**.** *Given geometric theories* $\mathtt{T}$ *and* $\mathtt{U}$ *on* At*, the space* $\mathbf{Mod}_{\mathtt{T}}(\mathtt{U})$ *is equal to the subspace induced by the inclusion* $\mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \subseteq \mathbf{Mod}(\mathtt{T})$.

Proof. Write Ω for the topology $\Omega(\mathbf{Mod}(\mathtt{T}))$ and let P be the subset $\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})$ of $\mathbf{Mod}(\mathtt{T})$. We just have to check that $\Omega(\mathbf{Mod}_{\mathtt{T}}(\mathtt{U}))$ is the subspace topology $\Omega{\restriction}P$. We have

$$\begin{array}{rcl} \Omega{\restriction}P & = & \{V \cap P \mid V \in \Omega\} \\ & = & \left\{\mathsf{mod}_{\mathbf{Mod}(\mathtt{T})}(\varphi) \cap P \mid \varphi \in \mathtt{Geom}(At)\right\} \\ & = & \left\{\mathsf{mod}_{\mathbf{Mod}(\mathtt{T})}(\varphi) \cap \mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \mid \varphi \in \mathtt{Geom}(At)\right\} \end{array}$$

On the other hand, for each $\varphi \in \mathtt{Geom}(At)$,

$$\begin{array}{lll} \mathsf{mod}_{\mathbf{Mod}(\mathtt{T})}(\varphi) \cap \mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) & = & \{\nu \in \mathbf{Mod}(\mathtt{T}) \mid \nu \models \varphi\} \cap \mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \\ & = & \{\nu \in \mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \mid \nu \models \varphi\} \\ & = & \mathsf{mod}_{\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})}(\varphi) \end{array}$$

It follows that

$$\begin{array}{lll} \Omega \restriction P & = & \left\{\mathsf{mod}_{\mathbf{Mod}(\mathtt{T})}(\varphi) \cap \mathbf{Mod}(\mathtt{T} \cup \mathtt{U}) \mid \varphi \in \mathtt{Geom}(At)\right\} \\ & = & \left\{\mathsf{mod}_{\mathbf{Mod}(\mathtt{T} \cup \mathtt{U})}(\varphi) \mid \varphi \in \mathtt{Geom}(At)\right\} \\ & = & \Omega(\mathbf{Mod}_{\mathtt{T}}(\mathtt{U})) \end{array}$$

□

C.3. Proofs of §4.3 (Operations on Theories)

Let $(\mathtt{T}_i \mid i \in I)$ be theories, all over At, with $\mathtt{T}_i = \{\psi_{i,j} \vdash \varphi_{i,j} \mid j \in J_i\}$.

(1) If I is finite, we let $\curlyvee_{i \in I} \mathtt{T}_i \coloneqq \left\{\bigwedge_{i \in I} \psi_{i,f(i)} \vdash \bigvee_{i \in I} \varphi_{i,f(i)} \mid f \in \prod_{i \in I} J_i\right\}$.

(2) If I is infinite, and all $\mathtt{T}_i$'s are antecedent-free, we let $\curlyvee_{i \in I} \mathtt{T}_i \coloneqq \left\{\vdash \bigvee_{i \in I} \varphi_{i,f(i)} \mid f \in \prod_{i \in I} J_i\right\}$.

Proposition 12 (Proposition 6)**.** *In both cases above, we have (using the Axiom of Choice when I is infinite)*

$$\mathbf{Mod}\left(\curlyvee_{i \in I} \mathtt{T}_i\right) \quad = \quad \bigcup_{i \in I} \mathbf{Mod}(\mathtt{T}_i)$$

Proof. We discuss each case separately.

(1) Assume that I is a finite set.

We first show that $\bigcup_{i \in I} \mathbf{Mod}(\mathtt{T}_i)$ is included in $\mathbf{Mod}(\curlyvee_{i \in I} \mathtt{T}_i)$. Let $\nu \in \mathbf{Mod}(\mathtt{T}_k)$ for some $k \in I$. Thus, given $f \in \prod_{i \in I} J_i$, if $\nu \models \bigwedge_{i \in I} \psi_{i,f(i)}$, then in particular $\nu \models \psi_{k,f(k)}$. Hence $\nu \models \varphi_{k,f(k)}$ since $\nu \in \mathbf{Mod}(\mathtt{T}_k)$. It follows that $\nu \models \bigvee_{i \in I} \varphi_{i,f(i)}$.

We now show the converse inclusion. Let ν such that $\nu \notin \bigcup_{i \in I} \mathbf{Mod}(\mathtt{T}_i)$. Hence, for all $i \in I$ we have $\nu \notin \mathbf{Mod}(\mathtt{T}_i)$. It follows that for all $i \in I$ there is some $j \in J_i$ such that ν is not a model of the sequent $\psi_{i,j} \vdash \varphi_{i,j}$. Since I is finite, this yields some $f \in \prod_{i \in I} J_i$ such that for all $i \in I$, ν is not a model of the sequent $\psi_{i,f(i)} \vdash \varphi_{i,f(i)}$ (see e.g. [Jec06, §5]). This implies that ν is not a model of the sequent $\bigwedge_{i \in I} \psi_{i,f(i)} \vdash \bigvee_{i \in I} \varphi_{i,f(i)}$. Hence $\nu \notin \mathbf{Mod}(\curlyvee_{i \in I} \mathtt{T}_i)$.

(2) The case when I is infinite (and all $\mathtt{T}_i$'s are antecedent-free) is proven similarly, excepted that now, for the inclusion $\mathbf{Mod}(\curlyvee_{i \in I} \mathtt{T}_i) \subseteq \bigcup_{i \in I} \mathbf{Mod}(\mathtt{T}_i)$ we use the full Axiom of Choice to obtain a suitable $f \in \prod_{i \in I} J_i$ (see e.g. [Jec06, §5]). □

C.3.1. Proofs of §4.3.1 (Translation of Negation-Free LTL Formulae)

Lemma 28 (Lemma 8)**.** *Fix set A.*

(1) The map $[\![\bigcirc]\!] \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ *preserves all unions and all intersections.*

(2) Given LTL *formulae* Φ, Ψ*, let* $H_{\Phi,\Psi} \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ *take S to* $[\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!](S))$*. Then*

$$[\![\Phi \mathbin{\mathsf{U}} \Psi]\!] \;=\; \bigcup_{n \in \mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{False}]\!]) \qquad \textit{and} \qquad [\![\Phi \mathbin{\mathsf{W}} \Psi]\!] \;=\; \bigcap_{n \in \mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{True}]\!])$$

Proof. Recall that $[\![\bigcirc]\!]$ takes $S \in \mathcal{P}([\![\mathsf{Str}A]\!])$ to $\{x \mid x{\restriction}1 \in S\}$.

(1) Let $(S_i \mid i \in I)$ be a set of subsets of $[\![\mathsf{Str}A]\!]$. Given $x \in [\![\mathsf{Str}A]\!]$, we have

$$\begin{array}{rll} x \in [\![\bigcirc]\!] \left(\bigcup_{i\in I} S_i\right) & \text{iff} & x{\restriction}1 \in \bigcup_{i\in I} S_i \\ & \text{iff} & \text{there is } i \in I \text{ such that } x{\restriction}1 \in S_i \\ & \text{iff} & \text{there is } i \in I \text{ such that } x \in [\![\bigcirc]\!]S_i \\ & \text{iff} & x \in \bigcup_{i\in I}[\![\bigcirc]\!]S_i \end{array}$$

Preservation of intersections is similar.

(2) Fix LTL formulae Φ and Ψ. We have seen in Remark 3 that $[\![\Phi \mathbin{\mathsf{U}} \Psi]\!]$ and $[\![\Phi \mathbin{\mathsf{W}} \Psi]\!]$ are respectively the least and the greatest fixpoints of $H_{\Phi,\Psi} \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$.

Now, it follows from item (1) that $H_{\Phi,\Psi}$ preserves all unions and all intersections. In particular, $H_{\Phi,\Psi}$ is Scott-continuous ($(\mathcal{P}([\![\mathsf{Str}A]\!]), \subseteq)$ is a complete lattice and thus in particular a cpo), and [DP02, Theorem 8.15] gives

$$[\![\Phi \mathbin{\mathsf{U}} \Psi]\!] \quad = \quad \bigcup_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{False}]\!])$$

The case of $[\![\Phi \mathbin{\mathsf{W}} \Psi]\!]$ is obtained dually (since $H_{\Phi,\Psi}$ preserves all intersections, it is a Scott-continuous endo-function on the cpo $([\![\mathsf{Str}A]\!], \supseteq)$). ☐

Remark 14. Note that given $d \in \mathrm{Fin}([\![\mathsf{Str}A]\!])$, we have $(\bot \cdot d) \leq_{[\![\mathsf{Str}A]\!]} x$ if, and only if, $d \leq_{[\![\mathsf{Str}A]\!]} x{\restriction}1$. Hence $[\![\bigcirc]\!](\uparrow d) = \uparrow(\bot \cdot d)$.

Now, recall from Example 7 that Scott-opens $U \subseteq [\![\mathsf{Str}A]\!]$ are unions of sets of the form $\uparrow d$ for $d \in \mathrm{Fin}([\![\mathsf{Str}A]\!])$. Hence, it follows from Lemma 28(1) that $[\![\bigcirc]\!](U)$ is Scott-open whenever so is U.

A second application of Lemma 28(1) implies that $[\![\bigcirc]\!](S)$ is a countable intersection of Scott-opens whenever so is S.

Lemma 29 (Remark 9)**.** *Fix a set A.*

(1) If $\Phi_1 \in G$, then $[\![\Phi_1]\!]$ is Scott-open in $[\![\mathsf{Str}A]\!]$.

(2) If $\Phi_2 \in G_\delta$, then $[\![\Phi_2]\!]$ is a countable intersection of Scott-opens (i.e. a G_δ subset of $[\![\mathsf{Str}A]\!]$).

Proof. We handle each case separately.

(1) We reason by induction on $\Phi_1 \in G$. In the case of the atomic formula $a \in A$, note that $[\![a]\!]$ is the basic open set with $\uparrow(a \cdot \bot^\omega)$. For the propositional connectives, use the induction hypothesis and the stability of open sets under (finite) unions and intersections. The case of $\bigcirc$ follows from Remark 14.

It remains to deal with $\Phi_1 \mathbin{\mathsf{U}} \Psi_1$. Assume $[\![\Phi_1]\!]$ and $[\![\Psi_1]\!]$ Scott-open. By Lemma 28(2), we have

$$[\![\Phi \mathbin{\mathsf{U}} \Psi]\!] \quad = \quad \bigcup_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{False}]\!])$$

where $H_{\Phi,\Psi} \colon \mathcal{P}([\![\mathsf{Str}A]\!]) \to \mathcal{P}([\![\mathsf{Str}A]\!])$ takes S to $[\![\Psi]\!] \cup ([\![\Phi]\!] \cap [\![\bigcirc]\!](S))$. Note that $H_{\Phi,\Psi}(U)$ is Scott-open whenever so is U. Since $[\![\mathsf{False}]\!] = \emptyset$ is Scott-open, it follows by induction on $n \in \mathbb{N}$ that each $H^n_{\Phi,\Psi}([\![\mathsf{False}]\!])$ is Scott-open. Hence $[\![\Phi \mathbin{\mathsf{U}} \Psi]\!]$ is Scott-open.

(2) We reason by induction on $\Phi_2 \in G_\delta$. The argument is similar to that of item (1) using that an open set is (trivially) an countable intersection of opens, and that countable intersections of opens are stable under finite unions and intersections.

In the case of $\Phi_2 \mathbin{\mathsf{W}} \Psi_2$, by Lemma 28(2) we have

$$[\![\Phi \mathbin{\mathsf{W}} \Psi]\!] \quad = \quad \bigcap_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{True}]\!])$$

Reasoning similarly as for item (1), since $[\![\mathsf{True}]\!] = [\![\mathsf{Str}A]\!]$ is open, we get that $H^n_{\Phi,\Psi}([\![\mathsf{True}]\!])$ is a countable intersection of opens for all $n \in \mathbb{N}$. But a countable intersection of countable intersections is a countable intersection. Hence $[\![\Phi \mathbin{\mathsf{W}} \Psi]\!]$ is a countable intersection of opens. ☐

Lemma 30 (Remark 11)**.** *Recall that* $\Phi \equiv \Psi$ *means* $[\![\Phi]\!] = [\![\Psi]\!]$*. Given* LTL *formulae* Φ *and* Ψ*, we have*

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl} \bigcirc\mathsf{False} & \equiv & \mathsf{False} & \bigcirc(\Phi \vee \Psi) & \equiv & \bigcirc\Phi \vee \bigcirc\Psi & \bigcirc(\Phi \mathbin{\mathsf{U}} \Psi) & \equiv & (\bigcirc\Phi) \mathbin{\mathsf{U}} (\bigcirc\Psi) \\ \bigcirc\mathsf{True} & \equiv & \mathsf{True} & \bigcirc(\Phi \wedge \Psi) & \equiv & \bigcirc\Phi \wedge \bigcirc\Psi & \bigcirc(\Phi \mathbin{\mathsf{W}} \Psi) & \equiv & (\bigcirc\Phi) \mathbin{\mathsf{W}} (\bigcirc\Psi) \end{array}$$

Proof. First, Lemma 28(1) directly yields the laws

$$\begin{array}{rcl@{\qquad}rcl} \bigcirc\mathsf{False} & \equiv & \mathsf{False} & \bigcirc(\Phi \vee \Psi) & \equiv & \bigcirc\Phi \vee \bigcirc\Psi \\ \bigcirc\mathsf{True} & \equiv & \mathsf{True} & \bigcirc(\Phi \wedge \Psi) & \equiv & \bigcirc\Phi \wedge \bigcirc\Psi \end{array}$$

We discuss

$$\bigcirc(\Phi \mathbin{\mathsf{U}} \Psi) \equiv (\bigcirc\Phi) \mathbin{\mathsf{U}} (\bigcirc\Psi) \qquad \text{and} \qquad \bigcirc(\Phi \mathbin{\mathsf{W}} \Psi) \equiv (\bigcirc\Phi) \mathbin{\mathsf{W}} (\bigcirc\Psi)$$

It follows from Lemma 28(1) that $[\![\bigcirc]\!] H_{\Phi,\Psi}(S) = H_{\bigcirc\Phi,\bigcirc\Psi}([\![\bigcirc]\!] S)$. Then, by induction on $n \in \mathbb{N}$ we obtain

$$\begin{array}{rcl@{\quad}c@{\quad}l} [\![\bigcirc]\!] H^n_{\Phi,\Psi}([\![\mathsf{False}]\!]) & = & H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\bigcirc\mathsf{False}]\!]) & = & H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\mathsf{False}]\!]) \\ [\![\bigcirc]\!] H^n_{\Phi,\Psi}([\![\mathsf{True}]\!]) & = & H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\bigcirc\mathsf{True}]\!]) & = & H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\mathsf{True}]\!]) \end{array}$$

Hence, using Lemma 28(1) and Lemma 28(2), we conclude that

$$\begin{array}{rclclcl} [\![\bigcirc(\Phi \mathbin{\mathsf{U}} \Psi)]\!] & = & [\![\bigcirc]\!] \bigcup_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{False}]\!]) & = & \bigcup_{n\in\mathbb{N}} H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\mathsf{False}]\!]) & = & [\![(\bigcirc\Phi) \mathbin{\mathsf{U}} (\bigcirc\Psi)]\!] \\ [\![\bigcirc(\Phi \mathbin{\mathsf{W}} \Psi)]\!] & = & [\![\bigcirc]\!] \bigcap_{n\in\mathbb{N}} H^n_{\Phi,\Psi}([\![\mathsf{True}]\!]) & = & \bigcap_{n\in\mathbb{N}} H^n_{\bigcirc\Phi,\bigcirc\Psi}([\![\mathsf{True}]\!]) & = & [\![(\bigcirc\Phi) \mathbin{\mathsf{W}} (\bigcirc\Psi)]\!] \end{array}$$

□

D. Proofs of §5 (Free Frames and Spatiality)

Lemma 31 (Lemma 11)**.** $(\mathtt{Geom}(At)/\mathtt{T}, \leq_{\mathtt{T}})$ *is a frame.*

Proof. The argument is mostly a direct inspection of Figure 2.

First, we have that $[\bigvee_i \varphi_i]_{\mathtt{T}}$ is an upper bound of $([\varphi_i]_{\mathtt{T}})_i$ since $\varphi_i \vdash \bigvee_i \varphi_i$ for all i (rule ($\bigvee$-R)). It is a least upper bound since $\bigvee_i \varphi_i \vdash \psi$ whenever $\varphi_i \vdash \psi$ for all i (rule ($\bigvee$-L)).

Similarly, $[\varphi \wedge \psi]_{\mathtt{T}}$ is a lower bound of $[\varphi]_{\mathtt{T}}$ and $[\psi]_{\mathtt{T}}$ since $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$ (rules ($\wedge$-L$_1$) and ($\wedge$-L$_2$)). It is a greatest lower bound since $\theta \vdash \varphi \wedge \psi$ whenever $\theta \vdash \varphi$ and $\theta \vdash \psi$ (rule ($\wedge$-R)).

Moreover, the rule ($\mathtt{true}$-R) yields that $[\varphi]_{\mathtt{T}} \leq_{\mathtt{T}} [\mathtt{true}]_{\mathtt{T}}$ for all φ.

Hence, $(\mathtt{Geom}(At)/\mathtt{T}, \leq_{\mathtt{T}})$ is a complete lattice whose sups and binary infs are respectively given by

$$\begin{array}{rcl} ([\varphi_i]_{\mathtt{T}})_i & \longmapsto & \bigvee_i [\varphi_i]_{\mathtt{T}} \quad = [\bigvee_i \varphi_i]_{\mathtt{T}} \\ [\varphi]_{\mathtt{T}},\ [\psi]_{\mathtt{T}} & \longmapsto & [\varphi]_{\mathtt{T}} \wedge [\psi]_{\mathtt{T}} = [\varphi \wedge \psi]_{\mathtt{T}} \end{array}$$

It remains to prove frame distributivity, namely

$$\bigvee_{i\in I} ([\psi]_{\mathtt{T}} \wedge [\varphi_i]_{\mathtt{T}}) \quad = \quad [\psi]_{\mathtt{T}} \wedge \bigvee_{i\in I} [\varphi_i]_{\mathtt{T}}$$

We reason on the syntax of geometric formulae. Assume $\psi = \bigvee\{\gamma'_k \mid k \in K\}$ and $\varphi_i = \bigvee\{\gamma_{i,j} \mid j \in J_i\}$ for each $i \in I$. Then, unfolding the notations of Remark 7 (i.e. Lemma 25), we have

$$\begin{array}{rcl} \psi \wedge \bigvee_{i\in I} \varphi_i & = & (\bigvee\{\gamma'_k \mid k \in K\}) \wedge (\bigvee\{\gamma_{i,j} \mid i \in I \text{ and } j \in J_i\}) \\ & = & \bigvee\{\gamma'_k \wedge \gamma_{i,j} \mid k \in K,\ i \in I \text{ and } j \in J_i\} \end{array}$$

On the other hand

$$\begin{array}{rcl} \psi \wedge \varphi_i & = & \bigvee\{\gamma'_k \wedge \gamma_{i,j} \mid k \in K \text{ and } j \in J_i\} \\ \bigvee_{i\in I} (\psi \wedge \varphi_i) & = & \bigvee\{\gamma'_k \wedge \gamma_{i,j} \mid i \in I,\ k \in K \text{ and } j \in J_i\} \end{array}$$

It follows that

$$\bigvee_{i\in I} (\psi \wedge \varphi_i) \quad = \quad \psi \wedge \bigvee_{i\in I} \varphi_i$$

and we are done. □

For some proofs in this §D, we make explicit that $\bigvee\{\gamma\}$ is the conjunctive formula γ seen as a conjunctive formula. The following observation will be useful several times.

Remark 15. Given a theory $\mathtt{T}$ and given $\varphi = \bigvee_{i\in I}\gamma_i$, we have

$$[\varphi]_{\mathtt{T}} \quad = \quad \bigvee_{i\in I}[\bigvee\{\gamma_i\}]_{\mathtt{T}}$$

Proof. Note that given $S \subseteq \mathtt{Conj}(At)$, making explicit the $\bigvee\{\gamma\}$'s in the rules ($\bigvee$-L) and ($\bigvee$-R) leads to the following instances

$$\frac{\text{for all } \gamma \in S,\ \bigvee\{\gamma\} \vdash_{\mathtt{T}} \psi}{\bigvee S \vdash_{\mathtt{T}} \psi} \qquad\qquad \frac{}{\bigvee\{\gamma\} \vdash_{\mathtt{T}} \bigvee S}\ (\text{if } \gamma \in S)$$

Hence by Lemma 31 (i.e. Lemma 11) we have

$$[\varphi]_{\mathtt{T}} \quad = \quad \bigvee_{i\in I}[\bigvee\{\gamma_i\}]_{\mathtt{T}}$$

□

The following simple property is mentioned in the text of §5.

Lemma 32. *Let* $\mathtt{T}$ *be a geometric theory over* At *and let* $M := \mathbf{Mod}(\mathtt{T})$*. Then for all* $\varphi, \psi \in \mathtt{Geom}(At)$*, we have*

$$\psi \vdash_{\mathtt{T}} \varphi \quad \Longrightarrow \quad \psi \preceq_M \varphi$$

Proof. The proof is a simple induction on $\psi \vdash_{\mathtt{T}} \varphi$, using Remark 7 (i.e. Lemma 25) for the logical rules. The case of the rule (TH) follows from the fact that $(\psi \vdash \varphi) \in \mathtt{T}$ implies $\psi \preceq_M \varphi$. The cases of (AX) and (CUT) follow from the fact that both $\vdash_{\mathtt{T}}$ and $\preceq_M$ are preorders. □

We now turn Lemma 12. Let $\mathtt{T}$ be a theory over At. Let $\mathcal{F}$ be a point of $\mathtt{Geom}(At)/\mathtt{T}$ such that $[\psi]_{\mathtt{T}} \in \mathcal{F}$ and $[\varphi]_{\mathtt{T}} \notin \mathcal{F}$. Let $\nu\colon At \to \mathbf{2}$ take $p \in At$ to 1 iff $[p]_{\mathtt{T}} \in \mathcal{F}$.

Lemma 33 (Lemma 12)**.** *Let* $\mathcal{F}$ *and* ν *as above. Then for every* $\theta \in \mathtt{Geom}(At)$*, we have* $\nu \models \theta$ *if, and only if,* $[\theta]_{\mathtt{T}} \in \mathcal{F}$*.*

In particular, ν *is a model of* $\mathtt{T}$ *with* $\nu \models \psi$ *and* $\nu \not\models \varphi$*.*

Proof. We first show that for every $\theta \in \mathtt{Geom}(At)$, we have $\nu \models \theta$ if, and only if, $[\theta]_{\mathtt{T}} \in \mathcal{F}$. The proof is similar to that of Lemma 27 (§C.2). We first show by induction on $\gamma \in \mathtt{Conj}(At)$ that $\nu \models \gamma$ if, and only if $[\bigvee\{\gamma\}]_{\mathtt{T}} \in \mathcal{F}$ (thus making explicit that $\bigvee\{\gamma\}$ is the conjunctive formula $\gamma \in \mathtt{Conj}(At)$ seen as a geometric formula).

Case of $p \in At$**.** Since by definition of ν, we have $\nu \models p$ if, and only if $[p]_{\mathtt{T}} = [\bigvee\{p\}]_{\mathtt{T}} \in \mathcal{F}$.

Case of $\mathtt{true}$**.** On the one hand, we have $\nu \models \mathtt{true}$. On the other hand, we have $[\bigvee\{\mathtt{true}\}]_{\mathtt{T}} \in \mathcal{F}$ by Lemma 31 (i.e. Lemma 11).

Case of $\gamma_1 \wedge \gamma_2$**.** We have $\nu \models \gamma_1 \wedge \gamma_2$ iff $\nu \models \gamma_1$ and $\nu \models \gamma_2$. By induction hypothesis, for $i = 1, 2$ we have $\nu \models \gamma_i$ iff $[\bigvee\{\gamma_i\}]_{\mathtt{T}} \in \mathcal{F}$. On the other hand, by Lemma 31 (i.e. Lemma 11) we have

$$[\bigvee\{\gamma\}]_{\mathtt{T}} \wedge [\bigvee\{\gamma'\}]_{\mathtt{T}} \quad = \quad [\bigvee\{\gamma \wedge \gamma'\}]_{\mathtt{T}}$$

Hence $[\bigvee\{\gamma \wedge \gamma'\}]_{\mathtt{T}} \in \mathcal{F}$ if, and only if, $[\bigvee\{\gamma\}]_{\mathtt{T}}, [\bigvee\{\gamma'\}]_{\mathtt{T}} \in \mathcal{F}$. It follows that $\nu \models \gamma_1 \wedge \gamma_2$ iff $[\bigvee\{\gamma \wedge \gamma'\}]_{\mathtt{T}} \in \mathcal{F}$.

We now consider the case of $\varphi = \bigvee_{i\in I}\gamma_i$. By Remark 15 we have

$$[\varphi]_{\mathtt{T}} \quad = \quad \bigvee_{i\in I}[\bigvee\{\gamma_i\}]_{\mathtt{T}}$$

Then, using the above inductive property on conjunctive formulae, we get

$$\begin{array}{lll} \nu \models \varphi & \text{iff} & \text{there is } i \in I \text{ such that } \nu \models \gamma_i \\ & \text{iff} & \text{there is } i \in I \text{ such that } [\bigvee\{\gamma_i\}]_{\mathtt{T}} \in \mathcal{F} \\ & \text{iff} & [\varphi]_{\mathtt{T}} \in \mathcal{F} \end{array}$$

For the second part of the statement, given $(\theta_1 \vdash \theta_2) \in \mathtt{T}$ we have $[\theta_1]_{\mathtt{T}} \leq_{\mathtt{T}} [\theta_2]_{\mathtt{T}}$. Hence, if $\nu \models \theta_1$, then $[\theta_1]_{\mathtt{T}} \in \mathcal{F}$, so that $[\theta_2]_{\mathtt{T}} \in \mathcal{F}$ and $\nu \models \theta_2$. It follows that ν is a model of $\mathtt{T}$.

Finally, we get $\nu \models \psi$ and $\nu \not\models \varphi$ since $[\psi]_{\mathtt{T}} \in \mathcal{F}$ and $[\varphi]_{\mathtt{T}} \notin \mathcal{F}$ by assumption. □

D.1. Proof of Proposition 7

Proposition 13 (Proposition 7). $(\mathtt{Geom}(At)/\mathtt{E}, \leq_{\mathtt{E}})$ *(together with the function $p \in At \mapsto [p]_{\mathtt{T}}$) is a free frame on At.*

Fix a set At. Consider a frame L and a function $f\colon At \to \mathbf{U}L$. We have to show that there is a unique frame morphism $f^\star\colon \mathtt{Geom}(At)/\mathtt{E} \to L$ such that the following commutes.

$$\begin{tikzcd} At \arrow[rr, "f"] \arrow[dr, "p\mapsto[p]_{\mathtt{E}}"'] & & \mathbf{U}L \\ & \mathbf{U}(\mathtt{Geom}(At)/\mathtt{E}) \arrow[ur, dashed, "\mathbf{U}f^\star"'] & \end{tikzcd}$$

We first extend f to the function

$$g_0 \quad : \quad \mathtt{Conj}(At) \quad \longrightarrow \quad \mathbf{U}L$$

defined by induction on $\gamma \in \mathtt{Conj}(At)$ as follows:

$$\begin{array}{rcl} g_0(p) & := & f(p) \\ g_0(\mathtt{true}) & := & \top_L \\ g_0(\gamma \wedge \gamma') & := & g_0(\gamma) \wedge_L g_0(\gamma') \end{array}$$

We then extend g_0 to a function

$$g \quad : \quad \mathtt{Geom}(At) \quad \longrightarrow \quad \mathbf{U}L$$

with

$$g\,(\bigvee S) \quad := \quad \bigvee\nolimits_L \{g_0(\gamma) \mid \gamma \in S\}$$

where $S \subseteq \mathtt{Conj}(At)$.

Lemma 34. *With the notation of Remark 7 (i.e. Lemma 25), we have*

(i) $g(\mathtt{true}) = \top_L$

(ii) $g(\varphi \wedge \psi) = g(\varphi) \wedge_L g(\psi)$

(iii) $g\,(\bigvee\{\varphi_i \mid i \in I\}) = \bigvee_L \{g(\varphi_i) \mid i \in I\}$

Proof. We discuss each case separately.

(i) Recall that we write $\mathtt{true}$ for the geometric formula $\bigvee\{\mathtt{true}\}$. Then we are done since $g(\bigvee\{\mathtt{true}\}) = \bigvee_L\{g_0(\mathtt{true})\} = \top_L$.

(ii) Write $\varphi = \bigvee_{i \in I} \gamma_i$ and $\psi = \bigvee_{j \in J} \gamma'_j$. We have

$$\begin{array}{rcl} g(\varphi) & = & \bigvee_L \{g_0(\gamma_i) \mid i \in I\} \\ g(\psi) & = & \bigvee_L \left\{g_0(\gamma'_j) \mid j \in J\right\} \end{array}$$

and

$$\begin{array}{rcl} \varphi \wedge \psi & = & \bigvee \left\{\gamma_i \wedge \gamma'_j \mid i \in I \text{ and } j \in J\right\} \\ g(\varphi \wedge \psi) & = & \bigvee_L \left\{g_0(\gamma_i) \wedge_L g_0(\gamma'_j) \mid i \in I \text{ and } j \in J\right\} \end{array}$$

Hence, using frame distributivity in L twice we get

$$g(\varphi \wedge \psi) \quad = \quad g(\varphi) \wedge_L g(\psi)$$

(iii) For each $i \in I$, write $\varphi_i = \bigvee\{\gamma_{i,j} \mid i \in I \text{ and } j \in J_i\}$. Then we are done since

$$\begin{array}{rcl} g(\varphi_i) & = & \bigvee_L \{g_0(\gamma_{i,j}) \mid j \in J_i\} \\ g\left(\bigvee_{i \in I} \varphi_i\right) & = & \bigvee_L \{g_0(\gamma_{i,j}) \mid i \in I \text{ and } j \in J_i\} \end{array}$$

□

Using Lemma 34, a straightforward induction on derivations in $\vdash_{\mathtt{E}}$ shows that

$$\psi \vdash_{\mathtt{E}} \varphi \quad \Longrightarrow \quad g(\psi) \leq_L g(\varphi)$$

It follows that $[\varphi]_{\mathtt{E}} = [\psi]_{\mathtt{E}}$ implies $g(\varphi) = g(\psi)$. This yields our function

$$\begin{array}{rcrcl} f^\star & : & \mathtt{Geom}(At)/\mathtt{E} & \longrightarrow & L \\ & & [\varphi]_{\mathtt{E}} & \longmapsto & g(\varphi) \end{array}$$

Lemma 34 implies that $f^\star$ is a frame morphism. Moreover, given $p \in At$ we have

$$\begin{array}{rcl} f([p]_{\mathtt{E}}) & = & g(p) \\ & = & g(\bigvee\{p\}) \\ & = & g_0(p) \\ & = & f(p) \end{array}$$

We can now conclude the proof of Proposition 13 (i.e. Proposition 7).

Proof of Proposition 13. It remains to show that $f^\star$ is the unique frame morphism such that $f^\star([p]_{\mathtt{E}}) = f(p)$ for all $p \in At$. Let $h \colon \mathtt{Geom}(At)/\mathtt{E} \to L$ be a frame morphism such that $h([p]_{\mathtt{E}}) = f(p)$ for all $p \in At$.

We show that $h = f$. We make explicit that $\bigvee\{\gamma\}$ is the conjunctive formula $\gamma \in \mathtt{Conj}(At)$ seen as a geometric formula.

We first show by induction on $\gamma \in \mathtt{Conj}(At)$ that

$$h([\bigvee\{\gamma\}]_{\mathtt{E}}) \quad = \quad g_0(\gamma)$$

Case of $p \in At$. Since $h([\bigvee\{p\}]_{\mathtt{E}}) = f(p)$ by assumption on h.

Case of $\mathtt{true}$. Since $h([\bigvee\{\mathtt{true}\}]_{\mathtt{E}}) = \top_L$ as h is a frame morphism.

Case of $\gamma \wedge \gamma'$. Note that we have

$$\bigvee\{\gamma\} \wedge \bigvee\{\gamma'\} \quad = \quad \bigvee\{\gamma \wedge \gamma'\}$$

Hence, the result follows from the induction hypothesis and the fact that h is a frame morphism.

Now, given $\varphi = \bigvee_i \gamma_i$, since h is a frame morphism, it follows from Remark 15 that

$$\begin{array}{rcl} h([\varphi]_{\mathtt{E}}) & = & h\left(\bigvee\{[\bigvee\{\gamma_i\}]_{\mathtt{E}} \mid i \in I\}\right) \\ & = & \bigvee_L \{h\left([\bigvee\{\gamma_i\}]_{\mathtt{E}}\right) \mid i \in I\} \\ & = & \bigvee_L \{g_0(\gamma_i) \mid i \in I\} \\ & = & g\left(\varphi\right) \\ & = & f\left([\varphi]_{\mathtt{E}}\right) \end{array}$$

□

D.2. Proof of Proposition 8

Proposition 14 (Proposition 8)**.** *Given geometric theories* $\mathtt{T}, \mathtt{U}$ *over* At*, let* $R \coloneqq \{([\varphi]_{\mathtt{T}}, [\psi]_{\mathtt{T}}) \mid (\varphi \vdash \psi) \in \mathtt{T} \cup \mathtt{U}\}$*. Then* $\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})$ *is isomorphic to the quotient of* $\mathtt{Geom}(At)/\mathtt{T}$ *by* $\sim_{\overline{R}}$*.*

Fix geometric theories $\mathtt{T}$ and $\mathtt{U}$ over At, and let R be as in the statement. Consider

$$\widetilde{R} \quad \coloneqq \quad \{([\varphi]_{\mathtt{T}}, [\psi]_{\mathtt{T}}) \mid \varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \psi\}$$

We are going to show that $\widetilde{R} = \overline{R}$, i.e. that $\widetilde{R}$ is the least congruence preorder containing R. It is trivial that $R \subseteq \widetilde{R}$.

Lemma 35. $\widetilde{R}$ *is a congruence preorder on* $\mathtt{Geom}(At)/\mathtt{T}$*.*

Proof. We first check that $\widetilde{R}$ is a preorder. Its reflexive since $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \varphi$. We now prove that $\widetilde{R}$ is transitive. Let $C \mathrel{\widetilde{R}} D$ and $D \mathrel{\widetilde{R}} E$. Hence there are φ, θ such that $C = [\varphi]_{\mathtt{T}}$, $D = [\theta]_{\mathtt{T}}$ and $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \theta$. Similarly, there are θ', ψ such that $D = [\theta']_{\mathtt{T}}$, $E = [\psi]_{\mathtt{T}}$ and $\theta' \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$. Since $[\theta]_{\mathtt{T}} = [\theta']_{\mathtt{T}}$, we get (say) $\theta \vdash_{\mathtt{T}} \theta'$. Hence $\theta \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$ and thus $C \mathrel{\widetilde{R}} E$.

Moreover, if $[\varphi]_{\mathtt{T}} \leq_{\mathtt{T}} [\psi]_{\mathtt{T}}$, then $\varphi \vdash_{\mathtt{T}} \psi$, so that $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$ and $[\varphi]_{\mathtt{T}} \mathrel{\widetilde{R}} [\psi]_{\mathtt{T}}$.

Let ψ and $(\varphi_i)_{i\in I}$ such that for all $i \in I$, we have

$$[\varphi_i]_{\mathtt{T}} \quad \widetilde{R} \quad [\psi]_{\mathtt{T}}$$

Since $\vdash_{\mathtt{T}} \subseteq \vdash_{\mathtt{T}\cup\mathtt{U}}$, this implies that for all $i \in I$, we have $\varphi_i \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$. Hence using the rule ($\bigvee$-L), we get $\bigvee_{i\in I} \varphi_i \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$, and it follows that

$$\bigvee_{i\in I} [\varphi_i]_{\mathtt{T}} \quad \widetilde{R} \quad [\psi]_{\mathtt{T}}$$

Note that for every φ, we have $\varphi \vdash_{\mathtt{T}} \mathtt{true}$, and thus

$$[\varphi]_{\mathtt{T}} \quad \widetilde{R} \quad [\mathtt{true}]_{\mathtt{T}}$$

Finally, assume

$$[\theta]_{\mathtt{T}} \quad \widetilde{R} \quad [\varphi]_{\mathtt{T}} \qquad \text{and} \qquad [\theta]_{\mathtt{T}} \quad \widetilde{R} \quad [\psi]_{\mathtt{T}}$$

Again since $\vdash_{\mathtt{T}} \subseteq \vdash_{\mathtt{T}\cup\mathtt{U}}$, this implies $\theta \vdash_{\mathtt{T}\cup\mathtt{U}} \varphi$ and $\theta \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$. Hence with the rule ($\wedge$-R) we get $\theta \vdash_{\mathtt{T}\cup\mathtt{U}} \varphi \wedge \psi$ and thus

$$[\theta]_{\mathtt{T}} \quad \widetilde{R} \quad [\varphi]_{\mathtt{T}} \wedge [\psi]_{\mathtt{T}}$$

□

Lemma 36. *Let* Q *be a congruence preorder on* $\mathtt{Geom}(At)/\mathtt{T}$ *such that* $R \subseteq Q$*. Then* $\widetilde{R} \subseteq Q$*.*

Proof. We show that if $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$ then $([\varphi]_{\mathtt{T}}, [\psi]_{\mathtt{T}}) \in Q$. We reason by induction on the derivation of $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$.

Case of (Th). Since $R \subseteq Q$.

Cases of (Ax) and (Cut) Since Q is a preorder.

Case of (true-R). Since Q is a preorder containing $\leq_{\mathtt{T}}$.

Case of ($\wedge$-L$_1$) and ($\wedge$-L$_2$). Since $[\varphi_1 \wedge \varphi_2]_{\mathtt{T}} \leq_{\mathtt{T}} [\varphi_i]_{\mathtt{T}}$ while Q contains $\leq_{\mathtt{T}}$.

Case of ($\bigvee$-L). Similar.

Case of ($\wedge$-R). Since Q is a congruence preorder, we have

$$[\theta]_{\mathtt{T}} \quad Q \quad [\varphi]_{\mathtt{T}} \wedge [\psi]_{\mathtt{T}}$$

whenever

$$[\theta]_{\mathtt{T}} \quad Q \quad [\varphi]_{\mathtt{T}} \qquad \text{and} \qquad [\theta]_{\mathtt{T}} \quad Q \quad [\psi]_{\mathtt{T}}$$

Then conclude with the induction hypothesis.

Case of ($\bigvee$-L). Similar.

Case of (Dist). Since by frame distributivity in $\mathtt{Geom}(At)/\mathtt{T}$, we have

$$[\psi \wedge \textstyle\bigvee_{i\in I} \varphi_i]_{\mathtt{T}} \quad = \quad [\textstyle\bigvee_{i\in I}(\psi \wedge \varphi_i)]_{\mathtt{T}}$$

□

Hence $\widetilde{R} = \overline{R}$, the least congruence preorder containing R. We can now conclude the proof of Proposition 14, (i.e. Proposition 8).

Proof of Proposition 14. Let $\sim$ be the equivalence relation induced by $\widetilde{R}$. We have to show that $\mathtt{Geom}(At)/(\mathtt{T} \cup \mathtt{U})$ is isomorphic to the quotient of $\mathtt{Geom}(At)/\mathtt{T}$ by $\sim$.

Recall that $\vdash_{\mathtt{T}} \subseteq \vdash_{\mathtt{T}\cup\mathtt{U}}$. Note that given $\varphi, \psi \in \mathtt{Geom}(At)$, we have $[\varphi]_{\mathtt{T}} \sim [\psi]_{\mathtt{T}}$ precisely when $\varphi \vdash_{\mathtt{T}\cup\mathtt{U}} \psi$ and $\psi \vdash_{\mathtt{T}\cup\mathtt{U}} \varphi$. In other words, for all $\varphi, \psi \in \mathtt{Geom}(At)$, we have

$$[\varphi]_{\mathtt{T}} \sim [\varphi]_{\mathtt{T}} \quad \text{if, and only if,} \quad [\varphi]_{\mathtt{T}\cup\mathtt{U}} = [\varphi]_{\mathtt{T}\cup\mathtt{U}}$$

and we are done. □

E. Proofs of §6 (A Specification for the Denotation of Filter)

Fix a finite set A and a Scott-continuous $p\colon [\![A]\!] \to [\![\mathsf{Bool}]\!]$ with $p(a) \neq \bot_{[\![\mathsf{Bool}]\!]}$ if $a \in A$. Let $\Psi = \Psi_p$ as in Example 5 and let $\Phi := \bigvee_{a\in A} a$.

Recall from Remark 2 that $[\![\mathsf{filter}]\!]p = \mathsf{Y}(f_p) = \bigvee_{n\in\mathbb{N}} f_p^n(\bot_X)$ where

$$f_p \quad := \quad \lambda g.\lambda x.\ \mathsf{if}\ p(x(0))\ \mathsf{then}\ x(0)\cdot g(x{\restriction}1)\ \mathsf{else}\ g(x{\restriction}1) \quad : \quad X \quad \longrightarrow_{\mathbf{CPO}} \quad X$$

and where X is the cpo $[\![\mathsf{Str}A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Str}A]\!]$.

Recall also the geometric formulae

$$\begin{array}{rrcl} & \psi_{n,k} & := & \bigvee \left\{ \bigwedge_{1\leq j\leq k} \bigcirc^{i_j} \mathtt{F}[\![\Psi]\!] \mid 0 \leq i_1 < \cdots < i_k < n \right\} \\ \text{and} & \varphi_k & := & \bigwedge_{m<k} \bigcirc^m \mathtt{F}[\![\Phi]\!] \end{array}$$

where $k \leq n$.

We begin with the following property, which is stated in the text of §6. Let $x \in [\![\mathsf{Str}A]\!]$.

Lemma 37. *We have $\nu(x) \in \mathbf{Mod}\left(\mathtt{T}[\![\Box\Diamond\Psi]\!]\right)$ if, and only if, $(\forall k \in \mathbb{N})(\exists n \geq k)(\nu(x) \models \psi_{n,k})$.*

Proof. Let $x \in [\![\mathsf{Str}A]\!]$ and write ν for $\nu(x)$. Recall from Example 14 that

$$\mathtt{T}[\![\Box\Diamond\Psi]\!] \quad = \quad \left\{\vdash \bigwedge_{n\leq N} \bigvee_{m\in\mathbb{N}} \bigcirc^{n}\mathtt{F}[\![\Psi]\!] \vee \bigcirc^{n+1}\mathtt{F}[\![\Psi]\!] \vee \cdots \vee \bigcirc^{n+m}\mathtt{F}[\![\Psi]\!] \mid N \in \mathbb{N}\right\}$$

Assume first $\nu \in \mathbf{Mod}\,(\mathtt{T}[\![\Box\Diamond\Psi]\!])$ and let $k \in \mathbb{N}$. Since

$$\nu \quad \models \quad \bigvee_{m\in\mathbb{N}} \mathtt{F}[\![\Psi]\!] \vee \bigcirc\mathtt{F}[\![\Psi]\!] \vee \cdots \vee \bigcirc^{m}\mathtt{F}[\![\Psi]\!]$$

we get some $i_1 \geq 0$ such that $\nu \models \bigcirc^{i_1}\mathtt{F}[\![\Psi]\!]$. Since

$$\nu \quad \models \quad \bigvee_{m\in\mathbb{N}} \bigcirc^{i_1+1}\mathtt{F}[\![\Psi]\!] \vee \bigcirc^{i_1+2}\mathtt{F}[\![\Psi]\!] \vee \cdots \vee \bigcirc^{i_1+m+1}\mathtt{F}[\![\Psi]\!]$$

we get some $i_2 > i_1$ such that $\nu \models \bigcirc^{i_2}\mathtt{F}[\![\Psi]\!]$. Iterating this up to $i_k > \cdots > i_2 > i_1$ yields the result with $n \coloneqq i_k + 1$.

For the converse, using the simplifications mentioned in Example 15, it is sufficient to show that ν is a model of

$$\left\{\vdash \bigvee_{m\in\mathbb{N}} \bigcirc^{N}\mathtt{F}[\![\Psi]\!] \vee \bigcirc^{N+1}\mathtt{F}[\![\Psi]\!] \vee \cdots \vee \bigcirc^{N+m}\mathtt{F}[\![\Psi]\!] \mid N \in \mathbb{N}\right\}$$

Let $N \in \mathbb{N}$. Given $k > N$, by assumption there is some $n \geq k$ such that $\nu \models \psi_{n,k}$. Hence, there are $0 \leq i_1 < \cdots < i_k < n$ such that $\nu \models \bigcirc^{i_j}\mathtt{F}[\![\Psi]\!]$ for all $j = 1, \ldots, k$. In particular, we have $\nu \models \bigcirc^{i_k}\mathtt{F}[\![\Psi]\!]$. But since $0 \leq i_1 < \cdots < i_k$, we necessarily have $i_k \geq k - 1$, and since $k > N$ we get $k - 1 \geq N$. It follows that $i_k \geq N$, and we are done. ☐

Let us now explain how

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total}, \qquad x \Vdash \Box\Diamond\Psi \quad \Longrightarrow \quad [\![\mathsf{filter}]\!]\ p\ x \Vdash \Box\Phi \tag{2}$$

can be obtained from

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total},\ \forall k \in \mathbb{N},\ \forall n \geq k, \quad \nu(x) \models \psi_{n,k} \quad \Longrightarrow \quad \nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k \tag{3}$$

First, it follows from Lemma 10 and Theorem 3 that condition (2) amounts to

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total}, \qquad \nu(x) \in \mathbf{Mod}(\mathtt{T}[\![\Box\Diamond\Psi]\!]) \quad \Longrightarrow \quad (\forall k \in \mathbb{N})\,(\nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k)$$

that is

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total},\ \forall k \in \mathbb{N}, \qquad \nu(x) \in \mathbf{Mod}(\mathtt{T}[\![\Box\Diamond\Psi]\!]) \quad \Longrightarrow \quad \nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k$$

Now, if $\nu(x) \in \mathbf{Mod}(\mathtt{T}[\![\Box\Diamond\Psi]\!])$, then by Lemma 37, for all $k \in \mathbb{N}$ we have $(\exists n \geq k)(\nu(x) \models \psi_{n,k})$. Hence, condition (2) follows from

$$\forall x \in [\![\mathsf{Str}A]\!],\ x \text{ total},\ \forall k \in \mathbb{N}, \qquad (\exists n \geq k)\,(\nu(x) \models \psi_{n,k}) \quad \Longrightarrow \quad \nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k$$

and the latter is equivalent to condition (3). Condition (3) is a direct consequence of the following.

Lemma 38 (Lemma 13)**.** *Write g_n for $f_p^n(\bot_X)\colon [\![\mathsf{Str}A]\!] \to_{\mathbf{CPO}} [\![\mathsf{Str}A]\!]$. Let $x \in [\![\mathsf{Str}A]\!]$ be a total stream.*

(1) Assume $k \leq n$. If $\nu(x) \models \psi_{n,k}$, then $\nu(g_n(x)) \models \varphi_k$.

(2) Let $n, k \in \mathbb{N}$. If $\nu(g_n(x)) \models \varphi_k$, then $\nu([\![\mathsf{filter}]\!]\ p\ x) \models \varphi_k$.

Proof.

(1) We show by induction on $n \in \mathbb{N}$ that for all $x \in [\![\mathsf{Str}A]\!]$ and all $k \leq n$, we have

$$\nu(x) \models \psi_{n,k} \quad \Longrightarrow \quad \nu(g_n(x)) \models \varphi_k$$

Base case $n = 0$**.** In this case we have also $k = 0$. But $\varphi_0 = \mathtt{true}$ and we are done.

Induction step. Note that

$$g_{n+1}(x) \quad = \quad \mathsf{if}\ p(x(0))\ \mathsf{then}\ x(0) \cdot g_n(x{\upharpoonright}1)\ \mathsf{else}\ g_n(x{\upharpoonright}1)$$

Let $k \leq n+1$ and assume $\nu(x) \models \psi_{n+1,k}$. Hence there are $0 \leq i_1 < \cdots < i_k < n+1$ such that $\nu(x) \models \bigcirc^{i_j}\mathtt{F}[\![\Psi]\!]$ for all $j = 1, \ldots, k$.

If $i_1 = 0$, then we have $p(x(0)) = \mathsf{tt}$ and $g_{n+1}(x) = x(0) \cdot g_n(x{\upharpoonright}1)$. Moreover, since $x{\upharpoonright}1 \models \psi_{n,k-1}$, the induction hypothesis gives $\nu(g_n(x{\upharpoonright}1)) \models \varphi_{k-1}$. Since $x{\upharpoonright}1 \models \Phi$, we obtain $\nu(x(0) \cdot g_n(x{\upharpoonright}1)) \models \varphi_k$ and we are done.

Otherwise, we have $i_1 > 0$. Hence $\nu(x{\upharpoonright}1) \models \psi_{n,k}$, so that $\nu(g_n(x{\upharpoonright}1)) \models \varphi_k$ by induction hypothesis.

Then if $p(x(0)) = \mathsf{tt}$, we have $g_{n+1}(x) = x(0) \cdot g_n(x{\upharpoonright}1)$. Since $x{\upharpoonright}1 \models \Phi$, we obtain $g_{n+1}(x) \models \varphi_{k+1}$. In particular, $\nu(g_{n+1}(x)) \models \varphi_k$ and we are done.

If $p(x(0)) = \mathsf{ff}$, then $g_{n+1}(x) = g_n(x{\upharpoonright}1)$ and we are done.

Note that the case of $p(x(0)) = \bot_{[\![\mathsf{Bool}]\!]}$ cannot happen since x is total and since we assumed $p(a) \neq \bot_{[\![\mathsf{Bool}]\!]}$ for all $a \in A$.

(2) Recall that $[\![\mathsf{filter}]\!]p$ is the sup of the chain $\bot_X \leq_X f_p(\bot_X) \leq_X \cdots \leq_X f_p^n(\bot_X) \leq \cdots$, so that $g_n = f_p^n(\bot_X) \leq_X [\![\mathsf{filter}]\!]p$. Hence $g_n(x) \leq_{[\![\mathsf{Str}A]\!]} [\![\mathsf{filter}]\!]\ p\ x$.

On the other hand, it follows from Proposition 4 (i.e. Proposition 10) that the set of all $x \in [\![\mathsf{Str}A]\!]$ such that $\nu(x) \models \varphi_k$ is Scott-open and thus upward-closed. Hence $\nu([\![\mathsf{filter}]\!]\ p\ x)$ is a model of φ_k whenever so is $\nu(g_n(x))$. $\square$

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