The Strength of Safra's Construction
Master Internship

This internship subject aims at calibrating the mathematical strength required to reason on automata on infinite words.

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1 Scientific Context

Automata on Infinite Words. Automata running on infinite words (such as *non-deterministic Büchi automata*) provide an established framework for the specification and verification of non-terminating programs (in particular *via* the model checking technique).

However, some of their basic properties are known to require non-trivial reasoning principles. This is most notably the case of *closure under complement* of non-deterministic Büchi automata: Given an automaton $\mathcal{A}$, to construct an automaton $\tilde{\mathcal{A}}$ which accepts exactly the infinite words rejected by $\mathcal{A}$. There are different known constructions of $\tilde{\mathcal{A}}$ from $\mathcal{A}$, but all of them require non-trivial non-constructive mathematical principles to be proven correct.

Complementation can be achieved *via determinization*: Given a non-deterministic Büchi automaton, to construct an equivalent deterministic automaton (with a different type of acceptance condition). As it is usually trivial to complement a deterministic automaton, determinization constructions for automata on infinite words are at least as hard as complementation of Büchi automata. We are primarily interested in *Safra's construction* for determinization (see e.g. [PP04] for details), which gives (asymptotically) optimal automata for complementation. Safra's construction relies on *Weak König Lemma*, which states that every infinite tree on a finite alphabet has an infinite path.

Subsystems of Second-Order Arithmetic. On the other hand, logic provides an adequate framework in which to compare mathematical theorems w.r.t. the mathematical power their proof require. Second-order arithmetic (and its subsystems) proved to be a particularly sucessful tool for that, see e.g. [Sim10].

The relevant subsystems of second-order arithmetic are mainly characterized by the sets of natural numbers which they suppose to exist (technically, this corresponds to specific restrictions of the *comprehension scheme* of second-order logic). For instance, in the system called WKL_0 (for *Weak König Lemma*), one assumes that for every infinite tree T on a finite alphabet, there exists an infinite set representing an infinite path in T.

2 Description of the Internship

The goal of this internship is to calibrate the strength of Safra's construction for determinization of Büchi automata. We propose to study its correctness proof relatively to the system WKL_0.

Prerequisites. There is no formal prerequisite other than undergraduate logic and automata. Basic knowledge of Büchi automata (in part. Safra construction) and/or of second-order arithmetic is a plus, but not a requirement.

3 Possible Extensions

The above question has different possible extensions, including:

- The comparison of WKL_0 with *Additive Ramsey's theorem*, which is used in direct complementation constructions for non-determinisitic Büchi automata.
- The calibration, w.r.t. WKL_0, of an operation on strategies in parity games called *positionalization*: Safra's construction provides a way to obtain *positional* winning strategies in these games, assuming the existence of winning strategies [Jut97] (showing the mere existence of winning strategies in such games is known to require a much stronger form of comprehension than WKL_0, see e.g. [MT07]).

These extensions, among other, may lead to a PhD, part of a wider project aiming at applying proof-theoretical method to variants of the *Monadic Second Order Logic* on infinite words and trees.

References

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[MT07] M. O. MedSalem and K. Tanaka. Δ^0_3-determinacy, comprehension and induction. *J. Symb. Log.*, 72(2):452–462, 2007.

[PP04] D. Perrin and J.-É. Pin. *Infinite Words: Automata, Semigroups, Logic and Games.* Pure and Applied Mathematics. Elsevier, 2004.

[Sim10] S.G. Simpson. *Subsystems of Second Order Arithmetic.* Perspectives in Logic. Cambridge University Press, 2nd edition, 2010.