Advanced Semantics of Programming Languages

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LIP - ENS de Lyon

Course 03
09/25

Gödel's System T

(recap)

Motivation

General Idea:

- Devise models of programming languages . . .

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- Booleans + Divergence (Ω).
- Set-theoretic model with

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Gödel's System T.

- Restrict to recursion over $\mathbb{N}$:

$$\begin{array}{lcl} \mathbf{Rec}(u, v, \underline{0}) & \rhd & u \\ \mathbf{Rec}(u, v, t\mathbf{+1}) & \rhd & v\ t\ \mathbf{Rec}(u, v, t) \end{array}$$

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- Allows to see important techniques in a simple setting.

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Normalization.

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- $[\![\underline{0}]\!] := 0$.
- $[\![(-)\mathbf{+1}]\!] \; : \; (n \in \mathbb{N}) \;\longmapsto\; (n + 1 \in \mathbb{N})$.

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Recursor.

- Given $a \in [\![\sigma]\!]$ and

$$b \in [\![\mathtt{nat} \to \sigma \to \sigma]\!] = \left([\![\sigma]\!]^{[\![\sigma]\!]}\right)^{[\![\mathtt{nat}]\!]}$$

define $[\![\mathtt{Rec}^\sigma]\!](a, b, n) \in [\![\sigma]\!]$ by induction on $n \in \mathbb{N}$:

$$[\![\mathtt{Rec}^\sigma]\!](a, b, 0) := a \qquad \text{and} \qquad [\![\mathtt{Rec}^\sigma]\!](a, b, n+1) := b\ n\ [\![\mathtt{Rec}^\sigma]\!](a, b, n)$$

The Language PCF

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Notes.

- We assume an infinite set of variables $x, y, z, \ldots$
- We have one *numeral* $\underline{n}$ for each $n \in \mathbb{N}$.

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Typing Rules.

- Adaptation of System T with

$$\frac{}{\Gamma \vdash Y^\sigma : (\sigma \to \sigma) \to \sigma} \qquad \frac{}{\Gamma \vdash \underline{n} : \mathbf{nat}}\ (n \in \mathbb{N})$$

$$\frac{\Gamma \vdash t : \mathbf{nat} \qquad \Gamma \vdash u : \mathbf{nat} \qquad \Gamma \vdash v : \mathbf{nat}}{\Gamma \vdash \mathtt{if}\ t\ \mathtt{then}\ u\ \mathtt{else}\ v : \mathbf{nat}} \qquad \frac{\Gamma \vdash t : \mathbf{nat}}{\Gamma \vdash t\mathtt{-1} : \mathbf{nat}}$$

Operational Semantics

Weak Head Reduction.

- Usual (weak) call-by-name evaluation.

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$$\frac{}{(\lambda x.t)u \quad \triangleright \quad t[u/x]} \qquad \frac{}{\underline{n}\,\mathbf{+1} \quad \triangleright \quad \underline{n+1}} \qquad \frac{}{\underline{n+1}\,\mathbf{-1} \quad \triangleright \quad \underline{n}} \qquad \frac{}{\underline{0}\,\mathbf{-1} \quad \triangleright \quad \underline{0}}$$

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Congruence Rules.

$$\frac{t \;\triangleright\; u}{t\,v \;\triangleright\; u\,v} \qquad \frac{t \;\triangleright\; u}{t\mathbf{+1} \;\triangleright\; u\mathbf{+1}} \qquad \frac{t \;\triangleright\; u}{t\mathbf{-1} \;\triangleright\; u\mathbf{-1}}$$

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Substitution.

- If $\Gamma, x : \sigma \vdash t : \tau$ and $\Gamma \vdash u : \sigma$ then $\Gamma \vdash t[u/x] : \tau$.

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Normal Forms of Type `nat`.

- If $\vdash t : \mathtt{nat}$ with t in normal form w.r.t. $\triangleright$, then $t = \underline{n}$ for some $n \in \mathbb{N}$.

Examples

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- For each type σ we have

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- Note that

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- We can get for each type τ a set $[\![\tau]\!]$ with
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Difficulty:

$$[\![Y^\sigma]\!] \quad : \quad ([\![\sigma]\!] \to [\![\sigma]\!]) \quad \longrightarrow \quad [\![\sigma]\!]$$

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- For each type σ, a natural candidate for $[\![\Omega^\sigma]\!]$ is $\bot_\sigma$, where

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Idea: $\bot$ means "no information", reflecting $\Omega \rhd^+ \Omega \rhd^+ \dots$.

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