Advanced Semantics of Programming Languages

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LIP - ENS de Lyon

Course 13
12/11

The Polymorphic Lambda-Calculus

(a.k.a. *System* $\mathcal{F}$)

Introduction

Example (Map).

- Consider a *map* function

$$\begin{array}{lclcl} \mathtt{map}_{\sigma,\tau} & : & (\sigma \to \tau) \times \mathtt{list}(\sigma) & \longrightarrow & \mathtt{list}(\tau) \\ & & \big(f\,,\,[\mathtt{a}_0;\ldots;\mathtt{a}_n]\big) & \longmapsto & [f(\mathtt{a}_0)\,;\,\ldots\,;\,f(\mathtt{a}_n)] \end{array} \qquad (\sigma, \tau \text{ types})$$

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- Consider the typed terms

$$\mathrm{id}_\tau \;:=\; \boldsymbol{\lambda} x : \tau.x \;:\; \tau \to \tau \qquad \text{and} \qquad \mathrm{id}_\sigma \;:=\; \boldsymbol{\lambda} x : \sigma.x \;:\; \sigma \to \sigma$$

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Main Idea:

Polymorphism allows to express such uniformities
using explicit universal quantification over types.

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Terms.

$$t, u \quad ::= \quad x \quad | \quad t\, u \quad | \quad \lambda x : \sigma . t$$

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Typing Rules.

- Extension of the simply-typed systems with

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash \Lambda\alpha.t : \forall\alpha.\tau}\ (\alpha \text{ not free in } \Gamma) \qquad\qquad \frac{\Gamma \vdash t : \forall\alpha.\tau}{\Gamma \vdash t\,\sigma : \tau[\sigma/\alpha]}$$

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Reduction.

- We consider the full strong reduction $\triangleright$ defined as

$$\frac{}{(\lambda x : \tau.t)u \ \triangleright\ t[u/x]} \qquad \frac{t \ \triangleright\ u}{t\,v \ \triangleright\ u\,v} \qquad \frac{t \ \triangleright\ u}{v\,t \ \triangleright\ v\,u} \qquad \frac{t \ \triangleright\ u}{\lambda x : \sigma.t \ \triangleright\ \lambda x : \sigma.u}$$

$$\frac{}{(\Lambda\alpha.t)\sigma \ \triangleright\ t[\sigma/\alpha]} \qquad \frac{t \ \triangleright\ u}{t\sigma \ \triangleright\ u\sigma} \qquad \frac{t \ \triangleright\ u}{\Lambda\alpha.t \ \triangleright\ \Lambda\alpha.u}$$

- Let $=_\beta$ be the symmetric-reflexive-transitive closure of $\triangleright$.

Expressive Power

- The expressive power of System $\mathcal{F}$ is huge. We content ourselves with examples.

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Identity.

- Let

$$\mathrm{id} \quad := \quad \Lambda\alpha.\lambda x:\alpha.x \quad : \quad \underbrace{\forall\alpha(\alpha \to \alpha)}_{\mathrm{Id}}$$

For each τ we have

$$\mathrm{id}\,\tau \;\;\triangleright\;\; \mathrm{id}_\tau \;\;=\;\; \lambda x:\tau.x \;\;:\;\; \tau \to \tau$$

In particular

$$\mathrm{id}\,\mathrm{Id} \quad : \quad \mathrm{Id} \to \mathrm{Id}$$

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Void Type.

- Let $\perp := \forall\alpha.\alpha$.

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$$\mathrm{id}\,\mathrm{Id} \quad : \quad \mathrm{Id} \to \mathrm{Id}$$

Void Type.

- Let $\bot := \forall\alpha.\alpha$.
We have

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash t\,\tau : \tau}$$

- *Question.* Is there a closed term of type $\bot$?

Product Types

Let

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Moreover

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Moreover

$$\pi_1(\mathbf{pair}\, t\, u) \;\; \rhd^* \;\; \Big(\Lambda \alpha.\, \lambda p.\, p\, t\, u\Big) \tau (\lambda xy.x)$$

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and similarly

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Remark.

- We can also have polymorphic **pair** and π_i.

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$$\begin{array}{rcl} \tau + \sigma & := & \forall \alpha. \Big((\tau \to \alpha) \longrightarrow (\sigma \to \alpha) \longrightarrow \alpha \Big) \\ \mathtt{inl}\, t & := & \Lambda \alpha.\, \lambda \ell : \tau \to \alpha.\, \lambda r : \sigma \to \alpha.\, \ell\, t \\ \mathtt{inr}\, u & := & \Lambda \alpha.\, \lambda \ell : \tau \to \alpha.\, \lambda r : \sigma \to \alpha.\, r\, u \\ \mathtt{case}\, t\, u\, v & := & t\, \kappa\, u\, v \end{array}$$

so that

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash \mathtt{inl}\, t : \tau + \sigma} \qquad \frac{\Gamma \vdash u : \sigma}{\Gamma \vdash \mathtt{inr}\, u : \tau + \sigma}$$

$$\frac{\Gamma \vdash t : \tau + \sigma \qquad \Gamma \vdash u : \tau \to \kappa \qquad \Gamma \vdash v : \sigma \to \kappa}{\Gamma \vdash \mathtt{case}\, t\, u\, v : \kappa}$$

Note that

$$\mathtt{case}\,(\mathtt{inl}\, t)\, u\, v \;\rhd^*\; u\, t \qquad \text{and} \qquad \mathtt{case}\,(\mathtt{inr}\, t)\, u\, v \;\rhd^*\; v\, t$$

Sum Types

Let

$$\begin{array}{rcl} \tau + \sigma & := & \forall\alpha.\Big((\tau \to \alpha) \;\longrightarrow\; (\sigma \to \alpha) \;\longrightarrow\; \alpha\Big) \\ \mathtt{inl}\, t & := & \Lambda\alpha.\, \lambda\ell : \tau \to \alpha.\, \lambda r : \sigma \to \alpha.\, \ell\, t \\ \mathtt{inr}\, u & := & \Lambda\alpha.\, \lambda\ell : \tau \to \alpha.\, \lambda r : \sigma \to \alpha.\, r\, u \\ \mathtt{case}\, t\, u\, v & := & t\, \kappa\, u\, v \end{array}$$

so that

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash \mathtt{inl}\, t : \tau + \sigma} \qquad \frac{\Gamma \vdash u : \sigma}{\Gamma \vdash \mathtt{inr}\, u : \tau + \sigma}$$

$$\frac{\Gamma \vdash t : \tau + \sigma \qquad \Gamma \vdash u : \tau \to \kappa \qquad \Gamma \vdash v : \sigma \to \kappa}{\Gamma \vdash \mathtt{case}\, t\, u\, v : \kappa}$$

Note that

$$\mathtt{case}\,(\mathtt{inl}\, t)\, u\, v \;\; \triangleright^* \;\; u\, t \qquad \text{and} \qquad \mathtt{case}\,(\mathtt{inr}\, t)\, u\, v \;\; \triangleright^* \;\; v\, t$$

Remark.

- We can also have polymorphic **`inl`**, **`inr`**, **`case`**.

Natural Numbers

Let

$$\texttt{nat} \quad := \quad \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big)$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$\begin{array}{rcl} \mathbf{nat} & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\ \underline{n} & := & \end{array}$$

so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathbf{nat}}\ (n \in \mathbb{N})$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$\begin{array}{rcl} \mathbf{nat} & := & \forall\alpha.\Big(\alpha \longrightarrow (\alpha \to \alpha) \longrightarrow \alpha\Big) \\ \underline{n} & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s^n z \end{array}$$

so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathbf{nat}}\ (n \in \mathbb{N})$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$
\begin{array}{rcl}
\mathbf{nat} & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
\underline{n} & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s^n z \\
\mathbf{S}\, t & := &
\end{array}
$$

so that

$$
\frac{}{\Gamma \vdash \underline{n} : \mathbf{nat}}\ (n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathbf{nat}}{\Gamma \vdash \mathbf{S}\, t : \mathbf{nat}}
$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$
\begin{array}{rcl}
\mathtt{nat} & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
\underline{n} & := & \Lambda\alpha.\, \lambda z : \alpha.\, \lambda s : \alpha \to \alpha.\, s^n z \\
\mathtt{S}\, t & := & \Lambda\alpha.\, \lambda z : \alpha.\, \lambda s : \alpha \to \alpha.\, s\,(t\,\alpha\, z\, s)
\end{array}
$$

so that

$$
\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\,(n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}}
$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$
\begin{array}{rcl}
\mathtt{nat} & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
\underline{n} & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s^n z \\
\mathtt{S}\, t & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s\,(t\,\alpha\,z\,s) \\
\mathtt{iter}\, t\, u\, v & := &
\end{array}
$$

so that

$$
\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\ (n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, t\, u\, v : \sigma}
$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$
\begin{array}{rcl}
\mathbf{nat} & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
\underline{n} & := & \Lambda\alpha.\,\lambda z:\alpha.\,\lambda s:\alpha\to\alpha.\; s^n z \\
\mathbf{S}\,t & := & \Lambda\alpha.\,\lambda z:\alpha.\,\lambda s:\alpha\to\alpha.\; s\,(t\,\alpha\,z\,s) \\
\mathbf{iter}\,t\,u\,v & := & t\,\sigma\,u\,v
\end{array}
$$

so that

$$
\frac{}{\Gamma \vdash \underline{n} : \mathbf{nat}}\;(n \in \mathbb{N})
\qquad
\frac{\Gamma \vdash t : \mathbf{nat}}{\Gamma \vdash \mathbf{S}\,t : \mathbf{nat}}
\qquad
\frac{\Gamma \vdash t : \mathbf{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathbf{iter}\,t\,u\,v : \sigma}
$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

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so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\ (n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, t\, u\, v : \sigma}$$

Moreover

$$\mathtt{S}\, \underline{n} \ \rhd^* \ \underline{n+1}$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$\begin{array}{rcl} \mathtt{nat} & := & \forall\alpha.\Big(\alpha \longrightarrow (\alpha \to \alpha) \longrightarrow \alpha\Big) \\ \underline{n} & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s^n z \\ \mathtt{S}\, t & := & \Lambda\alpha.\ \lambda z : \alpha.\ \lambda s : \alpha \to \alpha.\ s\,(t\,\alpha\, z\, s) \\ \mathtt{iter}\, t\, u\, v & := & t\,\sigma\, u\, v \end{array}$$

so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\,(n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, t\, u\, v : \sigma}$$

Moreover

$$\begin{array}{rcl} \mathtt{S}\, \underline{n} & \rhd^* & \underline{n+1} \\ \mathtt{iter}\, \underline{0}\, u\, v & \rhd^* & u \end{array}$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

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\mathtt{S}\, t & := & \Lambda\alpha.\, \lambda z : \alpha.\, \lambda s : \alpha \to \alpha.\, s\,(t\, \alpha\, z\, s) \\
\mathtt{iter}\, t\, u\, v & := & t\, \sigma\, u\, v
\end{array}$$

so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\ (n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, t\, u\, v : \sigma}$$

Moreover

$$\begin{array}{rcl}
\mathtt{S}\, \underline{n} & \rhd^* & \underline{n+1} \\
\mathtt{iter}\, \underline{0}\, u\, v & \rhd^* & u \\
\mathtt{iter}\,(\mathtt{S}\, t)\, u\, v & =_\beta & v\,(\mathtt{iter}\, t\, u\, v)
\end{array}$$

Natural Numbers

Let (for $n \in \mathbb{N}$)

$$\begin{array}{rcl}
\mathtt{nat} & := & \forall \alpha. \Big(\alpha \longrightarrow (\alpha \to \alpha) \longrightarrow \alpha \Big) \\
\underline{n} & := & \Lambda \alpha.\, \lambda z : \alpha.\, \lambda s : \alpha \to \alpha.\, s^n z \\
\mathtt{S}\, t & := & \Lambda \alpha.\, \lambda z : \alpha.\, \lambda s : \alpha \to \alpha.\, s\,(t\, \alpha\, z\, s) \\
\mathtt{iter}\, t\, u\, v & := & t\, \sigma\, u\, v
\end{array}$$

so that

$$\frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\ (n \in \mathbb{N}) \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{S}\, t : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, t\, u\, v : \sigma}$$

Moreover

$$\begin{array}{rcl}
\mathtt{S}\, \underline{n} & \rhd^* & \underline{n+1} \\
\mathtt{iter}\, \underline{0}\, u\, v & \rhd^* & u \\
\mathtt{iter}\, (\mathtt{S}\, t)\, u\, v & =_\beta & v\, (\mathtt{iter}\, t\, u\, v)
\end{array}$$

Remark.

- The iterator $\mathtt{iter}$ can be polymorphic.
- The computational power of System $\mathcal{F}$ is huge: the definable functions $\mathtt{nat} \to \mathtt{nat}$ are exactly those provably total in second-order arithmetic !

Lists

Let

$$\texttt{list}(\tau) \quad := \quad \forall\alpha.\Big(\alpha \;\longrightarrow\; (\tau \to \alpha \to \alpha) \;\longrightarrow\; \alpha\Big)$$

Lists

Let

$$\begin{array}{rcl} \texttt{list}(\tau) & := & \forall\alpha.\Big(\alpha \longrightarrow (\tau \to \alpha \to \alpha) \longrightarrow \alpha\Big) \\ [t_0; \ldots; t_n] & := & \end{array}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \texttt{list}(\tau)}$$

Lists

Let

$$\begin{array}{rcl} \mathtt{list}(\tau) & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\tau \to \alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\ [t_0; \ldots; t_n] & := & \Lambda\alpha.\,\lambda e : \alpha.\,\lambda c : \tau \to \alpha \to \alpha.\; c\,t_n\,(\ldots(c\,t_0\,e)) \end{array}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \qquad \ldots \qquad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \mathtt{list}(\tau)}$$

Lists

Let

$$
\begin{array}{rcl}
\texttt{list}(\tau) & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\tau \to \alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
[t_0; \ldots; t_n] & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t_n\, (\ldots (c\, t_0\, e)) \\
\texttt{nil} & := &
\end{array}
$$

so that

$$
\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \texttt{list}(\tau)} \qquad \frac{}{\Gamma \vdash \texttt{nil} : \texttt{list}(\tau)}
$$

Lists

Let

$$\begin{array}{rcl}\texttt{list}(\tau) & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\tau \to \alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\ [t_0; \dots ; t_n] & := & \Lambda\alpha.\,\lambda e : \alpha.\,\lambda c : \tau \to \alpha \to \alpha.\; c\,t_n\,(\dots(c\,t_0\,e)) \\ \texttt{nil} & := & \Lambda\alpha.\,\lambda e : \alpha.\,\lambda c : \tau \to \alpha \to \alpha.\; e\end{array}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \quad \dots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \dots ; t_n] : \texttt{list}(\tau)} \qquad \frac{}{\Gamma \vdash \texttt{nil} : \texttt{list}(\tau)}$$

Lists

Let

$$\begin{aligned}
\mathbf{list}(\tau) &:= \forall\alpha.\Big(\alpha \longrightarrow (\tau \to \alpha \to \alpha) \longrightarrow \alpha\Big) \\
[t_0; \ldots; t_n] &:= \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\; c\, t_n\, (\ldots (c\, t_0\, e)) \\
\mathbf{nil} &:= \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\; e \\
\mathbf{cons}\, t\, \ell &:=
\end{aligned}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \mathbf{list}(\tau)} \qquad \frac{}{\Gamma \vdash \mathbf{nil} : \mathbf{list}(\tau)} \qquad \frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash \ell : \mathbf{list}(\tau)}{\Gamma \vdash \mathbf{cons}\, t\, \ell : \mathbf{list}(\tau)}$$

Lists

Let

$$\begin{array}{rcl} \texttt{list}(\tau) & := & \forall\alpha.\Big(\alpha \ \longrightarrow\ (\tau \to \alpha \to \alpha) \ \longrightarrow\ \alpha\Big) \\ [t_0; \ldots; t_n] & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t_n\, (\ldots(c\, t_0\, e)) \\ \texttt{nil} & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, e \\ \texttt{cons}\, t\, \ell & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t\, (\ell\, \alpha\, e\, c) \end{array}$$

so that

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Lists

Let

$$
\begin{array}{rcl}
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\mathtt{nil} & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, e \\
\mathtt{cons}\, t\, \ell & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t\, (\ell\, \alpha\, e\, c) \\
\mathtt{iter}\, \ell\, u\, v & := &
\end{array}
$$

so that

$$
\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \mathtt{list}(\tau)}
\qquad
\frac{}{\Gamma \vdash \mathtt{nil} : \mathtt{list}(\tau)}
\qquad
\frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash \ell : \mathtt{list}(\tau)}{\Gamma \vdash \mathtt{cons}\, t\, \ell : \mathtt{list}(\tau)}
$$

$$
\frac{\Gamma \vdash \ell : \mathtt{list}(\tau) \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \tau \to \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, \ell\, u\, v : \sigma}
$$

Lists

Let

$$
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\mathbf{cons}\, t\, \ell & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t\, (\ell\, \alpha\, e\, c) \\
\mathbf{iter}\, \ell\, u\, v & := & \ell\, \sigma\, u\, v
\end{array}
$$

so that

$$
\frac{\Gamma \vdash t_0 : \tau \quad \dots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \dots ; t_n] : \mathbf{list}(\tau)}
\qquad
\frac{}{\Gamma \vdash \mathbf{nil} : \mathbf{list}(\tau)}
\qquad
\frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash \ell : \mathbf{list}(\tau)}{\Gamma \vdash \mathbf{cons}\, t\, \ell : \mathbf{list}(\tau)}
$$

$$
\frac{\Gamma \vdash \ell : \mathbf{list}(\tau) \qquad \Gamma \vdash u : \sigma \qquad \Gamma \vdash v : \tau \to \sigma \to \sigma}{\Gamma \vdash \mathbf{iter}\, \ell\, u\, v : \sigma}
$$

Lists

Let

$$\begin{array}{rcl} \mathtt{list}(\tau) & := & \forall\alpha.\Big(\alpha \longrightarrow (\tau \to \alpha \to \alpha) \longrightarrow \alpha\Big) \\ [t_0; \ldots; t_n] & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t_n\, (\ldots (c\, t_0\, e)) \\ \mathtt{nil} & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, e \\ \mathtt{cons}\, t\, \ell & := & \Lambda\alpha.\, \lambda e : \alpha.\, \lambda c : \tau \to \alpha \to \alpha.\, c\, t\, (\ell\, \alpha\, e\, c) \\ \mathtt{iter}\, \ell\, u\, v & := & \ell\, \sigma\, u\, v \end{array}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \mathtt{list}(\tau)} \qquad \frac{}{\Gamma \vdash \mathtt{nil} : \mathtt{list}(\tau)} \qquad \frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \ell : \mathtt{list}(\tau)}{\Gamma \vdash \mathtt{cons}\, t\, \ell : \mathtt{list}(\tau)}$$

$$\frac{\Gamma \vdash \ell : \mathtt{list}(\tau) \quad \Gamma \vdash u : \sigma \quad \Gamma \vdash v : \tau \to \sigma \to \sigma}{\Gamma \vdash \mathtt{iter}\, \ell\, u\, v : \sigma}$$

Moreover

$$\mathtt{cons}\, t\, \mathtt{nil} \;\triangleright^*\; [t]$$

Lists

Let

$$
\begin{array}{rcl}
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\mathtt{cons}\,t\,\ell & := & \Lambda\alpha.\,\lambda e:\alpha.\,\lambda c:\tau\to\alpha\to\alpha.\; c\,t\,(\ell\,\alpha\,e\,c) \\
\mathtt{iter}\,\ell\,u\,v & := & \ell\,\sigma\,u\,v
\end{array}
$$

so that

$$
\frac{\Gamma\vdash t_0:\tau \quad \ldots \quad \Gamma\vdash t_n:\tau}{\Gamma\vdash [t_0;\ldots;t_n]:\mathtt{list}(\tau)}
\qquad
\frac{}{\Gamma\vdash \mathtt{nil}:\mathtt{list}(\tau)}
\qquad
\frac{\Gamma\vdash t:\tau \qquad \Gamma\vdash \ell:\mathtt{list}(\tau)}{\Gamma\vdash \mathtt{cons}\,t\,\ell:\mathtt{list}(\tau)}
$$

$$
\frac{\Gamma\vdash \ell:\mathtt{list}(\tau) \qquad \Gamma\vdash u:\sigma \qquad \Gamma\vdash v:\tau\to\sigma\to\sigma}{\Gamma\vdash \mathtt{iter}\,\ell\,u\,v:\sigma}
$$

Moreover

$$
\begin{array}{rcl}
\mathtt{cons}\,t\,\mathtt{nil} & \rhd^* & [t] \\
\mathtt{cons}\,t\,[t_0;\ldots;t_n] & \rhd^* & [t_0;\ldots;t_n;t]
\end{array}
$$

Lists

Let

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\texttt{nil} & := & \Lambda\alpha.\,\lambda e:\alpha.\,\lambda c:\tau\to\alpha\to\alpha.\, e \\
\texttt{cons}\,t\,\ell & := & \Lambda\alpha.\,\lambda e:\alpha.\,\lambda c:\tau\to\alpha\to\alpha.\, c\,t\,(\ell\,\alpha\,e\,c) \\
\texttt{iter}\,\ell\,u\,v & := & \ell\,\sigma\,u\,v
\end{array}$$

so that

$$\frac{\Gamma\vdash t_0:\tau \quad \ldots \quad \Gamma\vdash t_n:\tau}{\Gamma\vdash[t_0;\ldots;t_n]:\texttt{list}(\tau)} \qquad \frac{}{\Gamma\vdash\texttt{nil}:\texttt{list}(\tau)} \qquad \frac{\Gamma\vdash t:\tau \quad \Gamma\vdash\ell:\texttt{list}(\tau)}{\Gamma\vdash\texttt{cons}\,t\,\ell:\texttt{list}(\tau)}$$

$$\frac{\Gamma\vdash\ell:\texttt{list}(\tau) \quad \Gamma\vdash u:\sigma \quad \Gamma\vdash v:\tau\to\sigma\to\sigma}{\Gamma\vdash\texttt{iter}\,\ell\,u\,v:\sigma}$$

Moreover

$$\begin{array}{rcl}
\texttt{cons}\,t\,\texttt{nil} & \rhd^* & [t] \\
\texttt{cons}\,t\,[t_0;\ldots;t_n] & \rhd^* & [t_0;\ldots;t_n;t] \\
\texttt{iter}\,\texttt{nil}\,u\,v & \rhd^* & u
\end{array}$$

Lists

Let

$$\begin{array}{rcl}
\mathtt{list}(\tau) & := & \forall\alpha.\Big(\alpha \;\longrightarrow\; (\tau \to \alpha \to \alpha) \;\longrightarrow\; \alpha\Big) \\
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\mathtt{iter}\, \ell\, u\, v & := & \ell\, \sigma\, u\, v
\end{array}$$

so that

$$\frac{\Gamma \vdash t_0 : \tau \quad \ldots \quad \Gamma \vdash t_n : \tau}{\Gamma \vdash [t_0; \ldots; t_n] : \mathtt{list}(\tau)} \qquad \frac{}{\Gamma \vdash \mathtt{nil} : \mathtt{list}(\tau)} \qquad \frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \ell : \mathtt{list}(\tau)}{\Gamma \vdash \mathtt{cons}\, t\, \ell : \mathtt{list}(\tau)}$$

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- For polymorphic lists, just take $\texttt{list} := \forall\alpha.\,\texttt{list}(\alpha)$.
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  It is then easy to define a polymorphic **map** function.
- Any first-order structure over a finite signature is representable (with its iterator).

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Subject Reduction.

- If $t \rhd u$ and $\Gamma \vdash t : \sigma$ then $\Gamma \vdash u : \sigma$.

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$$\forall u \Big(t \rhd u \;\Rightarrow\; u \in \mathcal{SN} \Big) \quad\Longrightarrow\quad t \in \mathcal{SN}$$

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Theorem (Girard '71)

If t is typable then $t \in \mathcal{SN}$.

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Remark.

- There is no (natural) set-theoretic model of System $\mathcal{F}$.

Curry Style System $\mathcal{F}$

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Undecidability of Typing.

- It is undecidable whether a term is typable (and whether it has a given type).

The Erasure Map

The erasure map $|-|$ from Church to Curry terms is defined by induction as follows:

$$
\begin{array}{rcl}
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Lemma

Consider a (possibly untyped) Church term t. If $|t|$ is S.N., then t is S.N.

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Corollary

If Curry's style System $\mathcal{F}$ is S.N., then Church's style System $\mathcal{F}$ is S.N.