Course CR03

Logical Foundations of Programming Languages

Homework

The solutions must be sent either in paper or by email at `colin.riba@ens-lyon.fr` before the course of 28th of October. The questions marked with an asterisk $(*)$ are optional and will not be graded.

1 Algebraicity and Scott Domains

Basic definitions from the course are recalled in Appendix A. Our goal here is to study the following notion.

Definition 1.1 (Finite Element of a CPO)**.** *Let $(A, \leq)$ be a CPO. We say that $a \in A$ is* **finite** *if for each directed $X \subseteq A$ such that $a \leq \bigvee X$, we have $a \leq b$ for some $b \in X$.*

Definition 1.2 (Algebraic CPO)**.** *A CPO $(A, \leq)$ is* **algebraic** *if for every element $a \in A$, the set*

$$\{b \in A \mid b \text{ is finite and } \leq a\}$$

is directed and has supremum a.

Question 1. Q. 1

(1) Show that each element of $[\![\mathtt{nat}]\!]$ is finite.

(2) Show that $[\![\mathtt{nat}]\!]$ is algebraic.

We are going to prove that each $[\![\tau]\!]$ is algebraic. This relies on the following fundamental notion.

Definition 1.3. *Consider two types σ and τ, and two* **finite** *elements $a \in [\![\sigma]\!]$ and $b \in [\![\tau]\!]$. The* **step function** *$(a \Rightarrow b) : [\![\sigma]\!] \to [\![\tau]\!]$ is defined as*

$$(a \Rightarrow b)(x) \quad := \quad \begin{cases} b & \text{if } a \sqsubseteq_{[\![\sigma]\!]} x \\ \bot_{[\![\tau]\!]} & \text{otherwise} \end{cases}$$

Question 2. *Consider two types σ and τ, and two* **finite** *elements $a \in [\![\sigma]\!]$ and $b \in [\![\tau]\!]$.* Q. 2

(1) Show that $(a \Rightarrow b)$ is Scott-continuous.

(2) Show that $(a \Rightarrow b)$ is a finite element of $[\![\sigma \to \tau]\!]$.

In order to show that the interpretation of a function type $\sigma \to \tau$ is an algebraic CPO, we need a description of the finite elements of $[\![\sigma \to \tau]\!]$. The latter will turn out to be some finite supremums of steps functions. But not every finite set of step functions admits a supremum. We shall thus be concerned with the existence of **some** finite supremums.

Definition 1.4 (Bounded-Complete CPO)**.** *A CPO $(A, \leq)$ is* **bounded-complete** *if for each* **finite subset** *$X \subseteq A$, if X has an upper bound in A (i.e. if there is some $a \in A$ such that $a \geq x$ for all $x \in X$), then X has a supremum (i.e. a least upper bound) $\bigvee X \in A$.*

Question 3. *Show that for each type τ, the CPO $[\![\tau]\!]$ is bounded-complete.* Q. 3

Question 4. *Fix two types σ and τ. Consider, for some $k \geq 1$, some finite elements $a_1, \ldots, a_k \in [\![\sigma]\!]$ and $b_1, \ldots, b_k \in [\![\tau]\!]$. Show that the function* Q. 4

$$(a_1 \Rightarrow b_1) \sqcup \cdots \sqcup (a_k \Rightarrow b_k)$$

is defined if for each $I \subseteq \{1, \ldots, k\}$ such that $\{a_i \mid i \in I\}$ has an upper bound, the set $\{b_i \mid i \in I\}$ has an upper bound as well.

Definition 1.5. *Fix two types σ and τ. Given $f \in [\![\sigma \to \tau]\!]$, define*

$$\Downarrow(f) \quad := \quad \{(a \Rightarrow b) \mid a \in [\![\sigma]\!] \text{ and } b \in [\![\tau]\!] \text{ are finite and } (a \Rightarrow b) \sqsubseteq f\}$$

Question 5. *Let $f \in [\![\sigma \to \tau]\!]$. Show that for each finite subset $F \subseteq \Downarrow(f)$, the function $\bigsqcup F$ is defined and finite.* Q. 5

We are now going to see that for each $f \in [\![\sigma \to \tau]\!]$ we have

$$f \quad = \quad \bigsqcup\{\bigsqcup F \mid F \text{ is a finite subset of } \Downarrow(f)\}$$

Question 6. *Let $f \in [\![\sigma \to \tau]\!]$, where $[\![\sigma]\!]$ and $[\![\tau]\!]$ are* **algebraic**. Q. 6

(1) Show that the set

$$\{\bigsqcup F \mid F \text{ is a finite subset of } \Downarrow(f)\}$$

is directed has supremum f.

(2) Assume that f is a finite element of $[\![\sigma \to \tau]\!]$. Show that $f = \bigsqcup F$ for some finite set F of step functions.

Question 7. *Show that for each τ, the CPO $[\![\tau]\!]$ is algebraic.* Q. 7

2 Definability in PCF

We are now going to discuss whether some $a \in [\![\tau]\!]$ can be defined by a PCF term $\vdash t : \tau$. We rely on some results of §1.

Definition 2.1. *We say that an element $a \in [\![\tau]\!]$ is* **definable** *if there is some $\vdash t : \tau$ such that $[\![t]\!] = a$.*

Question* 8. *Q. 8

(1) Show that for each type τ, the least element $\bot_{[\![\tau]\!]}$ of $[\![\tau]\!]$ is definable.

(2) Show that each element of $[\![\mathtt{nat}]\!]$ is definable.

Question* 9. *Give some $f \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$ which is not definable.* *Q. 9

We shall only be interested in the definability of the **finite** $a \in [\![\tau]\!]$ (in the sense of Def. 1.1).

Question* 10. *Show that each step function $(a \Rightarrow b) \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$ is definable.* *Q. 10

Let $f \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$. We are going to show that f is definable whenever it is finite. This relies on the following. Define the **graph** of f to be the set

$$\mathrm{Gph}(f) \quad := \quad \{(n, f(n)) \mid n \in \mathbb{N} \text{ and } f(n) \in \mathbb{N}\}$$

Question* 11. *Consider a function $f \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$ which is strict and different from $\bot_{[\![\mathtt{nat}\to\mathtt{nat}]\!]}$. Show that the following are equivalent:* *Q. 11

(a) $\mathrm{Gph}(f)$ *is a finite set.*

(b) f *is of the form*

$$(n_1 \Rightarrow m_1) \sqcup \cdots \sqcup (n_k \Rightarrow m_k)$$

for some $k \geq 1$ *and with* $n_i \neq n_j$ *whenever* $i \neq j$.

(c) f *is finite.*

Question* 12. *Give an* $f \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$ *which is finite, different from* $\bot_{[\![\mathtt{nat}\to\mathtt{nat}]\!]}$ *but such that* $\mathrm{Gph}(f)$ *is infinite.* *Q. 12

Question* 13. *Show that if* $f \in [\![\mathtt{nat} \to \mathtt{nat}]\!]$ *is finite, then* f *is definable.* *Q. 13

We would like to extend this result to each type of PCF. This is however **not** possible, because of the famous "parallel or" function.

Definition 2.2 (Parallel Or)**.** *The function* $\mathtt{por} : [\![\mathtt{nat}]\!] \to [\![\mathtt{nat}]\!] \to [\![\mathtt{nat}]\!]$ *is defined as*

$$\mathtt{por}\, a\, b \quad := \quad \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0 \\ 1 & \text{if } a = 1 \text{ and } b = 1 \\ \bot_{[\![\mathtt{nat}]\!]} & \text{otherwise} \end{cases}$$

Question* 14. *Show that* $\mathtt{por}$ *is a finite element of* $[\![\mathtt{nat}]\!] \to [\![\mathtt{nat}]\!] \to [\![\mathtt{nat}]\!]$. *Q. 14

It is well-known that $\mathtt{por}$ is not definable in PCF. However, if we extend PCF with a constant for $\mathtt{por}$ then we can obtain a language in which every finite element is definable.

The bounded-complete algebraic CPOs are called **Scott domains**. They are of fundamental importance for denotational semantics, in particular for the following reason. Say that two (closed) terms t, u of type τ are **observationally equivalent** (notation $t \equiv u$) if for every closed term C of type $\tau \to \mathtt{nat}$,

$$\forall n \in \mathbb{N}\left(C t \rhd^* \underline{n} \quad \Longleftrightarrow \quad C u \rhd^* \underline{n}\right)$$

Consider an extension PCF^* of PCF, together with an interpretation $[\![-]\!]^*$ which takes each type τ of PCF^* to a Scott domain $[\![\tau]\!]^*$ and each closed term t of type τ to some $[\![t]\!]^* \in [\![\tau]\!]^*$. We assume tat $[\![\mathtt{nat}]\!]^* = [\![\mathtt{nat}]\!]$ and that $[\![\sigma \to \tau]\!]^*$ is a set of Scott-continuous functions $[\![\sigma]\!]^* \to [\![\tau]\!]^*$ ordered pointwise. We furthermore require $[\![-]\!]^*$ to be compositional, to be preserved by evaluation and to be computationaly adequate. The algebraicity of Scott domains implies that if the finite elements of each $[\![\tau]\!]^*$ are definable, then $[\![-]\!]^*$ is **fully abstract**, in the sense that for all terms t, u of the same type, we have

$$t \equiv u \quad \Longleftrightarrow \quad [\![t]\!]^* = [\![u]\!]^*$$

Question* 15. *Prove the last assertion above.* *Q. 15

A PCF and its Denotational Semantics

We recall here the definition of the language PCF and its Scott-continuous denotational semantics. The **types** of PCF are given by the grammar:

$$\tau, \sigma \quad ::= \quad \texttt{nat} \quad | \quad \sigma \to \tau$$

The **terms** of PCF are given by the grammar:

$$\begin{array}{rcl} t, u & ::= & x \quad | \quad \lambda x : \sigma . t \quad | \quad t\, u \quad | \quad Y^\sigma \quad | \quad t\texttt{+1} \quad | \quad t\texttt{-1} \quad | \quad \underline{n} \\ & | & \texttt{if } t \texttt{ then } u \texttt{ else } v \end{array}$$

where $n \in \mathbb{N}$ (so that we have a **numeral** $\underline{n}$ for each natural number $n \in \mathbb{N}$). The evaluation relation $\triangleright$ is defined by the following rules:

$$\frac{}{(\lambda x : \sigma . t) u \quad \triangleright \quad t[u/x]} \qquad \frac{}{\underline{n}\texttt{+1} \quad \triangleright \quad \underline{n+1}} \qquad \frac{}{\underline{n+1}\texttt{-1} \quad \triangleright \quad \underline{n}} \qquad \frac{}{\underline{0}\texttt{-1} \quad \triangleright \quad \underline{0}}$$

$$\frac{}{\texttt{if } \underline{0} \texttt{ then } u \texttt{ else } v \quad \triangleright \quad u} \qquad \frac{}{\texttt{if } \underline{n+1} \texttt{ then } u \texttt{ else } v \quad \triangleright \quad v} \qquad \frac{}{Y t \quad \triangleright \quad t(Y t)}$$

$$\frac{t \quad \triangleright \quad u}{t\, v \quad \triangleright \quad u\, v} \qquad \frac{t \quad \triangleright \quad u}{t\texttt{+1} \quad \triangleright \quad u\texttt{+1}} \qquad \frac{t \quad \triangleright \quad u}{t\texttt{-1} \quad \triangleright \quad u\texttt{-1}}$$

$$\frac{t \quad \triangleright \quad u}{\texttt{if } t \texttt{ then } v \texttt{ else } w \quad \triangleright \quad \texttt{if } u \texttt{ then } v \texttt{ else } w}$$

We write $\triangleright^*$ for the reflexive-transitive closure of $\triangleright$. The **typing rules** of PCF are the following:

$$\frac{(x : \sigma) \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x : \sigma . t : \tau} \qquad \frac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t u : \tau}$$

$$\frac{}{\Gamma \vdash \underline{n} : \texttt{nat}} \qquad \frac{\Gamma \vdash t : \texttt{nat}}{\Gamma \vdash t\texttt{+1} : \texttt{nat}} \qquad \frac{\Gamma \vdash t : \texttt{nat}}{\Gamma \vdash t\texttt{-1} : \texttt{nat}}$$

$$\frac{}{\Gamma \vdash Y^\sigma : (\sigma \to \sigma) \to \sigma} \qquad \frac{\Gamma \vdash t : \texttt{nat} \qquad \Gamma \vdash u : \texttt{nat} \qquad \Gamma \vdash v : \texttt{nat}}{\Gamma \vdash \texttt{if } t \texttt{ then } u \texttt{ else } v : \texttt{nat}}$$

We recall a basic property of PCF.

Lemma A.1 (Subject Reduction)**.** *If $\Gamma \vdash t : \tau$ and $t \triangleright u$ then $\Gamma \vdash u : \tau$.*

We now turn to the Scott-continuous denotational semantics of PCF.

Definition A.2 (CPO)**.** *Let $(A, \leq)$ be a poset.*

- *Fix some $X \subseteq A$. The* **supremum** *$\bigvee X \in A$ of X is (if it exists) the least upper bound of X:*
$$\forall a \in X.\ a \leq \bigvee X \qquad \textit{and} \qquad \forall b \in A \Big((\forall a \in X.\ a \leq b) \quad \Longrightarrow \quad \bigvee X \leq b \Big)$$
- *A subset $X \subseteq A$ is* **directed** *if it is non-empty and if for all $a, b \in X$ there is some $c \in X$ such that $a \leq c$ and $b \leq c$.*
- *We say that A is a* **complete partial order** *(CPO) if A has a least element (often denoted $\bot_A$ or $\bot$) and if every directed $X \subseteq A$ has a supremum $\bigvee X \in A$.*

Definition A.3 (Scott-Continuous Function). *Let $(A, \leq_A)$ and $(B, \leq_B)$ be CPOs. A function $f : A \to A$ is* **Scott-continuous** *if it is monotone ($a \leq_A a'$ implies $f(a) \leq_B f(a')$) and if for each directed $X \subseteq A$ we have*

$$f(\bigvee X) \quad = \quad \bigvee f(X)$$

Definition A.4 (Interpretation of PCF). *We define the CPO $(\llbracket \tau \rrbracket, \sqsubseteq_\tau)$ by induction on the type τ as follows:*

- $\llbracket \mathtt{nat} \rrbracket = \mathbb{N} + \{\bot\}$ *and $a \sqsubseteq_{\mathtt{nat}} b$ iff either $a = b$ or $a = \bot$.*
- $\llbracket \sigma \to \tau \rrbracket$ *is the set of Scott-continuous functions from $(\llbracket \sigma \rrbracket, \sqsubseteq_\sigma)$ to $(\llbracket \tau \rrbracket, \sqsubseteq_\tau)$ equipped with the* **pointwise** *ordering:*

$$f \sqsubseteq_{\sigma \to \tau} g \qquad \Longleftrightarrow \qquad \forall a \in \llbracket \sigma \rrbracket.\ f(a) \sqsubseteq_\tau g(a)$$

Each term $\vdash t : \sigma$ is interpreted by an element $\llbracket t \rrbracket \in \llbracket \sigma \rrbracket$.

We recall the interpretation of the constants of PCF. *Each numeral $\vdash \underline{n} : \mathtt{nat}$ is interpreted by $\llbracket \underline{n} \rrbracket := n \in \llbracket \mathtt{nat} \rrbracket$. The other constants of* PCF *are interpreted by the following Scott-continuous functions:*

- $\llbracket Y^\sigma \rrbracket : \llbracket \sigma \to \sigma \rrbracket \to \llbracket \sigma \rrbracket$ *is given by*

$$\llbracket Y^\sigma \rrbracket(f) \quad := \quad \bigvee_{n \in \mathbb{N}} f^n(\bot)$$

- $\llbracket (-)\mathtt{+1} \rrbracket : \llbracket \mathtt{nat} \rrbracket \to \llbracket \mathtt{nat} \rrbracket$ *is given by*

$$\llbracket (-)\mathtt{+1} \rrbracket(a) \quad = \quad \llbracket a\mathtt{+1} \rrbracket \quad := \quad \begin{cases} \bot & \textit{if } a = \bot \\ a + 1 & \textit{if } a \in \mathbb{N} \end{cases}$$

- $\llbracket (-)\mathtt{-1} \rrbracket : \llbracket \mathtt{nat} \rrbracket \to \llbracket \mathtt{nat} \rrbracket$ *is given by*

$$\llbracket (-)\mathtt{-1} \rrbracket(a) \quad = \quad \llbracket a\mathtt{-1} \rrbracket \quad := \quad \begin{cases} \bot & \textit{if } a = \bot \\ 0 & \textit{if } a = 0 \\ a - 1 & \textit{if } a > 0 \end{cases}$$

- $\llbracket \mathtt{if}\ (-)\ \mathtt{then}\ (-)\ \mathtt{else}\ (-) \rrbracket : \llbracket \mathtt{nat} \rrbracket \times \llbracket \mathtt{nat} \rrbracket \times \llbracket \mathtt{nat} \rrbracket \to \llbracket \mathtt{nat} \rrbracket$ *is given by*

$$\llbracket \mathtt{if}\ (-)\ \mathtt{then}\ (-)\ \mathtt{else}\ (-) \rrbracket(a, b, c) \quad = \quad \llbracket \mathtt{if}\ a\ \mathtt{then}\ b\ \mathtt{else}\ c \rrbracket \quad := \quad \begin{cases} \bot & \textit{if } a = \bot \\ b & \textit{if } a = 0 \\ c & \textit{if } a > 0 \end{cases}$$

We recall some basic properties.

Lemma A.5. *If $\vdash t : \tau$ and $t \rhd^* u$ then $\llbracket t \rrbracket = \llbracket u \rrbracket$.*

Theorem A.6 (Computational Adequacy). *Given $\vdash t : \mathtt{nat}$, for each $n \in \mathbb{N}$ we have*

$$t \rhd^* \underline{n} \quad \Longleftrightarrow \quad \llbracket t \rrbracket = n$$

*   *
*