Logical Foundations of Programming Languages

Olivier LAURENT & Colin RIBA

LIP - ENS de Lyon

Course 01

A Naive Introduction

A First Example

Consider

foo(f:int->int):
 return f((10^10)!)*0

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- ► They are **not** equivalent if **f** does not terminate.
- > They are **not** equivalent in terms of exection time.

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- ► They are **not** equivalent if **f** does not terminate.
- > They are **not** equivalent in terms of exection time.
- They are equivalent if f behaves as a function

$$\llbracket \texttt{f} \rrbracket$$
 : $\llbracket \texttt{int} \rrbracket \longrightarrow \llbracket \texttt{int} \rrbracket$

where [int] is the set of integers \mathbb{Z} .

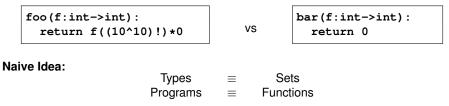
Mathematical Models of Programming Languages

foo(f:int->int):
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Mathematical Models of Programming Languages



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Curry-Howard-Lambek Corresponde		
Prog. Languages	Logic	
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Logical Foundations of Programming Languages

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Logical Foundations of Programming Languages

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Key	ywords.				

► Type systems, proof theory, denotational semantics, category theory.

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Now:

A naive introduction to some basic ideas.

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Assume types, say int, bool, are to be interpreted as sets [int], [bool].

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Question.

Can we assume

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$$bool$$
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Consider the non-terminating program

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loop (b:bool):
  while true:
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We shall therefore represent divergence and assume

$$\llbracket \texttt{bool} \rrbracket := \{ \bot, \texttt{true}, \texttt{false} \}$$
 (\bot "=" divergence

We can then have, as expected:

$$[\texttt{loop}](a) = \bot$$
 (for all $a \in [\texttt{bool}]$)

Motivation.

A simple language to discuss

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The Language of Finitary PCF.

- au, σ ::= bool $\mid \sigma
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 - t, u ::= $t u \mid fun x \rightarrow t \mid true \mid false \mid if then u else v \mid \Omega$

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We assume *call-by-name* evaluation:

$$(\texttt{fun} x o t) u = t[u/x]$$

if true then t else $u = t$
if false then t else $u = u$

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or_l := fun a, b -> if a then a else b

vs

or_r := fun a, b ->
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Example.

Consider, for $b \in \{true, false\}$, the program

```
taste_b := fun f ->
if f(true, Ω) and
f(Ω, true) and
not(f(false, false))
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Questions.

- Do we have [[taste_true]] = [[taste_false]] ?
- Are taste_true and taste_false equivalent ?

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Motivation.

- Extend Finitary PCF with general recursion (= fixpoint combinator).
- Mathematically cleaner if an infinite type is assumed (say the natural numbers).

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 au, σ ::= ... | nat

 $t, u ::= \dots | t+1 | t-1 | z? | Y | \underline{n}$ (for each $n \in \mathbb{N}$)

► Y is a *fixpoint* combinator:

$$Yt = t(Yt)$$

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can be defined as

where

add_rec := fun f, x, y -> if (z? x) then y else (f x-1 y)+1

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Traditional Solution.

► Restrict S → S to the continuous functions for a suitable topology (cpos, Scott domains, etc).

Gödel's System T.

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- ► A kernel language without recursion.

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Indicative Outline

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Gödel's System T.

Set-Theoretic Semantics.

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Set-Theoretic Semantics.

Categorical Semantics of the Simply Typed $\lambda\text{-Calculus.}$

- Basic Notions of Category Theory.
- Cartesian Closed Categories.
- (Curry-Howard Correspondence.)

Logical Foundations of Programming Languages

(C. RIBA)

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PCF.

- CPOs and Scott-Continuity.
- Denotational Semantics.

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PCF.

- CPOs and Scott-Continuity.
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Category Theory.

- Adjunctions.
- Monads.

. . .