Logical Foundations of Programming Languages

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LIP - ENS de Lyon

Course 05

The Language PCF

The Syntax of PCF

Motivation.

- A simple functional language with general recursion.

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Typing Rules.

- Adaptation of System T with

$$\frac{}{\Gamma \vdash Y^\sigma : (\sigma \to \sigma) \to \sigma} \qquad \frac{}{\Gamma \vdash \underline{n} : \mathtt{nat}}\ (n \in \mathbb{N})$$

$$\frac{\Gamma \vdash t : \mathtt{nat} \qquad \Gamma \vdash u : \mathtt{nat} \qquad \Gamma \vdash v : \mathtt{nat}}{\Gamma \vdash \mathtt{if}\ t\ \mathtt{then}\ u\ \mathtt{else}\ v : \mathtt{nat}} \qquad \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash t\mathtt{-1} : \mathtt{nat}}$$

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Weak Head Reduction.

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$$\frac{}{(\lambda x : \sigma.t)u \;\triangleright\; t[u/x]} \qquad \frac{}{\underline{n}\texttt{+1} \;\triangleright\; \underline{n+1}} \qquad \frac{}{\underline{n+1}\texttt{-1} \;\triangleright\; \underline{n}} \qquad \frac{}{\underline{0}\texttt{-1} \;\triangleright\; \underline{0}}$$

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Congruence Rules.

$$\frac{t \;\triangleright\; u}{t\,v \;\triangleright\; u\,v} \qquad \frac{t \;\triangleright\; u}{t\mathbf{+1} \;\triangleright\; u\mathbf{+1}} \qquad \frac{t \;\triangleright\; u}{t\mathbf{-1} \;\triangleright\; u\mathbf{-1}}$$

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Substitution.

- If $\Gamma, x : \sigma \vdash t : \tau$ and $\Gamma \vdash u : \sigma$ then $\Gamma \vdash t[u/x] : \tau$.

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Normal Forms of Type `nat`.

- If $\vdash t : \mathtt{nat}$ with t in normal form w.r.t. $\rhd$, then $t = \underline{n}$ for some $n \in \mathbb{N}$.

Examples

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- For each type σ we let

$$\Omega^\sigma \quad := \quad Y^\sigma(\lambda x : \sigma.x)$$

  so that $\Omega \triangleright^+ \Omega \triangleright^+ \ldots$.

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- Note that

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- Because of $\Omega^{\mathtt{nat}}$, we let

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Difficulty:
$$[\![Y^\sigma]\!] \;:\; ([\![\sigma]\!] \to [\![\sigma]\!]) \;\longrightarrow\; [\![\sigma]\!]$$

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$$\begin{aligned} [\![\mathtt{add\_rec}]\!]\ \bot\ a\ b \quad &= \quad [\![\mathtt{if}\ a\ \mathtt{then}\ b\ \mathtt{else}\ \Omega]\!] \\ [\![\mathtt{add\_rec}]\!]\ ([\![\mathtt{add\_rec}]\!]\bot)\ a\ b \quad &= \\ &[\![\mathtt{if}\ a\ \mathtt{then}\ b\ \mathtt{else}\ (\mathtt{if}\ a\mathtt{-1}\ \mathtt{then}\ b\ \mathtt{else}\ \Omega)\mathtt{+1}]\!] \end{aligned}$$

- We obtain, for $k, m \in \mathbb{N}$ and $n > 0$:

$$([\![\mathtt{add\_rec}]\!]^n\bot)\ k\ m = \begin{cases} k+m & \text{if } k < n \\ \bot & \text{otherwise} \end{cases}$$

- This suggests

$$[\![\mathtt{add}]\!]\ a\ b \quad = \quad [\![Y\mathtt{add\_rec}]\!]\ a\ b \quad := \quad \bigvee([\![\mathtt{add\_rec}]\!]^n\bot)\ a\ b$$

Idea:

- $a \in [\![\tau]\!]$ is "more defined" than $\bot_\tau \in [\![\tau]\!]$.
- $([\![\mathtt{add\_rec}]\!]^{n+1}\bot)$ is "more defined" than $([\![\mathtt{add\_rec}]\!]^n\bot)$.

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Definition (The Information Order)

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- $a \sqsubseteq_{\mathtt{nat}} b$ iff $a = \bot$ or $a = b$.
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(b) We will moreover require a form of continuity.