Alternating Tree Automata

This homework has to be returned to Colin Riba (possibly by email colin.riba@ens-lyon.fr) on Thursday 18th Dec. 2014.

Questions marked with an asterisk (*) are supposed to be more difficult.

This homework is about alternating tree automata. The goal is to show some of their basic properties: direct proofs of closure under Boolean operations and equivalence with non-deterministic tree automata.

1 Preliminaries

Notations. We write $\Sigma$ for alphabets i.e. for finite non-empty sets, and $T^{\Sigma}_*$ for the set of full binary trees on $\Sigma$, i.e. the set of all $t : 2^* \rightarrow \Sigma$.

Given a set $S \subseteq A \times B$, we write $\text{proj}_1(S) \subseteq A$ and $\text{proj}_2(S) \subseteq B$ for resp. the first and second projections of $S$:

$$\text{proj}_1(S) := \{ a \in A \mid \exists b \in B, (a,b) \in S \}$$
$$\text{proj}_2(S) := \{ b \in B \mid \exists a \in A, (a,b) \in S \}$$

Games. Recall that a graph game $G$ is a bipartite graph together with a winning condition: $G = (V_0, V_1, E, W)$ with $V_0 \cap V_1 = \emptyset$, $E \subseteq (V_0 \times V_1) \cup (V_1 \times V_0)$ and $W \subseteq V^\omega$ where $V = V_0 \cup V_1$. Let $W_0 := W$ and $W_1 := V^\omega \setminus W$.

Given $v_0 \in V_0$, a play from $v_0$ is a finite or infinite sequence $\rho \in V^* \cup V^\omega$ such that $\rho(0) = v_0$ and $(\rho(i), \rho(i+1)) \in E$ whenever $i + 1$ is in the domain of $\rho$.

A Player 0 strategy from $v_0 \in V_0$ is a set $\sigma$ of plays from $v_0$, which is closed under prefix ($p \in \sigma$ and $p'$ prefix of $p$ imply $p' \in \sigma$) and which branches only at even-length plays: if $pv, pv' \in \sigma$, with $p$ even-length then $v = v'$. Similarly, a Player 1 strategy from $v_0$ is a set $\tau$ of plays from $v_0$, which is closed under prefix and which branches only at odd-length plays: if $pv, pv' \in \tau$, with $p$ odd-length then $v = v'$. 
We say that a Player 0 strategy $\sigma$ is positional if given even-length plays $p,p'$ ending in the same node, for all $v,v' \in V_1$, if $pv,p'v' \in \sigma$ then $v = v'$.

Similarly, a Player 1 strategy $\tau$ is positional if given odd-length plays $p,p'$ ending in the same node, for all $v,v' \in V_0$, if $pv,p'v' \in \tau$ then $v = v'$.

Fix $v_0 \in V_0$ and consider a finite play $p$ from $v_0$. Assume that $p$ is maximal in $G$ (i.e. there is no play $q$ from $v_0$ strictly extending $p$). If it is Player $i$'s turn to play, then Player $i$ can actually not play and hence should lose. We thus say that a maximal finite $p$ is winning for Player 0 iff $p$ ends with a vertex in $V_1$, and that $p$ is winning for Player 1 otherwise. Consider now an infinite path $\pi$ in $G$ from $v_0$. We say that $\pi$ is winning for Player $i$ if $\pi \in W_i$.

A Player $i$ strategy $\sigma_i$ from $v_0 \in V_0$ is winning (for Player $i$) if all its finite maximal plays are winning for Player $i$, and for all infinite play $\pi$ such that $\pi(0) \cdot \ldots \cdot \pi(n) \in \sigma$ for all $n \in \mathbb{N}$, we have $\pi \in W_i$. The game $G$ from $v_0$ is determined if either Player 0 or Player 1 has a winning strategy from $v_0$.

### 2 Non-Deterministic Tree Automata

We recall here the definition of tree automata. Fix an alphabet $\Sigma$. Recall that a non-deterministic tree automaton on $\Sigma$ is a tuple $A = (Q, \Delta, q_\iota, \Omega)$ where $Q$ is the finite set of states, $q_\iota \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation and $\Omega \subseteq Q^\omega$ is the acceptance condition.

Recall also that a run of $A$ on $t \in T_\Sigma$ is a tree $\rho \in T_Q$, such that $\rho(\varepsilon) = q'$ and $(\rho(p), t(p), \rho(p0), \rho(p1)) \in \Delta$ for all $p \in 2^*$. We say that $\rho$ is accepting if $\rho|_\pi \in \Omega$ for all path $\pi \in 2^\omega$; and we let $L(A)$ be the set of $t \in T_\Sigma$ accepted by $A$, i.e. on which $A$ has an accepting run.

We will consider the following modified version of the acceptance game. Given $A$ and $t \in T_\Sigma$, let

$$\Gamma(A,t) := (V_0, V_1, E, W)$$

where:

- $V_0 = 2^* \times Q$,
- $V_1 = 2^* \times (Q \times Q)$,
- the edge relation $E$ is given by:
  
  in $V_0 \times V_1$: from $(p, q)$ to $(p, q_0, q_1)$ if $(q, t(p), q_0, q_1) \in \Delta$
  in $V_1 \times V_0$: from $(p, q_0, q_1)$ to $(p0, q_0)$
  from $(p, q_0, q_1)$ to $(p1, q_1)$
- $W$ is the set of
  $$(\varepsilon, q_0) \cdot (\varepsilon, q_1^0, q_1^1) \cdot \ldots \cdot (p_n, q_n) \cdot (p_n, q_n^0, q_n^1) \cdot \ldots$$
  such that $q_0 \ldots q_n \ldots \in \Omega$.  

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Convention 2.1. We write $\Gamma(A, t)$ for the game $\Gamma(A, t)$ from the node $(\varepsilon, q') \in V_0$.

Question 2.2. Given $A$ and $t \in T_{\Sigma}$, show that:

(i) Player 0 wins $\Gamma(A, t)$ iff $A$ accepts $t$,
(ii) if $\Gamma(A, t)$ is determined, then Player 1 wins $\Gamma(A, t)$ iff $A$ rejects $t$.

3 Alternating Tree Automata

Alternating tree automata generalize non-deterministic tree automata by generalizing their transition relation.

Definition 3.1 (Alternating Tree Automaton). Let $\Sigma$ be an alphabet. An alternating tree automaton on $\Sigma$ is a tuple $A = (Q, \Delta, q^i, \Omega)$ where $Q$ is the finite set of states, $\Delta \subseteq Q \times \Sigma \times \mathcal{P}(Q \times 2)$ is the transition relation, $q^i \in Q$ is the initial state and $\Omega \subseteq Q^\omega$ is the acceptance condition.

The Acceptance Game. The acceptance game for alternating automata allows more general positions for Player 1. For non-deterministic automata, from $(p, q_0, q_1) \in V_1$, Player 1 essentially chooses a direction $d \in \{0, 1\}$, the corresponding state $q_d$ in $(q_d, d)$ being determined by $d$. In the acceptance games for alternating automata, Player 1’s positions contain sets of pairs $f \in \mathcal{P}(Q \times 2)$, in which he has to choose some $(q, d)$. On the other hand, Player 0’s positions will still be pairs $(p, q) \in 2^* \times Q$. It follows that Player’s 0 moves are now from $Q \times 2$ to $\mathcal{P}(Q \times 2)$.

Let $A = (Q, \Delta, q^i, \Omega)$ be an alternating tree automaton on the alphabet $\Sigma$. Fix $t \in T_{\Sigma}$. The acceptance game $\Gamma_{\text{Alt}}(A, t)$ is the graph game $(V_0, V_1, E, W)$ defined as follows:

- $V_0 = 2^* \times Q$,
- $V_1 = 2^* \times \mathcal{P}(Q \times 2)$,
- the edge relation $E$ is given by:

  in $V_0 \times V_1$: from $(p, q)$ to $(p, f)$ if $(q, t(p), f) \in \Delta$

  in $V_1 \times V_0$: from $(p, f)$ to $(p.d, q)$ if $(q, d) \in f$

- for the winning condition, we let

\[ (p_0, q_0)(p_0, f_0)(p_1, q_1)(p_1, f_1) \cdots (p_n, q_n)(p_n, f_n) \cdots \in W \]

iff $q_0 q_1 \cdots q_n \cdots \in \Omega$.

Convention 3.1. We write $\Gamma_{\text{Alt}}(A, t)$ for the game $\Gamma_{\text{Alt}}(A, t)$ from the node $(\varepsilon, q') \in V_0$. 

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**Definition 3.2** (Acceptance for Alternating Automata). Let $A = (Q, \Delta, q^f, \Omega)$ be an alternating tree automaton on the alphabet $\Sigma$.

We say that $A$ accepts $t \in T_\Sigma$ if Player 0 has winning strategy in the game $\Gamma_{\text{Alt}}(A, t)$, and we let $L(A)$ be the set of trees accepted by $A$.

Let us consider an example.

**Question 3.2** (Example). Consider the following automaton $A$ on the alphabet $\Sigma = \{a, b\}$.

Its states are $q_0$ and $q_1$, with $q_0$ initial. Its transitions are, for $i \in 2$:

$(q_i, a, \{(q_0, 0)\})$ $(q_i, b, \{(q_1, 0)\})$

$(q_i, a, \{(q_0, 1)\})$ $(q_i, b, \{(q_1, 1)\})$

The acceptance condition of $A$ is generated by the Muller family $\{\{q_0\}, \{q_0, q_1\}\}$.

Show that $t \in L(A)$ if and only if there is $\pi \in 2^\omega$ such that $t|\pi$ contains infinitely many occurrences of the letter ‘a’.

It is not difficult to show that alternating automata generalize non-deterministic automata.

**Question 3.3.** Show that for any non-deterministic automaton $A$ on the alphabet $\Sigma$, there is an equivalent alternating automaton $A'$ (i.e. $L(A') = L(A)$).

The (more difficult) converse direction will be discussed in Sect. 6.

### 4 Closure Under Boolean Operations

We first check that alternating tree automata are closed under finite Boolean operations.

**Closure Under Intersection and Union.** We first check that alternating automata are closed under finite intersections and union. For the closure under intersections, a direct construction can be given using alternation.

**Question 4.1** (Closure under Intersection). Fix an alphabet $\Sigma$ and consider alternating automata $A_1 = (Q_1, \Delta_1, q^f_1, \Omega_1)$ and $A_2 = (Q_2, \Delta_2, q^f_2, \Omega_2)$ on $\Sigma$.

Assume that $Q_1$ and $Q_2$ are disjoint, and let $q^f \notin Q_1 \cup Q_2$.

Let $A = (Q, \Delta, q^f, \Omega)$ be such that

- $Q = Q_1 \cup Q_2 \cup \{q^f\}$
- $\Omega = \{q^f \cdot \rho \mid q^f_1 \cdot \rho \in \Omega_1\} \cup \{q^f \cdot \rho \mid q^f_2 \cdot \rho \in \Omega_2\}$
- $(q, a, f) \in \Delta$ iff either $(q, a, f) \in \Delta_1 \cup \Delta_2$, or $q = q^f$ and $f = f_1 \cup f_1$ for some $f_1 \subseteq Q_1 \times 2$, $f_2 \subseteq Q_2 \times 2$ such that $(q^f_1, a, f_i) \in \Delta_i$ for $i = 1, 2$.

Show that $L(A) = L(A_1) \cap L(A_2)$.

**Question 4.2** (Closure under Union). Using a direct construction, show that alternating automata are closed under union.
Closure under Complement. We now turn to complementation. Given an alternating automaton $A = (Q, \Delta, q_I, \Omega)$ on $\Sigma$, we will define an alternating automaton $\overline{A}$ on $\Sigma$ such that for every input tree $t \in T_\Sigma$, each Player 1 winning strategy on $\Gamma_{Alt}(A, t)$ can be mapped to a Player 0 winning strategy on $\Gamma_{Alt}(\overline{A}, t)$, and vice-versa. If all acceptance games $\Gamma_{Alt}(A, t)$ (for $t \in T_\Sigma$) are determined, it will follow that $\overline{A}$ recognizes the complement of $L(A)$.

Let $t \in T_\Sigma$ and assume that Player 1 has a winning strategy in $\Gamma_{Alt}(A, t)$ (so, in particular $A$ rejects $t$). In order to see how $\overline{A}$ should behave, it is convenient to assume that this winning strategy is positional\(^1\). From every position $(p, f)$ in $\Gamma_{Alt}(A, t)$, Player 1 is able to choose some $(q, d) \in f$ so that to produce a play loosing for Player 0. Hence, for every Player 0 position $(p, q)$ in $\Gamma_{Alt}(A, t)$ and every $f$ such that $(q, t(p), f) \in \Delta$, Player 1 can choose some $(q, d) \in f$ so that to win the game. In other words, a Player 1 strategy can be described by assigning to each $(p, q)$ a set $f \in P(Q \times 2)$ such that $f \cap f \neq \emptyset$ for every $f$ with $(q, t(p), f) \in \Delta$.

This leads to the automaton:

\[
\overline{A} := (Q, \Delta, q_I, \overline{\Omega})
\]

where $\overline{\Omega} := Q^\omega \setminus \Omega$ and $(q, a, \overline{f}) \in \overline{\Delta}$ iff for all $f \in P(Q \times 2)$,

\[(q, a, f) \in \Delta \implies \overline{f} \cap f \neq \emptyset
\]

Note that given $t \in T_\Sigma$, the graphs of the acceptance games $\Gamma_{Alt}(A, t)$ and $\Gamma_{Alt}(\overline{A}, t)$ have the same Player 0 vertices.

We are now going to check that $\overline{A}$ indeed recognizes the complement of $L(A)$: for all $t \in T_\Sigma$, $\overline{A}$ accepts $t$ if $A$ rejects $t$.

Let us begin with an example.

**Question 4.3 (Example).** Consider the following automaton $A$ on the alphabet $\Sigma = \{a, b\}$.

Its states are $q_0$ and $q_1$, with $q_0$ initial. Its transitions are, for $i \in 2$,

\[(q_i, a, \{(q_0, 0), (q_0, 1)\}) \quad \text{and} \quad (q_i, b, \{(q_1, 0), (q_1, 1)\})
\]

Its acceptance condition is generated by the Muller family $\{\{q_1\}\}$.

(i) Give the complement automaton $\overline{A}$ built according to the above rules.

(ii) What are the languages $L(A)$ and $L(\overline{A})$ ?

We now check that $L(\overline{A}) \subseteq T_\Sigma \setminus L(A)$. The idea is to associate to each winning Player 0 strategy in $\Gamma_{Alt}(\overline{A}, t)$ a winning Player 1 strategy in $\Gamma_{Alt}(A, t)$.

\(^1\)Although positionality is not required to complement alternating automata.
Question 4.4. Let \( \mathcal{A} = (Q, \Delta, q^0, \Omega) \) be an alternating automaton on \( \Sigma \) and let \( \overline{\mathcal{A}} \) be defined as above. Let also \( t \in T_\Sigma \).

(i) Consider two finite plays \( h \in \Gamma_{\text{Alt}}(\mathcal{A}, t) \) and \( \overline{h} \in \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) with the same Player 0 positions, and ending with the Player 0 position \((p, q_0)\).

Show the following:

For every Player 0 move \( v_0 = (p, f) \) in \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) available from \( h \), and every Player 0 move \( v_0 = (p, f) \) in \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) available from \( h \), there is a Player 1 move \( v_1 \) in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) available from \( h \) such that the plays \( h' = h \cdot v_0 \cdot v_1 \) and \( \overline{h'} = \overline{h} \cdot \overline{v_0} \cdot \overline{v_1} \) have the same projection on \( V_0 \).

(ii) Deduce that \( \overline{\mathcal{A}} \) rejects \( t \) if \( \mathcal{A} \) accepts \( t \).

For the converse direction (\( T_\Sigma \setminus L(\mathcal{A}) \subseteq L(\overline{\mathcal{A}}) \)) we actually reason as if Player 1 wins \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) whenever \( \mathcal{A} \) rejects \( t \). The idea is then to associate a winning Player 0 strategy in \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) to each winning Player 1 strategy in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \).

Question 4.5. Let \( \mathcal{A} = (Q, \Delta, q^0, \Omega) \) be an alternating automaton on \( \Sigma \) and let \( \overline{\mathcal{A}} \) be defined as above. Let also \( t \in T_\Sigma \).

(i) Consider two finite plays \( h \in \Gamma_{\text{Alt}}(\mathcal{A}, t) \) and \( \overline{h} \in \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) with the same Player 0 positions, and ending with a Player 0 position.

Assume given a function \( g \) mapping Player 0 moves \( v_0 \in V_1 \) in \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) available from \( h \), to Player 1 moves \( v_1 \in V_0 \) in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) available from \( h \cdot v_0 \).

Show the following:

There is a Player 0 move \( \overline{v_0} \) in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) available from \( \overline{h} \) such that for every Player 1 move \( \overline{v_1} \) in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \) available from \( \overline{h} \cdot \overline{v_0} \), there is a Player 0 move \( v_0 \) in \( \Gamma_{\text{Alt}}(\mathcal{A}, t) \) available from \( h \) such that \( g(v_0) = v_1 \).

(ii) Deduce that \( \overline{\mathcal{A}} \) accepts \( t \) whenever Player 1 has a winning strategy in \( \Gamma_{\text{Alt}}(\overline{\mathcal{A}}, t) \).

5 Projection

We now discuss the operation of projection. This is the operation used for instance to interpret the existential quantifiers of MSO.

Assume that the alphabet \( \Sigma \) is of the form \( \Sigma_0 \times \Sigma_1 \) and consider an alternating automaton \( \mathcal{A} = (Q, \Delta, q^0, \Omega) \) on \( \Sigma \). The projection of \( L := L(\mathcal{A}) \) on \( \Sigma_1 \) is the set \( \Pi_{\Sigma_1}(L) \) of trees \( u \in T_{\Sigma_1} \) such that

\[
\exists t \in L(\mathcal{A}), \forall p \in 2^*, \exists a \in \Sigma_0, \ t(p) = (a, u(p))
\]
A natural way to build a projection automaton $A_{\Sigma_1}$ recognizing $\Pi_{\Sigma_1}(L)$ could be to proceed as follows: let $A_{\Sigma_1}$ be $(Q, \Delta_{\Sigma_1}, q', \Omega)$, with

$$(q, b, f) \in \Delta_{\Sigma_1} \iff \text{there is } a \in \Sigma_0 \text{ s.t. } (q, (a, b), f) \in \Delta$$

**Question 5.1** (Example). Consider the following alternating automaton $A$ on the alphabet $\Sigma = \{a, b\} \times 2$.

It states are $q_0$ and $q_1$, with $q_0$ initial. Its transitions are, for $i, j \in 2$:

$$(q_i, (a, j), \{(q_0, 0), (q_0, 1)\}) \quad \text{and} \quad (q_i, (b, j), \{(q_1, 0), (q_1, 1)\})$$

The acceptance condition of $A$ is generated by the Muller family $\{\{q_0, q_1\}\}$.

(i) What is the language $L(A)$?

(ii) Build the automaton $A_2$ according to the above rules.

The above construction for projection work in some situations, but not all.

**Question 5.2.** Let $A = (Q, \Delta, q', \Omega)$ be an alternating automaton on $\Sigma = \Sigma_0 \times \Sigma_1$.

Show that $\Pi_{\Sigma_1}(L(A)) \subseteq L(A_{\Sigma_1})$.

**Question 5.3.** Let $A = (Q, \Delta, q', \Omega)$ be an alternating automaton on $\Sigma = \Sigma_0 \times \Sigma_1$.

Assume that in $A$, for all $(q, (a, b), f) \in \Delta$, there is at most one $q_0$ such that $(q_0, 0) \in f$ and at most one $q_1$ such that $(q_1, 1) \in f$.

Show that $L(A_{\Sigma_1})$ is the projection of $L(A)$ on $\Sigma_1$.

**Question* 5.4** (Example). Consider the following automaton $A$ on the alphabet $\Sigma = \{a, b\} \times (2 \times 2)$.

Its state set is $\{q_0, q_1, q_0', q_1'\}$, with $q_0$ initial. For the transition relation $\Delta$ we consider three cases. First, for $i \in 2$ we let $(q_i, (x, y, z), f) \in \Delta$ iff

$$x = a \iff \{(q_0, 0), (q_1, 1)\} \subseteq f$$

and $$x = b \iff \{(q_1, 0), (q_0, 1)\} \subseteq f$$

and $$x = a \iff \{(q_0', 0), (q_1', 1)\} \subseteq f$$

Second, $(q_0, (x, y, z), f) \in \Delta$ iff $f = \emptyset$ and $(y = 1 \Rightarrow x = a)$. And third, $(q_1, (x, y, z), f) \in \Delta$ iff $f = \emptyset$ and $(z = 1 \Rightarrow x = b)$. We let the acceptance condition of $A$ be given by the Muller family $\{\{q_0, q_1\}\}$.

(i) Show that if $t \in L(A)$ then no node of $t$ labeled by $(a, y, z)$ has a direct son labeled by $(x, 1, 1)$.

(ii) Deduce that the constant tree $c_{(1, 1)} : p \in 2^* \mapsto (1, 1)$ does not belong to $\Pi_{2 \times 2}(L(A))$.

(iii) Show that $c_{(1, 1)} \in L(A_{2 \times 2})$. 7
6 Equivalence with Non-Deterministic Automata

Let \( A = (Q, \Delta, q', \Omega) \) be an alternating tree automaton on the alphabet \( \Sigma \).

We build an equivalent non-deterministic automaton \( A_{ND} \) on \( \Sigma \). We define it as follows:

- The set of states \( Q_{ND} \) of \( A_{ND} \) is
  \[
  Q_{ND} := \mathcal{P}(Q \times Q)
  \]

- The initial state of \( A_{ND} \) is \( \{(q', q')\} \)

- The transition relation \( \Delta_{ND} \subseteq Q_{ND} \times \Sigma \times Q_{ND} \times Q_{ND} \) of \( A_{ND} \) is defined as follows.
  Consider some \( (S, a, S_0, S_1) \in Q_{ND} \times \Sigma \times Q_{ND} \times Q_{ND} \) and assume that \( \text{proj}_2(S) = \{q_1, \ldots, q_k\} \). We let \( (S, a, S_0, S_1) \in \Delta_{ND} \) if there are \( f_1, \ldots, f_k \) with \( (q_i, a, f_i) \in \Delta \) for \( 1 \leq i \leq k \), and such that
  \[
  S_0 = \{(q_1, q) | (q, 0) \in f_1\} \cup \cdots \cup \{(q_k, q) | (q, 0) \in f_k\}
  \]
  \[
  S_1 = \{(q_1, q) | (q, 1) \in f_1\} \cup \cdots \cup \{(q_k, q) | (q, 1) \in f_k\}
  \]

- The acceptance condition \( \Omega_{ND} \) is defined as follows.
  First, we say that a sequence of \( A \)-states \( (q_i)_{i \in \mathbb{N}} \) is a trace in a sequence of \( A_{ND} \)-states \( (S_i)_{i \in \mathbb{N}} \) if \( (q_i, q_{i+1}) \in S_{i+1} \) for all \( i \in \mathbb{N} \).
  Then we let \( \Omega_{ND} \) be the set of sequences \( (S_i)_{i \in \mathbb{N}} \) whose traces belong all to \( \Omega \).

Question* 6.1. Let \( A_{ND} = (Q_{ND}, \Delta_{ND}, \{(q', q')\}, \Omega_{ND}) \) be as above.
Show that \( \Omega_{ND} \) is \( \omega \)-regular if \( \Omega \) is \( \omega \)-regular.

We will now show that \( A_{ND} \) is equivalent to \( A \) when the games \( \Gamma_{Alt}(A, t) \) (for \( t \in T_\Sigma \)) are positionally determined. We begin by showing that \( L(A_{ND}) \subseteq L(A) \), which requires no positionality assumption.

Question* 6.2. Let \( A \) and \( A_{ND} \) be as above, and let \( t \in T_\Sigma \).
Show that \( A \) accepts \( t \) whenever \( A_{ND} \) accepts \( t \).

The converse direction requires positionality.

Question* 6.3. Let \( A \) and \( A_{ND} \) be as above, and let \( t \in T_\Sigma \).
Show that Player 0 has a positional winning strategy in \( \Gamma(A_{ND}, t) \) if Player 0 has a positional winning strategy in \( \Gamma_{Alt}(A, t) \).