Church’s Synthesis Problem

This homework has to be returned to Colin Riba (possibly by email at colin.riba@ens-lyon.fr) on Monday 24th Nov. 2014.

1 Introduction

This homework is about Church’s Synthesis Problem.

Consider a formula $\psi[X,Y]$ of MSO S1S, where $X$ and $Y$ are two distinct monadic second-order variables. Assume that $\psi$ has no parameters and that $\text{FV}(\psi) \subseteq \{X,Y\}$. This formula defines a binary relation $\tilde{\psi}$ on $2^\omega \times 2^\omega$:

$$\tilde{\psi} := \{(\alpha, \beta) \mid \models \psi[\alpha, \beta]\}$$

Church’s Synthesis Problem takes as input such a formula $\psi[X,Y]$ and asks for a finite-state synchronous uniformization of the relation $\tilde{\psi}$, i.e. for a function $F : 2^\omega \rightarrow 2^\omega$ such that

- $F : 2^\omega \rightarrow 2^\omega$ uniformizes $\tilde{\psi}$:

  $$\forall \alpha \in 2^\omega, (\alpha, F(\alpha)) \in \tilde{\psi}$$

- $F : 2^2 \rightarrow 2^2$ is synchronous: there exists a function $f : 2^* \rightarrow 2^*$ such that
  - $|f(w)| = |w|$ for all $w \in 2^*$ (where $|w|$ denotes the length of $w$), and
  - for all $\alpha \in 2^\omega$ and all $n \in \mathbb{N}$,
    $$F(\alpha)[n] = f(\alpha|n)$$

  where $\alpha|n$ is the finite word $\alpha(0) \ldots \alpha(n-1)$ (and similarly for $F(\alpha)|n$).
• And moreover $f : 2^* \to 2^*$ can be realized by a finite-state Mealy machine. There is a finite state automaton $(Q, \delta, q_\text{ι})$ on the alphabet $2$, equipped with an output function $o : Q \times 2 \to 2$ such that for every finite word $w = a_0 \cdot \ldots \cdot a_k \in 2^+$,

$$f(w) = o(\delta^*(\varepsilon), a_0) \cdot o(\delta^*(a_0), a_1) \cdot \ldots \cdot o(\delta^*(a_0 \cdot \ldots \cdot a_{k-1}), a_k)$$

where $\delta^* : 2^* \to Q$ is defined as $\delta^*(\varepsilon) := q_\text{ι}$ and $\delta^*(u \cdot a) := \delta(\delta^*(u), a)$.

The goal of this homework is to show the following result:

**Theorem 1.1**

(i) Given an MSO S1S formula $\psi[X,Y]$ as above, if there is a synchronous uniformization of $\tilde{\psi}$, then there is a finite-state synchronous uniformization of $\tilde{\psi}$.

(ii) There is an algorithm which takes as input an MSO S1S formula $\psi[X,Y]$, decides if there is a synchronous uniformization of $\tilde{\psi}$, and if there is one, provides a Mealy machine realizing it.

2 From MSO S1S to Automata

In this section, we fix an MSO S1S formula $\psi[X,Y]$.

**Question 2.1** Justify the existence of a complete Muller automaton $M$ on the alphabet $2 \times 2$, such that for all $\alpha, \beta \in 2^\omega$, $M$ accepts the $\omega$-word $\alpha \times \beta := (n \mapsto (\alpha(n), \beta(n)))$ if and only if $\models \psi[\alpha, \beta]$.

The next step in the solution of Church’s Synthesis problem is to turn a Muller automaton $M$ as obtained in Question 2.1 to an equivalent parity automaton.

This is achieved thanks to the following device. Fix a Muller automaton $M = (Q, \delta, q_\text{ι}, T)$. A Latest Appearance Record (LAR) on $Q$ is a pair $(u,v)$ of finite words $u \in Q^*$ and $v \in Q^+$, such that the word $uv$ contains no repetition of a state $q \in Q$.

We will build a (max) parity automaton $A = (S, \delta_A, s_\text{ι}, c)$ on $2 \times 2$ whose states $s \in S$ are LARs on $Q$. The transition function $\delta_A : S \times (2 \times 2) \to S$ is defined as follows: Consider a LAR $(u,v)$ on $Q$. Let $q$ be the last state of $v$ (recall that $v \in Q^+$ by assumption) and given $a \in 2 \times 2$, let $q' := \delta(q,a)$. If $q'$ does not occur in $uv$, then let

$$\delta_A((u,v), a) := (uv, q')$$

Otherwise, write

$$uv := q_1 \cdot \ldots \cdot q_n$$

We have $q' = q_k$ for a unique $k \in \{1, \ldots, n\}$. Let

$$\delta_A((u,v), a) := (q_1 \cdot \ldots \cdot q_{k-1}, q_{k+1} \cdot \ldots \cdot q_n \cdot q')$$
For the initial state $s_i$, take
\[ s_i := (\varepsilon, q_i) \]

The acceptance condition of $A$ is given by a max-parity condition issued from the coloring
\[ c : S \to 2 \cdot |Q| \]
defined as follows:
\[ c(u, v) := \begin{cases} 
2 \cdot |v| & \text{if } v \in T \\
2 \cdot |v| - 1 & \text{otherwise}
\end{cases} \]

A run $\rho \in S^\omega$ of $A$ is accepting if the maximal color occurring infinitely often in $c \circ \rho$ is even.

**Question 2.2** Consider an input $\omega$-word $\gamma \in (2 \times 2)^\omega$ and the corresponding runs $\rho \in Q^\omega$ of $M$ and $\rho_A \in S^\omega$ of $A$ on $\gamma$.

Consider the infinite sequence $(k_n)_{n \in \mathbb{N}}$ such that $k_n = |v_n|$ where $\rho_A(n) = (u_n, v_n)$. Let $k$ be the maximal number occurring infinitely often in $(k_n)_{n \in \mathbb{N}}$.

Show the following:
- There is some $n \in \mathbb{N}$ such that for all $t \geq n$, if $\rho_A(t) = (u_t, v_t)$ with $|v_t| = k$ then $\text{Inf}(\rho) = v_t$ (i.e. $q \in \text{Inf}(\rho)$ iff $q$ occurs in $v_t$).

**Question 2.3** Show that $M$ and $A$ accept the same $\omega$-words $\gamma \in (2 \times 2)^\omega$.

## 3 From Automata to Games

Consider a parity automaton $A = (S, \delta_A, s_A, c)$ on $2 \times 2$ as obtained in Sect. 2. Assume that $c : S \to [n] = \{0, \ldots, n\}$.

We consider the following finite graph game
\[ G := (V_0, V_1, E, W) \]
where
- $V_0 := S$
- $V_1 := S \times 2$
- There is an edge $s \to_G (s, a)$ for all $s \in S$ and all $a \in 2$.
- There is an edge $(s, a) \to_G s'$ if and only if there is $b \in 2$ such that $s' = \delta_A(s, (a, b))$. 

3
Let $V := V_0 \cup V_1$. The winning condition $\mathcal{W}$ of $G$ is generated from the coloring $c_G : V \to [n]$ defined as

\[ c_G((s, a)) := c(s) \quad \text{and} \quad c_G(s) := c(s) \]

Given an infinite path $\pi \in V^\omega$, we let $\pi \in \mathcal{W}$ if and only if the maximal color occurring infinitely often in $c_G \circ \rho$ is even.

**Question 3.1** Show that Player 1 has a positional winning strategy in $G$ from $s_i$ if and only if there is a synchronous map realizing $\tilde{\psi}$.

**Question 3.2** Show Thm. 1.1. (i).

## 4 Effective Positional Determinacy of Finite Parity Games

We now discuss the effective positional determinacy of finite parity games. We follow Sect. 3 and consider max-parity games. They are presented as structures

\[(G, c)\]

where $G = (V_0, V_1, E, \mathcal{W})$ is a finite graph game, and writing $V := V_0 \cup V_1$:

- $c : V \to [n] = \{0, \ldots, n\}$
- $\mathcal{W} \subseteq V^\omega$ is the set of infinite paths $\pi$ such that the maximal color occurring infinitely often in $c \circ \pi$ is even.

We assume that all games $G = (V_0, V_1, E, \mathcal{W})$ considered here have no dead-ends: for all $v \in V$, there is $w \in V$ such that $v \rightarrow_G w$.

### 4.1 A Naive Algorithm

Using positional determinacy of parity games, together with the finiteness of the graph $G$ obtained in Sect. 3, we have everything at hand to show Thm. 1.1.

**Question 4.1** Give an algorithm which takes as input a finite parity game and a position $v$ in that game, and decides whether Player 0 or Player 1 wins from $v$.

**Question 4.2** Show Thm. 1.1.

### 4.2 An Inductive Algorithm for Finite Parity Games

We will now look at an algorithm which follows the proof of positional determinacy of parity games by induction on the number of colors.
4.2.1 Attractor Sets

We first introduce the following notion: Given a finite graph game \( G = (V_0, V_1, E, W), U \subseteq V \) and \( i \in \{0, 1\} \), define:

\[
\begin{align*}
\text{Att}_0^i(G, U) & := U \\
\text{Att}_k^i(G, U) & := \{v \in V_i \mid \exists w. \ v \rightarrow_G w \text{ and } w \in \text{Att}_k^i(G, U)\} \\
& \quad \cup \{v \in V_{1-i} \mid \forall w. \ v \rightarrow_G w \text{ implies } w \in \text{Att}_k^i(G, U)\} \\
& \quad \cup \text{Att}_k^i(G, U)
\end{align*}
\]

**Question 4.3** Show that there is some \( k \in \mathbb{N} \) such that \( \text{Att}_k^i(G, U) = \text{Att}_{k+1}^i(G, U) \).

We let \( \text{Att}_k^i(G, U) \) be \( \text{Att}_k^i(GmU) \) for the \( k \) obtained in Question 4.3.

Consider the game \( G_U = (V_0, V_1, E, W_U) \) where \( \pi \in W_U \) iff there is \( k \in \mathbb{N} \) such that \( \pi(k) \in U \).

**Question 4.4** Show that \( \text{Att}_k^i(G, U) \) is the set of positions from which Player \( i \) has a winning positional strategy for \( G_U \).

**Question 4.5** Show that there is a positional strategy for Player \( i \) which is winning for \( G_U \) from any position \( v \in \text{Att}_k^i(G, U) \).

4.2.2 Merging Winning Strategies

We now discuss the merging of winning strategies in the context of finite parity games.

Consider, in a finite parity game \( (G, c) \), a finite collection of vertices \( (U_j)_{j \in J} \). Fix \( i \in \{0, 1\} \) and assume given for all \( j \in J \), a Player \( i \) positional strategy \( \sigma_j \) which is winning for \( (G, c) \) from any position \( v \in U_j \).

**Question 4.7** Give an algorithm which given \( (G, c) \), \( (U_j)_{j \in J}, i \in \{0, 1\} \) and \( (\sigma_j)_{j \in J} \) as above, computes a positional strategy for Player \( i \) which is winning for \( G \) from any position \( v \in \bigcup_{j \in J} U_j \).

4.2.3 The Algorithm

We can now proceed to the description of the algorithm \( \text{WinPar} \). Its specification is as follows:

- If \( (G, c) \) is a finite parity game and \( n \) is the maximal color occurring in \( G \), then \( \text{WinPar}(G, c, n) \) returns the sets \( W_0 \) and \( W_1 \) of positions from which Player 0 (resp. Player 1) has a winning positional strategy for \( G \), together with positional strategies \( (\sigma_0, \sigma_1) \) for Player 0, resp. Player 1, such that \( \sigma_i \) is winning for Player \( i \) in \( G \) from all \( v \in W_i \).
The algorithm proceeds by recursion on the maximal color $n$ occurring in $G$.

**Question 4.8** Describe $\text{WinPar}(G,c,n)$ when $n$ is the only color occurring in $(G,c)$.

**The Inductive Step.** When more than one color occurs in $(G,c)$, $\text{WinPar}(G,c,n)$ proceeds as in the proof of positional determinacy\(^1\), excepted that winning regions (and the corresponding positional winning strategies) are computed inductively.

Assume given $(G,c)$ with $c : V \to [n]$ and let

\[
i := n \mod 2
\]

We will define an increasing family $(W_{1-i}^k)_{k \in \mathbb{N}}$ of sets of winning positions for Player $(1-i)$ in $G$, together with positional strategies $\tau_{1-i}^k$ for Player $(1-i)$ such that $\tau_{1-i}^k$ is winning in $G$ from any position of $W_{1-i}^k$.

First, let

\[
W_{0}^{1-i} := \emptyset
\]

and take for $\tau_{0}^{1-i}$ any total strategy for Player $(1-i)$.

Assuming that $W_{k}^{1-i}$ and $\tau_{k}^{1-i}$ are defined, let

\[
X_{k}^{1-i} := \text{Att}(G,W_{k}^{1-i}) \quad \text{and} \quad X_{k}^{i} := V \setminus X_{k}^{1-i}
\]

**Question 4.9** Define a positional strategy $\tau_{1-i}$ for Player $(1-i)$ which is winning for $G$ from any position in $X_{k}^{1-i}$.

Let $G[X_k^i]$ be the restriction of $G$ to $X_k^i$:

\[
G[X_k^i] := (V_0 \cap X_k^i, V_1 \cap X_k^i, E \cap (X_k^i \times X_k^i), W \cap X_k^i \omega)
\]

**Question 4.10** Show that $G[X_k^i]$ has no dead-end.

The graph game $G[X_k^i]$ is equipped with the coloring function $c|_{X_k^i}$. Consider the sets

\[
N := c_{|X_k^i}^{-1}([n]) \quad \text{and} \quad Y_k^i := \text{Att}^i(G[X_k^i],N)
\]

Let moreover

\[
Z := X_k^i \setminus Y_k^i
\]

and consider the graph $G' := G[Z]$, equipped with the coloring function $c' := c|Z$.

**Question 4.11** Show that $G'$ has no dead-end.

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\(^{1}\)Following the proof of positional determinacy given in the Course 06.
Assume now given a partition of the set of vertices $V'$ of $G'$ into winning regions $W'_0$ and $W'_1$, together with positional strategies $\sigma'_0$ and $\sigma'_1$ for resp. Player 0 and 1 in $(G', c')$, such that $\sigma'_0$ is winning for Player 0 in $G'$ from any position in $W'_0$, and similarly for $\sigma'_1$.

Let
\[ W^{1-i}_{k+1} := X^{1-i}_k \cup W'_1 \]

**Question 4.12** Define a positional strategy $\tau^{1-i}_{k+1}$ for Player $(1 - i)$ which is winning for $G$ from any position in $W^{1-i}_{k+1}$.

This completes the definition of the sequences $(W^{1-i}_k)_{k \in \mathbb{N}}$ and $(\tau^{1-i}_k)_{k \in \mathbb{N}}$. We let $W^{1-i} := W^{1-i}_k$ for the first $k$ such that $W^{1-i}_k = W^{1-i}_{k+1}$.

The winning region of Player $i$ is defined as expected: let
\[ W^i := V \setminus W^{1-i} \]

**Question 4.13** Define a positional Player $i$ strategy $\sigma_i$ such that $\sigma_i$ wins $G$ from any position in $W^i$.

**Question 4.14** Describe a procedure $\text{WinPar}(G, c, n)$ following the above inductive constructions.