Recognizability and Model-Checking
from Automata to the Lambda-Calculus

Colin RIBA
LIP - ENS de Lyon

Course 03
22nd Sept. 2014
Deterministic Automata
Context: Complementation of Büchi Automata

Theorem (Complementation of Büchi Automata – Büchi 1962)

Given a Büchi automaton $A$ on $\Sigma$, one can effectively build an automaton $\tilde{A}$ such that $L(\tilde{A}) = \Sigma^\omega \setminus L(A)$.

Non-trivial construction.
Context: Complementation of Büchi Automata

Theorem (Complementation of Büchi Automata – Büchi 1962)

Given a Büchi automaton $A$ on $\Sigma$, one can effectively build an automaton $\tilde{A}$ such that $L(\tilde{A}) = \Sigma^\omega \setminus L(A)$.

Non-trivial construction.

Different methods:
Theorem (Complementation of Büchi Automata – Büchi 1962)

Given a Büchi automaton $A$ on $\Sigma$, one can effectively build an automaton $\tilde{A}$ such that $L(\tilde{A}) = \Sigma^\omega \setminus L(A)$.

Non-trivial construction.

Different methods:
- Direct complementation.
  Various approaches, in part. using Ramsey’s Theorem.
Context: Complementation of Büchi Automata

Theorem (Complementation of Büchi Automata – Büchi 1962)

Given a Büchi automaton $A$ on $\Sigma$, one can effectively build an automaton $\tilde{A}$ such that $L(\tilde{A}) = \Sigma^\omega \setminus L(A)$.

Non-trivial construction.

Different methods:

- Direct complementation. Various approaches, in part. using Ramsey’s Theorem.
- Translation to deterministic automata on $\omega$-words (McNaughton’s Theorem, 1966). Different kinds: Muller, Rabin, Streett. Differ from Büchi by their acceptance condition.
Deterministic Automata

Context: Complementation of Büchi Automata

**Theorem (Complementation of Büchi Automata – Büchi 1962)**

Given a Büchi automaton $A$ on $\Sigma$, one can effectively build an automaton $\tilde{A}$ such that $L(\tilde{A}) = \Sigma^\omega \setminus L(A)$.

Non-trivial construction.

**Different methods:**

- Direct complementation.
  Various approaches, in part. using Ramsey’s Theorem.

- Translation to **deterministic** automata on $\omega$-words
  (McNaughton’s Theorem, 1966).
  Different kinds: Muller, Rabin, Streett.
  Differ from Büchi by their acceptance condition.
Deterministic Büchi Automata

Given $W \subseteq \Sigma^+$, let

$$\lim(W) := \{ \alpha \in \Sigma^\omega | \alpha \upharpoonright n \in W \text{ for infinitely many } n \in \mathbb{N} \}$$

where $\alpha \upharpoonright n = \alpha(0)\alpha(1)\ldots\alpha(n-1)$

Definition (Deterministic Set)

$A \subseteq \Sigma^\omega$ is deterministic if $A = \lim(U)$ for some regular $U \subseteq \Sigma^+$. 
Deterministic Büchi Automata

Given $W \subseteq \Sigma^+$, let

$$\text{lim}(W) := \{ \alpha \in \Sigma^\omega \mid \alpha\upharpoonright n \in W \text{ for infinitely many } n \in \mathbb{N} \}$$

where $\alpha\upharpoonright n = \alpha(0)\alpha(1) \ldots \alpha(n - 1)$

**Definition (Deterministic Set)**

A $\subseteq \Sigma^\omega$ is **deterministic** if $A = \text{lim}(U)$ for some regular $U \subseteq \Sigma^+$.

**Examples.**

- Regular Safety properties.
- More generally closed $\omega$-regular languages. (Admitted.)
- On $\Sigma = \{a, b\}$, the set

$$P := \{ \alpha \in \Sigma^\omega \mid \exists t. \alpha(t) = a \}$$

- More generally $\Pi^0_2 \omega$-regular languages. (Admitted.)
Deterministic Büchi Automata

Given $W \subseteq \Sigma^+$, let

$$\lim(W) := \{ \alpha \in \Sigma^\omega \mid \alpha|_n \in W \text{ for infinitely many } n \in \mathbb{N} \}$$

where $\alpha|_n = \alpha(0)\alpha(1)\ldots\alpha(n-1)$

Definition (Deterministic Set)

$A \subseteq \Sigma^\omega$ is deterministic if $A = \lim(U)$ for some regular $U \subseteq \Sigma^+$.

Examples.

- Regular Safety properties.
- More generally closed $\omega$-regular languages.  (Admitted.)
- On $\Sigma = \{a, b\}$, the set
  
  $$P := \{ \alpha \in \Sigma^\omega \mid \exists \omega t. \alpha(t) = a \}$$

- More generally $\Pi_2^0$ $\omega$-regular languages.  (Admitted.)

Theorem (Deterministic Sets and Automata)

$A \subseteq \Sigma^\omega$ is deterministic iff it is recognizable by a deterministic Büchi automaton.

Proof:
Theorem (Non-Equivalence with Deterministic Büchi Automata)

There is an \( \omega \)-regular set \( A \subseteq \{0,1\}^\omega \) which is not deterministic. (Moreover we have \( A = U \cdot V \omega \).)

Proof.

On \( \Sigma = \{a, b\} \), the set \( S := \Sigma^\omega \setminus P \) is not recognizable by a deterministic Büchi automaton.

Corollary

Deterministic languages are not closed under complement.

Remark.

Deterministic Büchi automata are not closed under projection.

Lemma

Deterministic sets are closed under union.

Deterministic sets are closed under intersection.
Some Properties of Deterministic Büchi Automata

Theorem (Non-Equivalence with Deterministic Büchi Automata)

There is an $\omega$-regular $A \subseteq 2^\omega$ which is not deterministic.
(Moreover we have $A = U \cdot V^\omega$.)

Proof.

- On $\Sigma = \{a, b\}$, the set
  
  $$S := \Sigma^\omega \setminus P$$

  is not recognizable by a deterministic Büchi automaton.
Some Properties of Deterministic Büchi Automata

Theorem (Non-Equivalence with Deterministic Büchi Automata)

There is an \( \omega \)-regular \( A \subseteq 2^\omega \) which is not deterministic.

(Moreover we have \( A = U \cdot V^\omega \).)

Proof.

- On \( \Sigma = \{a, b\} \), the set

\[
S := \Sigma^\omega \setminus P
\]

is not recognizable by a deterministic Büchi automaton.

Corollary

Deterministic languages are not closed under complement.
Some Properties of Deterministic Büchi Automata

Theorem (Non-Equivalence with Deterministic Büchi Automata)

There is an \( \omega \)-regular \( A \subseteq 2^\omega \) which is not deterministic.

(Moreover we have \( A = U \cdot V^\omega \).)

Proof.

- On \( \Sigma = \{a, b\} \), the set
  \[
  S := \Sigma^\omega \setminus P
  \]
  is not recognizable by a deterministic Büchi automaton.

Corollary

Deterministic languages are not closed under complement.

Remark.

- Deterministic Büchi automata are not closed under projection.
Some Properties of Deterministic Büchi Automata

Theorem (Non-Equivalence with Deterministic Büchi Automata)

There is an $\omega$-regular $A \subseteq 2^\omega$ which is not deterministic. (Moreover we have $A = U \cdot V^\omega$.)

Proof.

- On $\Sigma = \{a, b\}$, the set
  
  $$S := \Sigma^\omega \setminus P$$

  is not recognizable by a deterministic Büchi automaton.

Corollary

Deterministic languages are not closed under complement.

Remark.

- Deterministic Büchi automata are not closed under projection.

Lemma

- Deterministic sets are closed under union.
- Deterministic sets are closed under intersection.
Weak MSO (WMSO)

**WMSO**: MSO where set quantification only range over *finite* sets:

\[ \models_{\text{WMSO}} \exists X . \phi[X] \quad \text{iff} \quad \text{there is a finite } A \in \mathcal{P}(\mathbb{N}) \text{ s.t. } \models_{\text{WMSO}} \phi[A] \]

(\(\phi\) can have infinite sets as parameters.)
Weak MSO (WMSO)

**WMSO:** MSO where set quantification only range over finite sets:

\[ \models_{\text{WMSO}} \exists X. \phi[X] \text{ iff there is a finite } A \in \mathcal{P}(\mathbb{N}) \text{ s.t. } \models_{\text{WMSO}} \phi[A] \]

(\( \phi \) can have infinite sets as parameters.)

MSO formulas representing Büchi automata have the form:

\[ \exists \overline{Z} (I[\overline{Z}(O)] \land \forall t H[\overline{Z}(t), \overline{X}(t), \overline{Z}(S(t))] \land \exists \omega t F[\overline{Z}(t)]) \]

- \( \overline{X} = X_1, \ldots, X_q \) represents the input \( \omega \)-word over \( 2^q \).
- Note that \( \exists \overline{Z} \) is the only quantification over possibly infinite sets.
Weak MSO (WMSO)

**WMSO:** MSO where set quantification only range over finite sets:

\[ \models_{\text{WMSO}} \exists X. \phi[X] \quad \text{iff} \quad \text{there is a finite } A \in \mathcal{P}(\mathbb{N}) \text{ s.t. } \models_{\text{WMSO}} \phi[A] \]

(\(\phi\) can have infinite sets as parameters.)

MSO formulas representing Büchi automata have the form:

\[ \exists \overline{Z} (I[\overline{Z}(O)] \land \forall t. H[\overline{Z}(t), \overline{X}(t), \overline{Z}(S(t))] \land \exists \omega t. F[\overline{Z}(t)]) \]

- \( \overline{X} = X_1, \ldots, X_q \) represents the input \( \omega \)-word over \( 2^q \).
- Note that \( \exists \overline{Z} \) is the only quantification over possibly infinite sets.

For a deterministic automaton, we can equivalently write

\[ \exists \omega u \exists \overline{Z} (I[\overline{Z}(O)] \land \forall t < u. H[\overline{Z}(t), \overline{X}(t), \overline{Z}(S(t))] \land F[\overline{Z}(u)]) \]
Weak MSO (WMSO)

**WMSO:** MSO where set quantification only range over finite sets:

\[ \models_{\text{WMSO}} \exists X . \phi[X] \quad \text{iff} \quad \text{there is a finite } A \in \mathcal{P}(\mathbb{N}) \text{ s.t. } \models_{\text{WMSO}} \phi[A] \]

(\(\phi\) can have infinite sets as parameters.)

MSO formulas representing Büchi automata have the form:

\[ \exists \overline{Z} \left( I[\overline{Z}(O)] \land \forall t. \overline{H}[\overline{Z}(t), X(t), \overline{Z}(S(t))] \land \exists \omega t. F[\overline{Z}(t)] \right) \]

- \(\overline{X} = X_1, \ldots, X_q\) represents the input \(\omega\)-word over \(2^q\).
- Note that \(\exists \overline{Z}\) is the only quantification over possibly infinite sets.

For a deterministic automaton, we can equivalently write

\[ \psi[\overline{X}] := \exists \omega u. \exists \overline{Z} \left( I[\overline{Z}(O)] \land \forall t < u. \overline{H}[\overline{Z}(t), X(t), \overline{Z}(S(t))] \land F[\overline{Z}(u)] \right) \]

Then given \(\overline{A} \in \mathcal{P}(\mathbb{N})\),

\[ \models_{\text{MSO}} \psi[\overline{A}] \quad \text{iff} \quad \models_{\text{WMSO}} \psi[\overline{A}] \]
Muller Automata

Muller Automata
Muller Automata

Fix $\Sigma$ alphabet.

A Muller automaton on $\Sigma$ is a finite deterministic automaton

$$\mathcal{A} = (Q, \delta, q_\iota, T)$$

where $q_\iota \in Q$, $\delta : Q \times \Sigma \rightarrow Q$ and $T \subseteq \mathcal{P}(Q)$ is a family of sets of states.
**Muller Automata**

Fix $\Sigma$ alphabet.

A **Muller automaton** on $\Sigma$ is a finite *deterministic* automaton

$$
\mathcal{A} = (Q, \delta, q_\iota, \mathcal{T})
$$

where $q_\iota \in Q$, $\delta : Q \times \Sigma \rightarrow Q$ and $\mathcal{T} \subseteq \mathcal{P}(Q)$ is a family of sets of states.

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$.

A run $\rho$ is **accepting** iff

$$
\text{Inf}(\rho) \in \mathcal{T}
$$
Muller Automata

Fix $\Sigma$ alphabet.

A Muller automaton on $\Sigma$ is a finite *deterministic* automaton

$$\mathcal{A} = (Q, \delta, q_\iota, T)$$

where $q_\iota \in Q$, $\delta : Q \times \Sigma \rightarrow Q$ and $T \subseteq \mathcal{P}(Q)$ is a family of sets of states.

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$.

A run $\rho$ is **accepting** iff

$$\text{Inf}(\rho) \in T$$

Acceptance:

$\mathcal{A}$ accepts $\alpha \in \Sigma^\omega$ iff there exists an accepting run.

The language of $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) := \{ \alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha \}$$
Muller Automata

Fix $\Sigma$ alphabet.

A **Muller automaton** on $\Sigma$ is a finite *deterministic* automaton

$$\mathcal{A} = (Q, \delta, q_\iota, T)$$

where $q_\iota \in Q$, $\delta : Q \times \Sigma \rightarrow Q$ and $T \subseteq \mathcal{P}(Q)$ is a family of sets of states.

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$.

A run $\rho$ is **accepting** iff

$$\text{Inf}(\rho) \in T$$

Acceptance:

$\mathcal{A}$ accepts $\alpha \in \Sigma^\omega$ iff there exists an accepting run.

The language of $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) := \{ \alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha \}$$

Remark. We only require $\delta$ to be partial, but a Muller aut. can always be turned to an equivalent Muller aut. with total transition function.
Muller Automata

Fix $\Sigma$ alphabet.

A **Muller automaton** on $\Sigma$ is a finite *deterministic* automaton

$$A = (Q, \delta, q_\iota, T)$$

where $q_\iota \in Q$, $\delta : Q \times \Sigma \rightarrow Q$ and $T \subseteq \mathcal{P}(Q)$ is a family of sets of states.

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$.

A run $\rho$ is **accepting** iff

$$\text{Inf}(\rho) \in T$$

Acceptance:

A *accepts* $\alpha \in \Sigma^\omega$ iff there exists an accepting run.

**The language** of $A$ is

$$\mathcal{L}(A) := \{ \alpha \in \Sigma^\omega \mid A \text{ accepts } \alpha \}$$

**Remark.** We only require $\delta$ to be partial, but a Muller aut. can always be turned to an equivalent Muller aut. with total transition function.

**Example.** *On Blackboard!*
Lemma (Closure Under Boolean Operations)

Given $A_1$ and $A_2$, Muller automata on $\Sigma$, one can build

- a Muller automaton $\tilde{A}$ such that
  \[
  \mathcal{L}(\tilde{A}) = \Sigma^\omega \setminus \mathcal{L}(A_1)
  \]

- a Muller automaton $A_\cup$ such that
  \[
  \mathcal{L}(A_\cup) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)
  \]

- a Muller automaton $A_\cap$ such that
  \[
  \mathcal{L}(A_\cap) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)
  \]

Proof.

*On Blackboard!*
Lemma

If $A$ is a Muller automaton, then $L(A)$ is $\omega$-regular.

Proof.

*On Blackboard!*
Lemma

If \( A \) is a Muller automaton, then \( \mathcal{L}(A) \) is \( \omega \)-regular.

Proof.

On Blackboard!

Lemma

Given \( X \subseteq \Sigma^\omega \), the following are equivalent:

(i) \( X \) is recognizable by a Muller automaton

(ii) \( X \) is of the form

\[
X = \bigcup_{1 \leq i \leq n} (U_i \setminus V_i)
\]

for deterministic \( U_i, V_i \).

(iii) \( X \) is a finite Boolean combination of deterministic sets.

Proof.

On Blackboard!
McNaughton’s Theorem
Summary: Complementation and Determinization

Büchi Automata.

- Equivalent to $\omega$-regular languages.
- Closed under $\cap$ and $\cup$.
- Closed under projection.

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Muller automaton.
Summary: Complementation and Determinization

**Büchi Automata.**
- Equivalent to $\omega$-regular languages.
- Closed under $\cap$ and $\cup$.
- Closed under projection.

**Deterministic Büchi Automata.**
- Correspond to $\omega$-regular $\Pi^0_2$ sets $X \subseteq \Sigma^\omega$.
- Closed under $\cap$ and $\cup$.
- **Not** closed under complement nor projection.
Summary: Complementation and Determinization

Büchi Automata.
- Equivalent to \( \omega \)-regular languages.
- Closed under \( \cap \) and \( \cup \).
- Closed under projection.

Deterministic Büchi Automata.
- Correspond to \( \omega \)-regular \( \Pi^0_2 \) sets \( X \subseteq \Sigma^\omega \).
- Closed under \( \cap \) and \( \cup \).
- **Not** closed under complement nor projection.

Muller Automata.
- Finite Boolean combinations of deterministic sets.
- Included in \( \omega \)-regular languages.
- Closed under finite Boolean operations.
Summary: Complementation and Determinization

Büchi Automata.
- Equivalent to $\omega$-regular languages.
- Closed under $\cap$ and $\cup$.
- Closed under projection.

Deterministic Büchi Automata.
- Correspond to $\omega$-regular $\Pi^0_2$ sets $X \subseteq \Sigma^\omega$.
- Closed under $\cap$ and $\cup$.
- **Not** closed under complement nor projection.

Muller Automata.
- Finite Boolean combinations of deterministic sets.
- Included in $\omega$-regular languages.
- Closed under finite Boolean operations.

**Theorem (McNaughton, 1966)**
*For every Büchi automaton, one can build an equivalent Muller automaton.*
Summary: Complementation and Determinization

Büchi Automata.
- Equivalent to $\omega$-regular languages.
- Closed under finite Boolean operations.
- Closed under projection.

Deterministic Büchi Automata.
- Correspond to $\omega$-regular $\Pi^0_2$ sets $X \subseteq \Sigma^\omega$.
- Closed under $\cap$ and $\cup$.
- Not closed under complement nor projection.

Muller Automata.
- Finite Boolean combinations of deterministic sets.
- Equivalent to $\omega$-regular languages.
- Closed under finite Boolean operations.

Theorem (McNaughton, 1966)
For every Büchi automaton, one can build an equivalent Muller automaton.
McNaughton’s Theorem

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Muller automaton.

Proof. Using Safra’s Construction.
McNaughton’s Theorem

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Muller automaton.

Proof. Using Safra’s Construction.

Some Important Consequences.
- The set of ω-regular languages is closed under complementation.
- If $X \subseteq \Sigma^\omega$ is ω-regular, then there are $n \in \mathbb{N}$ and regular $U_i, V_i \subseteq \Sigma^+$ ($1 \leq i \leq n$) s.t.

$$L = \bigcup_{1 \leq i \leq n} \text{lim}(U_i) \setminus \text{lim}(V_i)$$
McNaughton’s Theorem

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Muller automaton.

Proof. Using Safra’s Construction.

Some Important Consequences.

- The set of $\omega$-regular languages is closed under complementation.
- If $X \subseteq \Sigma^\omega$ is $\omega$-regular, then there are $n \in \mathbb{N}$ and regular $U_i, V_i \subseteq \Sigma^+$ $(1 \leq i \leq n)$ s.t.

$$L = \bigcup_{1 \leq i \leq n} \text{lim}(U_i) \setminus \text{lim}(V_i)$$

- $\omega$-Regular sets are finite Boolean combinations of $\Pi^0_2$ sets.
McNaughton’s Theorem

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Muller automaton.

Proof. Using Safra’s Construction.

Some Important Consequences.

- The set of \( \omega \)-regular languages is closed under complementation.
- If \( X \subseteq \Sigma^\omega \) is \( \omega \)-regular, then there are \( n \in \mathbb{N} \) and regular \( U_i, V_i \subseteq \Sigma^+ \) (\( 1 \leq i \leq n \)) s.t.

\[
L = \bigcup_{1 \leq i \leq n} \lim(U_i) \setminus \lim(V_i)
\]

- \( \omega \)-Regular sets are finite Boolean combinations of \( \Pi^0_2 \) sets.
- MSO on \( \omega \)-words is equivalent to WMSO on \( \omega \)-words:

For every \( \phi[X] \), one can build a \( \psi[X] \) s.t.:

\[
\text{for all } \bar{A} \in \mathcal{P}(\mathbb{N}), \quad (\models_{\text{MSO}} \phi[\bar{A}] \quad \text{iff} \quad \models_{\text{WMSO}} \psi[\bar{A}])
\]
McNaughton’s Theorem

Theorem (McNaughton, 1966)

For every Büchi automaton, one can build an equivalent Rabin automaton.

Proof. Using Safra’s Construction.

Some Important Consequences.

- The set of $\omega$-regular languages is closed under complementation.
- If $X \subseteq \Sigma^\omega$ is $\omega$-regular, then there are $n \in \mathbb{N}$ and regular $U_i, V_i \subseteq \Sigma^+$ ($1 \leq i \leq n$) s.t.
  \[
  L = \bigcup_{1 \leq i \leq n} \lim(U_i) \setminus \lim(V_i)
  \]
- $\omega$-Regular sets are finite Boolean combinations of $\Pi^0_2$ sets.
- MSO on $\omega$-words is equivalent to WMSO on $\omega$-words:
  For every $\phi[X]$, one can build a $\psi[X]$ s.t.:
  \[
  \text{for all } \bar{A} \in \mathcal{P}(\mathbb{N}), \quad (\models_{\text{MSO}} \phi[\bar{A}] \quad \text{iff} \quad \models_{\text{WMSO}} \psi[\bar{A}])
  \]
Rabin Automata
Rabin Automata

A Rabin automaton (with $n$ pairs) is a finite state deterministic automaton

$$A = (Q, \delta, q_0, R)$$

where $R = \{(L_i, U_i) \mid 1 \leq i \leq n\}$ is a family of pairs of sets of states:

$$L_i, U_i \subseteq Q$$

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$. 
Rabin Automata

A Rabin automaton (with $n$ pairs) is a finite state deterministic automaton

$$\mathcal{A} = (Q, \delta, q_\iota, \mathcal{R})$$

where $\mathcal{R} = \{(L_i, U_i) \mid 1 \leq i \leq n\}$ is a family of pairs of sets of states:

$$L_i, U_i \subseteq Q$$

Each $\omega$-word $\alpha \in \Sigma^\omega$ defines (at most) one run $\rho \in Q^\omega$.

A run $\rho$ is accepting iff

$$\bigvee_{1 \leq i \leq n} (\text{Inf}(\rho) \cap L_i = \emptyset \text{ and } \text{Inf}(\rho) \cap U_i \neq \emptyset)$$

Acceptance:

$\mathcal{A}$ accepts $\alpha \in \Sigma^\omega$ iff there exists an accepting run.

The language of $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) := \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}$$
Rabin Automata

A Rabin automaton (with \( n \) pairs) is a finite state deterministic automaton

\[ A = (Q, \delta, q_\iota, R) \]

where \( R = \{(L_i, U_i) \mid 1 \leq i \leq n\} \) is a family of pairs of sets of states:

\[ L_i, U_i \subseteq Q \]

Each \( \omega \)-word \( \alpha \in \Sigma^\omega \) defines (at most) one run \( \rho \in Q^\omega \).

A run \( \rho \) is accepting iff

\[ \bigvee_{1 \leq i \leq n} (\text{Inf}(\rho) \cap L_i = \emptyset \quad \text{and} \quad \text{Inf}(\rho) \cap U_i \neq \emptyset) \]

Acceptance:

\( A \) accepts \( \alpha \in \Sigma^\omega \) iff there exists an accepting run.

The language of \( A \) is

\[ \mathcal{L}(A) := \{ \alpha \in \Sigma^\omega \mid A \text{ accepts } \alpha \} \]

Example.  

On Blackboard!
Basic Properties

Lemma

(i) Each Rabin automaton is equivalent to a Muller automaton.

(ii) Each Muller automaton is equivalent to a Rabin automaton.

(iii) $X \subseteq \Sigma^\omega$ is recognized by a Rabin automaton with $n$ pairs iff $X$ is of the form

$$X = \bigcup_{1 \leq i \leq n} (U_i \setminus V_i)$$

for deterministic $U_i, V_i$.

Corollary. Rabin automata are closed under Boolean operations.
Strong Fairness Properties

Strong Fairness:

- Of the form

\[(\exists \omega_\infty t. \alpha(t) \in A) \implies (\exists \omega_\infty t. \alpha(t) \in B)\]

Correspond to Streett automata.
Strong Fairness Properties

Strong Fairness:

- Of the form
  \[(\exists \omega t. \alpha(t) \in A) \implies (\exists \omega t. \alpha(t) \in B)\]

Correspond to *Streett automata*.

- "Direct complement" of Rabin automata.
- Given a Rabin automaton \( \mathcal{A} = (Q, \delta, q_\iota, \{(L_1, U_1), \ldots, (L_n, U_n)\}) \)

  \[\alpha \in \mathcal{L}(\mathcal{A}) \text{ iff } \bigvee_{1 \leq i \leq n} (\text{Inf}(\rho_\alpha) \cap L_i = \emptyset \text{ and } \text{Inf}(\rho_\alpha) \cap U_i \neq \emptyset)\]
Strong Fairness Properties

Strong Fairness:

- Of the form

\[(\exists \omega . \alpha(t) \in A) \implies (\exists \omega . \alpha(t) \in B)\]

Correspond to Streett automata.

- “Direct complement” of Rabin automata.
- Given a Rabin automaton \( \mathcal{A} = (Q, \delta, q_\iota, \{(L_1, U_1), \ldots, (L_n, U_n)\}) \)

\[\alpha \in \mathcal{L}(\mathcal{A}) \iff \bigvee_{1 \leq i \leq n} (\text{Inf}(\rho_\alpha) \cap L_i = \emptyset \text{ and } \text{Inf}(\rho_\alpha) \cap U_i \neq \emptyset)\]

- The Streett automaton

\[\tilde{\mathcal{A}} = (Q, \delta, q_\iota, \{(L_1, U_1), \ldots, (L_n, U_n)\})\]

is such that

\[\alpha \in \mathcal{L}(\tilde{\mathcal{A}}) \iff \bigwedge_{1 \leq i \leq n} (\text{Inf}(\rho_\alpha) \cap U_i \neq \emptyset \implies \text{Inf}(\rho_\alpha) \cap L_i \neq \emptyset)\]