We discuss completeness and compactness w.r.t. Henkin models for Second-Order Predicate Logic. Most details can be found in [Sha91, vD04].

1 Preliminaries

We assume given a first-order language \((\mathcal{V^i}, \Sigma)\) consisting of an infinite set \(\mathcal{V^i} = \{x, y, \ldots \}\) of individual variables and a signature \(\Sigma\) of function symbols on individuals, each coming with an arity \(n \in \mathbb{N}\). We generally write \(f, g, h, \ldots\) for function symbols of arity \(> 0\) and \(c, d, \ldots\) for those of arity 0 (also called constants). We write \(\mathcal{Ter}(\mathcal{V^i}, \Sigma) = \{t, u, v, \ldots \}\) for the set of individuals (i.e. first-order terms) built with symbols from \(\Sigma\) and variables from \(\mathcal{V^i}\). We also assume a set \(\mathcal{R}\) of relation symbols, each coming with an arity \(n \in \mathbb{N}\).

For the second-order part of our language, we assume given for each \(n \in \mathbb{N}\) an infinite set \(\mathcal{V^o}_n = \{X^n, Y^n, Z^n, \ldots \}\) of predicate (or second-order) variables of arity \(n\). We sometimes write \(X\) for \(X^n\). We also assume given an atomic “membership” relation symbol \(\epsilon_n\) for each \(n \in \mathbb{N}\).

The formulae of second-order logic are given by

\[
A, B \;::=\; (t_1, \ldots, t_n) \epsilon_n X^n \mid R(t_1, \ldots, t_n) \\
\mid \top \mid \bot \mid A \land B \mid A \lor B \mid A \Rightarrow B \\
\mid (\forall x)A \mid (\exists x)A \mid (\forall X^n)A \mid (\exists X^n)A
\]

where \(X^n \in \mathcal{V^o}_n\), \(t_1, \ldots, t_n \in \mathcal{Ter}(\mathcal{V^i}, \Sigma)\) and \(R \in \mathcal{R}\) is of arity \(n\).

Notation 1.1. As usual, we let

\[
\neg A \;::=\; A \Rightarrow \bot \\
A \leftrightarrow B \;::=\; (A \Rightarrow B) \land (B \Rightarrow A) \\
X(t_1, \ldots, t_n) \;::=\; (t_1, \ldots, t_n) \epsilon_n X^n
\]
Basic (classical) second-order deduction ($\mathbf{NK}^0_2$) is given by the rules of Fig. 1 (where contexts $\Delta$ are finite lists of formulae), and (classical) Second-Order Predicate Logic ($\mathbf{NK}_2$) is $\mathbf{NK}^0_2$ augmented with the Comprehension Scheme:

\[
\frac{A \in \Delta}{\Delta \vdash A} \quad \frac{\Delta \vdash \top}{\Delta \vdash A} \quad \frac{\Delta \vdash A}{\Delta \vdash A \lor \neg A}
\]
\[
\frac{\Delta, A \vdash B}{\Delta \vdash A \Rightarrow B} \quad \frac{\Delta \vdash A \Rightarrow B}{\Delta \vdash B} \quad \frac{\Delta \vdash A}{\Delta \vdash A} \quad \frac{\Delta \vdash A \land B}{\Delta \vdash A} \quad \frac{\Delta \vdash A}{\Delta \vdash B} \quad \frac{\Delta \vdash B}{\Delta \vdash A \lor B} \quad \frac{\Delta \vdash A \lor B}{\Delta \vdash B}
\]
\[
\frac{\Delta \vdash A \land B}{\Delta \vdash A} \quad \frac{\Delta \vdash A}{\Delta \vdash B}
\]
\[
\frac{\Delta \vdash A}{\Delta \vdash (\forall x)A} \quad \frac{\Delta \vdash (\exists x)A}{\Delta, A \vdash B} \quad \frac{\Delta \vdash B}{\Delta \vdash C}
\]
\[
\frac{\Delta \vdash A[t/x]}{\Delta \vdash (\forall x)A} \quad \frac{\Delta \vdash (\exists x)A}{\Delta, A \vdash B} \quad (x \text{ not free in } \Delta, B)
\]
\[
\frac{\Delta \vdash A[1/x]}{\Delta \vdash (\forall x)A} \quad \frac{\Delta \vdash (\exists x)A}{\Delta, A \vdash B} \quad (x \text{ not free in } \Delta, B)
\]
\[
\frac{\Delta \vdash A[Y^n/X^n]}{\Delta \vdash (\exists X^n)A} \quad \frac{\Delta \vdash (\exists X^n)A}{\Delta, A \vdash B} \quad (X^n \text{ not free in } \Delta, B)
\]

Figure 1: Basic (Classical) Second-Order Deduction ($\mathbf{NK}^0_2$).

Basic (classical) second-order deduction ($\mathbf{NK}^0_2$) is given by the rules of Fig. 1 (where contexts $\Delta$ are finite lists of formulae), and (classical) Second-Order Predicate Logic ($\mathbf{NK}_2$) is $\mathbf{NK}^0_2$ augmented with the Comprehension Scheme:

\[
(X^n \text{ not free in } A) \quad \frac{\Delta \vdash (\exists X^n)(\forall x_1,\ldots,x_n)(X(x_1,\ldots,x_n) \iff A)}{A}
\]

Recall that equality on individuals is definable as

\[
(t \equiv u) := (\forall X^1)(X(t) \Rightarrow X(u))
\]

and that in $\mathbf{NK}_2$ we have

\[
\frac{\Delta \vdash (\forall x)(x \equiv x)}{} \quad \frac{\Delta \vdash (\forall x)(\forall y)(x \equiv y \Rightarrow A[x/z] \Rightarrow A[y/x])}{}
\]

2 Models of Second-Order Predicate Logic

We present here a notion of models for second-order logic which enjoys the same completeness and compactness properties as first-order logic. We mostly refer to [Sha91]. See also [vD04].
We write $M| (t_1, \ldots, t_n) \epsilon_n X^n$ iff $(\llbracket t_1 \rrbracket \rho, \ldots, \llbracket t_n \rrbracket \rho) \in \rho(X^n)$

$M, \rho \models R(t_1, \ldots, t_n)$ iff $(\llbracket t_1 \rrbracket \rho, \ldots, \llbracket t_n \rrbracket \rho) \in R^M$

$M, \rho \models \top$ iff $\rho \neq \bot$

$M, \rho \models A \land B$ iff $M, \rho \models A$ and $M, \rho \models B$

$M, \rho \models A \lor B$ iff $M, \rho \models A$ or $M, \rho \models B$

$M, \rho \models (\forall A)B$ iff $M, \rho \models A$ implies $M, \rho \models B$

$M, \rho \models (\exists x)A$ iff there exists $a \in M^i$ such that $M, \rho[a/x] \models A$

$M, \rho \models (\forall X^n)A$ iff for all $P \in M_n$, $M, \rho[P/X^n] \models A$

$M, \rho \models (\exists X^n)A$ iff there exists $P \in M_n$ such that $M, \rho[P/X^n] \models A$

Figure 2: The Satisfaction Relation $M, \rho \models A$.

2.1 First-Order Models

The starting point is that the second-order language described in §1 can be seen as a many-sorted first-order language. Formally, we have one sort $\iota$ of individuals and one sort $o(\iota^n)$ of n-ary predicates for each $n \in \mathbb{N}$. The individual terms $t \in \mathcal{T}_{\iota}(\mathcal{V}, \Sigma)$ are all of sort $\iota$, while the predicate variables $X^n \in \mathcal{V}_n$ are of sort $o(\iota^n)$. The relation symbols $R \in \mathcal{R}$ of arity $n$ are relations symbols over $\iota^n$ (i.e. $n$ times $\iota$), and the membership symbol $\epsilon_n$ is a relation symbol over $\iota^n \times o(\iota^n)$ (i.e. $(n$ times $\iota)) \times o(\iota^n)$.

Following the terminology of [Sha91, §3.3], a first-order model has the form

$$M = (M^i, (M_n^o)_{n \in \mathbb{N}}, \Sigma^M, R^M, (\epsilon_n^M)_{n \in \mathbb{N}})$$

where $M^i$ is a (non-empty) set of individuals, each $M_n^o$ is a (non-empty) set of n-ary predicates, $\Sigma^M$ assigns to each $f \in \Sigma$ of arity $n$ a function $f^M : (M^i)^n \to M^i$, $R^M$ assigns to each $R \in \mathcal{R}$ of arity $n$ a relation $R^M \subseteq (M^i)^n$, and $\epsilon^M_n \subseteq (M^i)^n \times M_n^o$.

The notion of satisfaction of a formula $A$ in a model $M$ under a valuation $\rho$ (notation $M, \rho \models A$) is standard. First, by a valuation $\rho$ we mean a pair $(\rho^i, (\rho^o_n)_{n \in \mathbb{N}})$ where $\rho^i : \mathcal{V} \to M^i$ and $\rho^o_n : \mathcal{V}_n \to M_n^o$. As usual, given $a \in M^i$ (resp. $P \in M_n^o$), we let $\rho[a/x]$ (resp. $\rho[P/X^n]$) be the valuation which takes $x$ to $a$ (resp. $X^n$ to $P$) and is equal to $\rho$ everywhere else. Each individual term $t$ induces an individual $\llbracket t \rrbracket \rho \in M^i$ in the obvious way. We define $M, \rho \models A$ by induction on $A$ in Fig. 2. If $A$ is a closed formula (i.e. a sentence) then we write $M \models A$ for $M, \rho \models A$.

Definition 2.1. Let $\Phi$ be a (possibly infinite) set of sentences and let $A$ be a sentence.

(a) We write $M \models \Phi$ if $M \models A$ for each $A \in \Phi$.

(b) We write $\Phi \models A$ if $M \models A$ whenever $M \models \Phi$.

(c) We write $\Phi \vdash A$ if there is a finite list $\Delta \subseteq \Phi$ (i.e. $A \in \Phi$ for each $A \in \Delta$) such that $\Delta \vdash A$ is derivable in $\text{NK}_2^\Omega$.

Usual soundness and completeness for first-order logic give the following.

Theorem 2.2 (Soundness [Sha91, Thm. 4.1]). If $\Phi \vdash A$ then $\Phi \models A$.

Theorem 2.3 (Completeness [Sha91, Thm. 4.2]). If $\Phi \models A$ then $\Phi \vdash A$. 

3
Corollary 2.4 (Compactness). Fix a set of sentences $\Phi$. If for every finite $\Phi_0 \subseteq \Phi$ there is a model $M$ such that $M \models \Phi_0$, then there is a model $M$ such that $M \models \Phi$.

Proof. Assume that there is no model $M$ such that $M \models \Phi$. This implies that $\Phi \models \bot$, so by Completeness there is a finite list $\Delta \subseteq \Phi$ such that $\Delta \vdash \bot$. Let $\Phi_0$ be the set of members of $\Delta$. By Soundness we have $\Phi_0 \models \bot$, hence there is no model $M$ such that $M \models \Phi_0$. \qed

2.2 Henkin Models

First-order models of second-order logic may seem not so relevant, since we may want to assume that each $P \in M^0_n$ is a subset of $(M^t)^n$, i.e. that $n$-ary predicates are indeed predicates on individuals.

Definition 2.5 (Henkin Model). A model $H$ is an Henkin Model if $H_n^0 \subseteq \mathcal{P}((H^0)^n)$ and $\epsilon_n^H$ is set membership (i.e. $(a_1, \ldots, a_n) \epsilon_n^H P$ iff $(a_1, \ldots, a_n) \in P$) for each $n \in \mathbb{N}$.

We usually write $H = (H^t, (H_n^0)_{n \in \mathbb{N}}, \Sigma^H, R^H)$ when we know that $H$ is an Henkin model. Each Henkin model is of course a first-order model. Conversely:

Proposition 2.6 ([Sha91, Thm. 3.5]). Fix a first-order model $M$. Let $H$ be the Henkin model

$$(M^t, (H_n^0)_{n \in \mathbb{N}}, \Sigma^M, R^M)$$

where $S \in H_n^0$ iff there is some $P \in M_n^0$ such that $S = \{(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n) \epsilon_n^M P\}$. Then for each sentence $A$, we have $M \models A$ if and only if $H \models A$.

Hence Soundness, Completeness and Compactness hold for Henkin models.

Definition 2.7 ((Henkin) Model of Second-Order Logic). An Henkin model $H$ is a model of Second-Order Logic if for each formula $A$ and each valuation $\rho$ we have

$$(H, \rho) \models (\exists X^n)(\forall x_1, \ldots, x_n)(X(x_1, \ldots, x_n) \iff A)$$

where $X^n$ is not free in $A$.

Write $\Phi \models A$ if $H \models A$ for each model of second-order logic $H$ such that $H \models \Phi$. We let $\Phi \models A$ if $\Delta \vdash A$ is derivable in NK$_2$ for some finite list $\Delta \subseteq \Phi$.

Corollary 2.8 ((Henkin) Models of Second-Order Logic).

Soundness. If $\Phi \models A$ then $\Phi \models A$.

Completeness. If $\Phi \models A$ then $\Phi \models A$.

Compactness. Fix a set of sentences $\Phi$. If for every finite $\Phi_0 \subseteq \Phi$ there is a model of second-order logic $H$ such that $H \models \Phi_0$, then there is a model of second-order logic $H$ such that $H \models \Phi$.

Remark 2.9. It is easy to see that if $H$ is an (Henkin) model of second-order logic then each $H_n^0$ is a sub-Boolean algebra of $\mathcal{P}((H^0)^n)$.

Remark 2.10 (Equality). Similarly as with first-order logic (see e.g. [vD04, Proof of Lem. 3.1.11]), we may always assume that equality $(-) = (-)$ in an Henkin model $H$ is interpreted as equality over $H$. More precisely, for each Henkin model $H$ there is an Henkin model $E$ such that $E, \rho \models (t = u)$ if and only if $[t] \rho = [u] \rho$, and such that for every sentence $A$ we have $E \models A$ if and only if $H \models A$. 

4
2.3 Full Models

Definition 2.11 (Full Model). An Henkin model $F$ is full if $F^n_0 = \mathcal{P}((F^n)^n)$ for each $n \in \mathbb{N}$.

We usually write $F = (F^\omega, \Sigma^F, R^F)$ when we know that $F$ is a full model.

Full models are automatically models of second-order logic in the sense of Def. 2.7 and they always interpret $\equiv$ by equality (see Rem. 2.10). Also, note that a finite model of second-order logic is full if it interprets $\equiv$ as equality.

Some authors (e.g. [BBJ07]) consider full models to be the right notion of model for second-order logic. However, Compactness (and thus Completeness) fails when one restricts to full models.

Example 2.12. Let $\Sigma$ consist of one constant symbol $c_k$ for each $k \in \mathbb{N}$ and let $R$ consist of a binary relation symbol $(-) \prec (-)$. Define the sentences $(A_k)_{k > 0}$ as

$$A_1 := (c_1 \prec c_0)$$

$$A_{k+1} := (c_{k+1} \prec c_k) \land A_k$$

Let $\Phi$ be the set of all the $A_k$’s together with the Axiom of Well-Founded Induction for $\prec$:

$$(\forall X)(\forall y)(\forall z)(z \prec y \Rightarrow X(z)) \Rightarrow X(y) \Rightarrow (\forall y)X(y)$$

Then every finite $\Phi_0 \subseteq \Phi$ has a full model but $\Phi$ has no full model.

Proof. First note that $\{0, \ldots, k\}$ is model of $A_k$ if one interprets $c_i$ by $i$ and if we let $i \prec j$ iff $i > j$ (the interpretation of $c_j$ for $j > k$ is irrelevant). Since $\{0, \ldots, k\}$ is finite and $> \succ$ is acyclic, it is clear that the full model induced by $\{(0, \ldots, k), (c_i \mapsto i \mid 0 \leq i \leq k), \succ\}$ satisfies well-founded induction. Since models of $A_k$ are models of $A_\ell$ for $\ell < k$, it follows that every finite $\Phi_0 \subseteq \Phi$ has a full model.

Consider now a full model $F$ of $\{A_k \mid k > 0\}$. We are going to show that $F$ does not satisfy well-founded induction. Write $k$ for the interpretation of $c_k$ in $F$. Let $S \subseteq F^\omega$ be the set of all $a \in F^\omega$ such that there is an infinite sequence $(a_n)_{n \in \mathbb{N}} \in F^\omega$ with $a_0 = a$ and $a_{n+1} \prec a_n$. Let $W := F^\omega \setminus S$. Consider some $b \in F^\omega$. If $b \in S$ then there is some $c \prec b$ such that $c \in S$. Hence, $b \in W$ if for all $c \prec b$ we have $c \in W$. However, for each $k \in \mathbb{N}$ we have $k \notin W$. □

References


Contents

1 Preliminaries 1

2 Models of Second-Order Predicate Logic 2

2.1 First-Order Models 3

2.2 Henkin Models 4

2.3 Full Models 5