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ON THE EXISTENCE OF PERIODIC INVARIANT CURVES FOR ANALYTIC FAMILIES OF TWIST MAPS AND BILLIARDS

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Abstract. In this paper we prove that in any analytic one-parameter family of twist maps of the annulus, homotopically invariant curves filled with periodic points corresponding to a given rotation number, either exist for all values of the parameters or at most for a discrete subset. This extends, in dimension 2, a previous result by Arnaud, Massetti and Sorrentino [2]. We then apply our result to rational caustics of billiards, considering several models such as Birkhoff billiards, outer billiards, symplectic billiards.

1. Introduction and main results

The existence of invariant curves in a two-dimensional dynamical system is a precious ally in the study of its stability property, *i.e.* how it behaves under small perturbations. This topic has been intensively studied since Poincaré and the breakthrough of KAM theory, impulsed by Kolmogorov [18], Arnol'd [3] and Moser [23]. KAM theory focuses on a point of view which takes into account invariant curves with highly irrational rotation number. It consists in considering small perturbations of a completely integrable system – a system foliated by invariant curves, and showing that there is a large (in a measure theoretic sense) number of these curves which persist after such perturbations. We recommend [26] for a precise overview of this theory.

Other results [10, 22, 30] proved that the other invariant curves (e.g. with rational rotation number) tend to be destroyed by an arbitrary small perturbation, and are replaced by zones of instability (hyperbolic periodic point with transverse intersection of their invariant manifolds).

Recent developments [4, 14, 15, 19] around integrable billiards and Birkhoff's conjecture suggest that the existence of invariant curves filled with periodic points is very rigid in the sense that the latter contain a lot of informations on the billiard shape (that is elliptic in the mentioned works). In [15], it is even suggested that to generalize their result to any domain, one could try to understand how invariant curves filled with periodic points behave as we embed the domain in a one-parameter family of domains converging to an ellipse (affine curve shortening flow, see [27] for more details).

In this work, we study how a fixed invariant curve of rational rotation number behaves in a given real-analytic one-parameter family of so-called twist maps (which will be defined later, and include billiard maps). This result works for any arbitrary real-analytic family of two-dimensional maps, thus generalizing a previous result of Arnaud, Massetti and Sorrentino [2] in the two-dimensional case. Our result can be roughly stated as follows:

Theorem (see Theorem 8 for a more precise statement). *Given an interval J and a real-analytic family of exact symplectic twist maps $(F_\varepsilon)_{\varepsilon\in J}$, the set of parameters $\varepsilon \in J$ such that F_ε has an invariant curve filled with periodic points of a given rotation number is either discrete or the whole interval J.*

In the rest of this section, we introduce the objects of study, and we state more precisely our theorem (see Theorem 8).

1.1. Exact-symplectic twist maps.

On the space $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ of pairs (p,q), consider $\pi_q, \pi_p : \mathbb{R}^2 \to \mathbb{R}$ the projections onto q and p.

Consider two continuous $\mathbb{Z}$-periodic maps $p^-, p^+ : \mathbb{R} \to \mathbb{R}\cup\{\pm\infty\}$ and assume that the inequality $p^-(q) < p^+(q)$ is satisfied for any $q \in \mathbb{R}$. We define the open set

$$\mathbb{A}_{p^\pm} = \{(q,p) \in \mathbb{R}^2 \,|\, p^-(q) < p < p^+(q)\}.$$

It is a bundle over $\mathbb{R}$, whose fibers are the intervals

$$\mathbb{A}_{p^\pm}\big|_q = \{p \in \mathbb{R} \,|\, p^-(q) < p < p^+(q)\}.$$

It projects onto an interval bundle over the torus $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ having the same fibers. Denote by graph (p^-) and graph (p^+), respectively, the graphs of p^- and p^+ in $\mathbb{R}^2$.

Definition 1. A diffeomorphism $F : \mathbb{A}_{p^\pm} \longrightarrow \mathbb{A}_{p^\pm}$, where $F(q,p) := (Q(q,p), P(q,p))$, is called an *exact-symplectic twist map* if it satisfies the following properties:

- **(i)** (*Periodicity*) $F(q+m,p) = F(q,p)+(m,0)$ for any $(q,p) \in \mathbb{A}_{p^\pm}$ and $m \in \mathbb{Z}$;

(ii) (*Twist condition*) for any $q \in \mathbb{R}$ the map

$$p \in \mathbb{A}_{p^\pm}|_q \longmapsto Q(q,p)$$

is a diffeomorphism onto its image;

(iii) (*Boundary preservation*) For any neighborhood V of graph $(p^-)\cup$ graph (p^+) in $\mathbb{R}^2$, there is another such neighborhood U satisfying

$$F(U \cap \mathbb{A}_{p^\pm}) \subseteq V \cap \mathbb{A}_{p^\pm};$$

(iv) (*Generating function)* There is an open set $\mathcal{D} \subseteq \mathbb{R}^2$ and a smooth map $S : \mathcal{D} \to \mathbb{R}$, called *generating function* of F, such that for any $(q,Q) \in \mathcal{D}$ and $m \in \mathbb{Z}$:

$$\begin{aligned} &(q,Q) \in \mathcal{D} \quad \Longrightarrow \quad (q+m, Q+m) \in \mathcal{D} \\ &S(p+m, Q+m) = S(q,Q), \end{aligned}$$

and

$$PdQ - pdq = dS(q,Q).$$

Remark 2. Given an exact symplectic twist map F, the map $p \in \mathbb{A}_{p^\pm}|_q \mapsto Q(q,p)$ is either strictly increasing for any q, or strictly decreasing for any q. In the first case, we say that F is *positive*, in the second one that it is *negative*. Note also that since $\partial^2_{12}S = -(\partial_p Q)^{-1}$ [16, Formula (9.2.4)], we observe that $\partial^2_{12}S < 0$ if F is positive, and $\partial^2_{12}S > 0$ if F is negative. In the proofs, we will often assume that F is positive to simplify the redaction, since the proofs in the negative case are analogous.

Remark 3. An exact symplectic twist map F induces a map f of the tangent bundle $T\mathbb{T}^1 \simeq \mathbb{T}^1 \times \mathbb{R}$. More precisely, if $\pi : \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}$ is the canonical projection, then $f : \pi(\mathbb{A}_{p^\pm}) \to \pi(\mathbb{A}_{p^\pm})$ is defined by $f \circ \pi = \pi \circ F$. We will sometimes use the same notations for both objects.

1.2. **Periodic and invariant graphs.** A *rotational invariant curve* of a symplectic twist map $F : \mathbb{A}_{p^\pm} \to \mathbb{A}_{p^\pm}$ is a curve $\Gamma \subset A_{p^\pm}$ such that $F(\Gamma) = \Gamma$ and such that $\mathbb{A}_{p^\pm} \setminus \Gamma$ consists of two connected components.

Remark 4. A famous theorem by Birkhoff (see for example [9, Theorem 15.1]) states that any rotational invariant curve Γ, if its exists, is the graph of a Lipschitz continuous 1-periodic map $\gamma : \mathbb{R} \to \mathbb{R}$. Moreover, the Lipschitz constant of γ only depends on $\inf_\Gamma \partial_p Q > 0$ ([9, Theorem 15.1] but also [13, Lemma 13.1.1] and [12, Proposition 12.3] for more details).

This leads to the following definition.

Definition 5. Let $F : \mathbb{A}_{p^\pm} \to \mathbb{A}_{p^\pm}$ be an exact-symplectic twist map, $\Gamma = \mathrm{graph}(\gamma) \subset \mathbb{A}_{p^\pm}$ be the graph of a 1-periodic Lipschitz-continuous map $\gamma : \mathbb{R} \to \mathbb{R}$, and $m \in \mathbb{Z}, n \in \mathbb{N}^*$ coprime. We say that Γ is

- **(i)** (m, n)*-periodic* if $F^n(q, \gamma(q)) = (q + m, \gamma(q))$ for any $q \in \mathbb{R}$;
- **(ii)** *invariant* under F if $F(\Gamma) \subseteq \Gamma$;
- **(iii)** $\mathcal{C}^k$-smooth (resp. analytic) if γ is a $\mathcal{C}^k$-smooth (resp. analytic).

We show in Proposition 13 that (m, n)-periodic graphs are automatically invariant by F and have the same regularity as F.

1.3. **Twist interval.** An (m, n)-periodic invariant graph Γ of F can be identified with $\mathbb{T}^1$ and $F|_\Gamma$ as the lift of a diffeomorphism of $\mathbb{T}^1$ whose rotation number is m/n. We introduce the set of all possible rotation numbers of orbits of F, and call it *twist interval* of F. It is defined as follows (see also [13, Definition 9.3.2]):

Definition 6. The *twist interval* of a symplectic twist map F is the set $\mathrm{TI}(F)$ of numbers $\alpha \in \mathbb{R}$ for which there is a neighborhood U^- of graph (p^-) and a neighborhood U^+ of graph (p^+) in $\mathbb{R}^2$ such that for $(q, p) \in \mathbb{A}_{p^\pm}$

$$(q, p) \in U^- \implies \pi_q \circ F(q, p) - q \leq \alpha$$

and

$$(q, p) \in U^+ \implies \pi_q \circ F(q, p) - q \geq \alpha.$$

Remark 7. The twist interval is by construction an open interval of $\mathbb{R}$. For example, given $\varepsilon \in \mathbb{R}$, the map $F_\varepsilon : \mathbb{R} \times (0, 1) \to \mathbb{R} \times (0, 1)$ defined for all $(q, p) \in \mathbb{R} \times (0, 1)$ by $F(q, p) = (q + p + \varepsilon, p)$ is an exact symplectic twist map whose twist interval is $\mathrm{TI}(F_\varepsilon) = (\varepsilon, 1 + \varepsilon)$.

Another example can be given in the case of the usual billiard map. Given a strictly convex domain Ω, the billiard map $F_\Omega : \mathbb{R} \times (-1, 1) \to \mathbb{R} \times (-1, 1)$ by $F_\Omega(q, -\cos\varphi) = (q_1, -\cos\varphi_1)$ where (q_1, φ_1) is the pair describing the point of impact and the angle of reflection after the bounce of a tractetory coming from q and making an angle φ with the boundary of Ω. The twist interval of F_Ω is given by $TI(F_\Omega) = (0, 1)$: indeed, when φ goes from 0 to π, the point of impact q_1 moves along the boundary $\partial\Omega$ from q to itself, winding exactly once around $\partial\Omega$.

1.4. **Main theorem.** We now consider analytic one-parameter families of symplectic twist maps. More specifically, consider an interval $I \subset \mathbb{R}$ and continuous maps $p^-, p^+ : I \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ that are 1-periodic in the second component and such that for any $(\varepsilon, q) \in I \times \mathbb{R}$

the inequality $p^-(\varepsilon,q) < p^+(\varepsilon,q)$.
One can introduce the open set

$$\mathbb{A}_{I,p^\pm} := \{(\varepsilon,q,p) \in I \times \mathbb{R}^2 \,|\, p^-(\varepsilon,q) < p < p^+(\varepsilon,q)\}$$

and denotes their closure by

$$\overline{\mathbb{A}}_{I,p^\pm} = \{(\varepsilon,q,p) \in I \times \mathbb{R}^2 \,|\, p^-(\varepsilon,q) \leq p \leq p^+(\varepsilon,q)\}.$$

Given $\varepsilon \in I$, we denote its ε-section by

$$\mathbb{A}^\varepsilon_{I,p^\pm} := \{(q,p) \in \mathbb{R}^2 \,|\, p^-(\varepsilon,q) < p < p^+(\varepsilon,q)\}.$$

Theorem 8 (Main Theorem)**.** *Assume that $I \subset \mathbb{R}$ is an interval and $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$. Suppose that for any $\varepsilon \in I$ we are given an exact symplectic twist map F_ε such that:*

- **(i)** *the map $(\varepsilon,q,p) \in \mathbb{A}_{I,p^\pm} \mapsto F_\varepsilon(q,p)$ is analytic;*
- **(ii)** *$m/n \in TI(F_\varepsilon)$ for every $\varepsilon \in I$.*

Then, the set

$$\mathcal{I}_{(m,n)}(\mathbb{R}) := \{\varepsilon \in I \,|\, F_\varepsilon \text{ has an } (m,n)\text{-periodic invariant graph}\}$$

is either discrete or consists of the whole I.

Remark 9. In the statement of Theorem 8, we do not need to precise the regularity of invariant graphs. In fact, it follows from Proposition 13 that they are *analytic*.

1.5. **Application to Birkhoff's billiards.** A billiard is a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with (piecewise) smooth boundary, in which one can study the behaviour of an infinitely small particle evolving inside Ω without friction. When reaching the boundary, the particle bounces on it according to the usual reflection law of geometrical optics *angle of incidence = angle of reflection*.

Let us consider the case when Ω is strictly convex with a smooth boundary, which defines a so-called *Birkhoff billiard*. The dynamics of a particle in Ω is described by a discrete map, *the billiard map*, acting on the space of oriented lines intersecting Ω: given such a line ℓ, the billiard map associates to it the line ℓ' naturally obtained by reflecting ℓ at the point of impact with $\partial\Omega$. The phase space is a cylinder, which in the case of dimension 2 can be parametrized by pairs $(s,\varphi) \in \mathbb{R}/|\partial\Omega|\mathbb{Z} \times [0,\pi]$, where s is an arc-length coordinate on the boundary $\partial\Omega$ and φ is the angle between the tangent line of Ω at s and the corresponding oriented line starting at s.

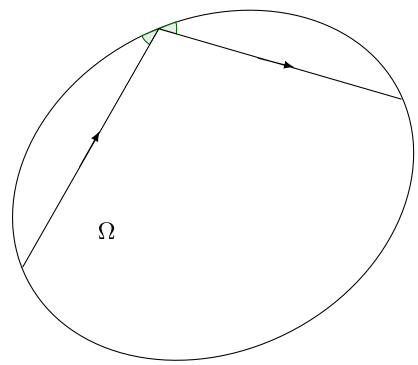

FIGURE 1. The classical reflection law of a particle inside a strictly convex domain Ω with smooth boundary.

As for dynamical systems in general, one can study the so-called *integrable billiards*: billiards whose phase space contains an open set foliated by curves which are invariant by the billiard map. Circles are examples of such billiards for which the whole phase space is foliated by invariant curves – in this case we speak about *globally* integrable billiards. Let us mention also the case of ellipses which are integrable, but not globally integrable. A famous conjecture, due to Birkhoff [6] and Poritsky [25], states that

Conjecture 10 (Birkhoff-Poritsky)**.** *The only integrable billiards are ellipses.*

Bialy [4] showed that the only globally integrable billiards are circles. Kaloshin-Sorrentino [15] proved that the only billiards close to ellipses having invariant curves of rotation number $1/q$ for any $q \geq 2$ are ellipses. Later, Koval [19] extended this result to billiards close to ellipses of small eccentricity having invariant curves of rotation number $r = p/q$, for any r lower than an arbitrarily small bound of the form $1/q_0$. The *rotation number* of an invariant curve is defined as the rotation number of the circle map obtained by restricting the billiard map to the corresponding curve.

In the case of non-rational rotation number, Lazutkin [20] showed that there is a Cantor set $C \subset [0, 1]$ of non-zero measure accumulating to 0 such that each $\omega \in C$ is the rotation number of an invariant curve. Popov showed [24] that these curve persists under a small deformation of the billiard. However the rotation numbers considered in these results are far from being rational: they are so-called Diophantine numbers, which are numbers badly approximated by rationals.

In the case of rational rotation numbers, it is expected that corresponding invariant curves are more fragile. Arnaud-Massetti-Sorrentino [2] recently proved a result in this direction for deformations of the standard map in the class of analytic twist-maps of the cylinder (in any dimension). Let us also mention the works of Kaloshin-Koudjinan [14] who are studying the case of billiard domains close to a disk having two invariant curves of rotation numbers $1/2$ and $1/2q+1$ for $q \geq 1$.

I do not know if this is the right place to mention [2] since it does not apply to billiards.

In this paper, we will extend [2] to the case of billiard maps. More precisely, we will prove the following

Theorem 11 (Invariant curves in families of Birkhoff billiards)**.** *Let I be a compact interval and $(\Omega_\varepsilon)_{\varepsilon \in I}$ be an analytic family of strictly convex analytic domains. Then given a pair $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$ of coprime integers such that $m/n \in (0,1)$, the set of $\varepsilon \in I$ such that the billiard map inside Ω_ε has an (m,n)-periodic invariant curve is either finite or consists of the whole I.*

This result doesn't answer Birkhoff's conjecture, but rather tells how fragile periodic invariant curves can be. In what follows we give more details on the objects we will consider. Theorem 11 on billiards is a consequence of Main Theorem (Theorem 8), as shown in Section 4.

1.6. **Application to dual billiards.** Given a strictly convex domain $\Omega \subset \mathbb{R}^2$ with a smooth oriented boundary, the *dual or outer billiard outside* Ω can be defined as follows (see Figure 2). For any point $p \in \mathbb{R}^2 \setminus \Omega$, there are at most two tangent lines to $\partial\Omega$ passing through p. Consider the unique one which is tangent to $\partial\Omega$ at a point q and such that the vector $\vec{pq}$ has the same orientation as the boundary $\partial\Omega$ at q. Define the image by p by the dual billiard map as the point $F(p)$ on the latter tangent line $T_q\partial\Omega$ such that q is the midpoint between p and $F(p)$.

Dual billiards were introduced by B. H. Newman (Reference?) and their properties were largely studied since then. They are known to be symplectic twist maps of the infinite annulus $\mathbb{T}^1 \times (0, +\infty)$, and thus can be investigated in the context of Aubry-Mather theory [7]. Douady [8] showed that if the boundary $\partial\Omega$ is sufficiently smooth (at least $\mathcal{C}^6$), then there is a positive measure set of invariant curves accumulating to the boundary, as well as a positive measure set of invariant curves accumulating at infinity. In paticular this gives a negative answer to

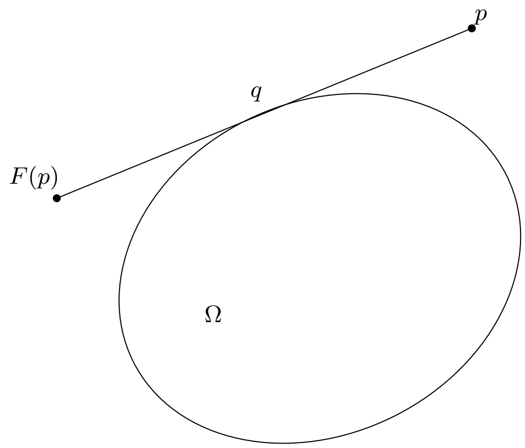

FIGURE 2. The point $F(p)$ is the image of p by the dual billiard map around the domain Ω. The point q is the midpoint between p and $F(p)$ which are supported by a line tangent to $\partial\Omega$ at q.

the famous question of the existence of unbounded orbits. Adapting a result of Mather for Birkhoff billiards (see [9]) Boyland [7] proved that in the context of simple convexity, if the curvature of the domain vanishes, or has jump discontinuities, then there is a neighborhood of the boundary without invariant curves.

A version of Birkhoff's conjecture is also studied for dual billiards. It is indeed known that the phase space of dual billiards around ellipses is foliated by invariant curves, induced by any bigger ellipse homothetically equivalent to the initial billiard. The question is to ask wether the converse is also true. Bialy [5] proved a total integrability result: if the phase space of a dual billiard is foliated by continuous invariant curves then the billiard is an ellipse. If we assume the foliation to be only in a open set of the phase space (local integrability), some partial positive results were given in [11, 29].

Theorem 12 (Invariant curves in families of outer billiards)**.** *Let I be a compact interval and $(\Omega_\varepsilon)_{\varepsilon\in I}$ be an analytic family of strictly convex analytic domains. Then given a pair $(m,n)\in\mathbb{Z}\times\mathbb{N}^*$ of coprime integers such that $m/n\in(0,1)$, the set of $\varepsilon\in I$ such that the outer billiard map associated to Ω_ε has an (m,n)-periodic invariant curve is either finite or consists of the whole I.*

Theorem 12 on billiards is a consequence of Main Theorem (Theorem 8), as shown in Section 5.

Maybe I would mention Symplectic billiards. As for the other billiard-like models, as in Albers-Tabachnikov, we could add a remark that there are many other models to which our result applies yielding to similar results, and refer to the article by Albers and Tab.

2. Preliminary results on periodic graphs

In this section we prove some preliminary results on invariant periodic graphs of a symplectic twist map $F : \mathbb{A}_{p^\pm} \to \mathbb{A}_{p^\pm}$ with generating function $S : \mathcal{D} \to \mathbb{R}$. The main results in this section are Proposition 13 and Proposition 15.

We first recall some basic notions related to the orbits of F, and we refer the reader to [12] for more details about them. The projection $\pi_q : \mathbb{R}^2 \to \mathbb{R}$ onto the q-component induces a bijection between *orbits* $(q_k, p_k)_{k\in\mathbb{Z}}$ of F, where for all $k \in \mathbb{Z}$

$$F(q_k, p_k) = (q_{k+1}, p_{k+1}),$$

and so-called *stationary configurations* $(q_k)_{k\in\mathbb{Z}}$, which are sequences satisfying for all $k \in \mathbb{Z}$

$$(q_k, q_{k+1}) \in \mathcal{D} \qquad \text{and} \qquad \partial_2 S(q_{k-1}, q_k) + \partial_1 S(q_k, q_{k+1}) = 0.$$

An orbit $(q_k, p_k)_{k\in\mathbb{Z}}$ or its associated stationary configuration $(q_k)_{k\in\mathbb{Z}}$ are called *minimal* if for any integers $u \le v$ the family $(q_k)_{u\le k\le v}$ minimizes the functionnal

$$\underline{x} = (x_k)_{u\le k\le v} \longmapsto \sum_{k=u}^{v-1} S(x_k, x_{k+1})$$

among all families $\underline{x} \in \mathbb{R}^{v-u+1}$ with $x_u = q_u$ and $x_v = q_v$.

It is known (see [4, proof of Theorem A], which applies to any symplectic twist maps, or [1] Proposition 6), that a minimal orbit $(q_k, p_k)_{k\in\mathbb{Z}}$ has no *conjugate points*, which means that any two points (q_k, p_k) and $(q_\ell, p_\ell) = F^{\ell-k}(q_k, p_k)$ along the orbit satisfy

$$\partial_p \left(\pi_q \circ F^{\ell-k}\right)(q_k, p_k) \neq 0.$$

Proposition 13. *Let $F : \mathbb{A}_{p^\pm} \longrightarrow \mathbb{A}_{p^\pm}$ be an exact-symplectic twist map and $\Gamma \subset \mathbb{A}_{p^\pm}$ be an (m, n)-periodic Lipschitz continuous graph of F, for some $m \in \mathbb{Z}, n \in \mathbb{N}^*$ coprime. Then:*

- **(i)** *Γ is invariant by F;*
- **(ii)** *the projection of an orbit intersecting Γ is a minimal configuration, and hence has no conjugate points;*
- **(iii)** *Γ is as smooth as F is;*

(iv) *F has no other (m, n)-periodic Lipschitz continuous graph.*

Remark 14. Note that in general two invariant graphs with the same rational rotation number might coexist; in this case, their intersection consists of periodic orbits and the two graphs should contain also non-periodic ones. For instance, consider the twist map corresponding to an elliptic billiard (non-circular): its phase space contains two invariant graphs of rotation number $1/2$. On the other hand, there might exist at most one invariant rotational curve for each irrational rotation number in the twist interval [16, Theorem 13.2.9].

Proof. It is well-known that Items (i) and (ii) are equivalent, see for example [9, Theorem 17.4] for the implication $(i) \Rightarrow (ii)$, and [2, Proposition 2.5] for the reverse implication. So to prove the result it is enough to show that items (i), (iii) and (iv) are satisfied.

(i) To show that Γ is invariant by F, we extend F to a symplectic twist map $G : \mathbb{R}^2 \to \mathbb{R}^2$, coinciding with F in a neighborhood of Γ and satisfying the superlinearity condition at infinity, namely

$$\lim_{|q-Q|\to+\infty} \frac{S_G(q, Q)}{|Q - q|} = +\infty,$$

where S_G denotes the generating function of G. If such a G exists, then the orbits of G intersecting Γ are minimal by [2, Proposition 2.5]; therefore, Γ is invariant by G, and hence by F.

The construction of such a G is quite standard, and we refer to [9, Section 8] or [12, Lemma 8.2]. It is however not necessarily analytic, but this does not affect the result. Let us assume that F is positive, and let S be the generating function of F: it is defined on an open set $\mathcal{D}$. Consider the set of pairs $(q, Q) \in \mathcal{D}$ such that there is a point $x \in \Gamma$ and an integer $k \in \{0, \ldots, n-1\}$ for which $\pi_q \circ F^k(x) = q$ and $\pi_q \circ F^{k+1}(x) = Q$; this set is contained in a compact set $K \subset \mathcal{D}$. On this compact set the twist condition is uniform, which means that there is $a > 0$ such that $\partial^2_{12} S_{|K} < -a$. Hence applying [9, Section 8], or [12, Lemma 8.2] we deduce the existence of a symplectic twist map $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ whose generating function S_G is defined on $\mathbb{R}^2$, coincides with S on K and has the uniform twist everywhere, that is $\partial^2_{12} S < -a'$ everywhere, for some $a' > 0$. This implies together with [12, Proposition 11.2] that the map G has the announced properties, which concludes the result.

(iii) The smoothness comes from the property that an orbit corresponding to an action-minimizing configuration has no conjugate points (see the proof of [4, Theorem A], which applies to any symplectic twist maps, or [1, Proposition 6]). Hence following the argument of [2, Proposition 2.5], this implies that the map $R : (q, p) \mapsto \pi_q \circ F^n(q, p) - q - m$ which vanishes on Γ satisfies $\partial_p R(q, p) \neq 0$ for $(q, p) \in \Gamma$. By the implicit mapping theorem, Γ can be described locally as a graph of a map which is as smooth as F, which proves the assertion.

(iv) We apply [13, Lemma 13.2.10]: if F has an invariant curve Γ of rotation number m/n, then any order-preserving orbit of F whose closure is distinct from Γ has rotation number $\neq m/n$ (the definition of order-preserving orbit is given in [13], and periodic orbits are a particular case of them). Hence any periodic orbit of rotation number m/n should have a point on Γ, and consequently be entirely contained in Γ since Γ is invariant by F. □

Let us show a localisation result when one consider a family of symplectic twist maps.

Proposition 15. *Assume that $I \subset \mathbb{R}$ is a compact interval and $(m, n) \in \mathbb{Z} \times \mathbb{N}^*$. Suppose that for any $\varepsilon \in I$ we are given an exact symplectic twist map F_ε such that:*

- **(i)** *the map $(\varepsilon, q, p) \in \mathbb{A}_{I,p^\pm} \longmapsto F_\varepsilon(q, p)$ is $\mathcal{C}^1$-smooth;*
- **(ii)** *$m/n \in TI(F_\varepsilon)$ for any $\varepsilon \in I$.*

Then, there is a neighborhood U of graph $(p^-) \cup$ graph (p^+) in $I \times \mathbb{R} \times \mathbb{R}$ such that for any $\varepsilon \in I$, any (m, n)-periodic orbit of F_ε is contained in $K := \mathbb{A}_{I,p^\pm} \setminus U$.

From this result together with Remark 4, we immediately deduce the following result.

Corollary 16. *Under the assumption of Proposition 15, there is a constant $k > 0$ depending only on $\inf_K \partial_p \pi_q F$ such that for any $\varepsilon > 0$, an (m, n)-periodic invariant graph Γ of F_ε is Lipschitz continuous with Lipschitz constant k.*

In order to prove Proposition 15, we need the following Lemma, whose proof can be found in [16, Theorem 9.3.7 and its proof].

Lemma 17. *Let F be an exact-symplectic twist map of the open annulus $\mathbb{A}_{p^\pm}$ and $m/n \in TI(F)$. Then, given $a, b \in TI(F)$ such that $a < m/n < b$, there exists a neighborhood U_- of graph (p^-) and a neighborhood U_+ of graph (p^+) such that for any $k \in \{0, \dots, n-1\}$*

$$(1) \qquad \forall (q, p) \in U_- \qquad \pi_q \circ F^{k+1}(q, p) - \pi_q \circ F^k(q, p) < a$$

and

$$(2) \qquad \forall (q,p) \in U_+ \qquad \pi_q \circ F^{k+1}(q,p) - \pi_q \circ F^k(q,p) > b.$$

We can now prove Proposition 15.

Proof of Proposition 15. Consider a decreasing sequence U_j of neighborhoods of $\mathrm{graph}\,(p^-) \cup \mathrm{graph}\,(p^+)$ in $I \times \mathbb{R} \times \mathbb{R}$ such that $\cap_j U_j = \mathrm{graph}\,(p^-) \cup \mathrm{graph}\,(p^+)$ and assume by contradiction that there are two sequences $(\underline{x}^{(j)})_j$ and $(\varepsilon_j)_j$ such that for each j, $\underline{x}^{(j)}$ is an (m,n)-periodic orbits of F_{ε_j} having a point in $(\{\varepsilon_j\} \times \mathbb{A}^{\varepsilon_j}_{I,p^\pm}) \cap U_j$.

Since I is compact, we can assume that the sequence $(\varepsilon_j)_j$ converges to a certain $\varepsilon \in I$. Consider the sets U_- and U_+ of Lemma 17 associated to the rotation number m/n and the twist map F_ε. Define $U = U_- \cup U_+$. By continuity of F, p^- and p^-, for ε' sufficiently close to ε, U remains a neighborhood of $\mathrm{graph}\,(p^-)(\varepsilon', \cdot) \cup \mathrm{graph}\,(p^+)(\varepsilon', \cdot)$ and Equations (1) and (2) are also satisfied by $F_{\varepsilon'}$. In particular, considering $\varepsilon' = \varepsilon_j$ for sufficiently large j, (m,n)-periodic orbits of F_{ε_j} have their points in $\mathbb{A}^{\varepsilon_j}_{I,p^\pm} \setminus U$. Now by intersection property on the sequence of U_j's, if j is sufficiently large $U_j \subset I \times U$, and the latter implies that $\underline{x}^{(j)}$ is an (m,n)-periodic orbit of F_{ε_j} having a point in U. This is contradictory and concludes the proof. $\square$

3. Proof of Main Theorem (Theorem 8)

In this section, we prove the Main Theorem, namely Theorem 8. Given a pair $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$ of coprime integers, we first recall some properties of the set

$$\mathcal{I}_{(m,n)}(\mathbb{R}) = \{\varepsilon \in I \,|\, F_\varepsilon \text{ has an } (m,n)\text{-periodic invariant graph}\}$$

defined in the statement of the theorem.

Lemma 18. *Under the assumptions of Theorem 8.*

- **(i)** *For $\varepsilon \in \mathcal{I}_{(m,n)}(\mathbb{R})$, there exists a unique 1-periodic continuous map $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that graph(γ_ε) is an (m,n)-periodic invariant graph invariant by F_ε.*
- **(ii)** *The map $(\varepsilon, q) \in \mathcal{I}_{(m,n)}(\mathbb{R}) \times \mathbb{R} \mapsto \gamma_\varepsilon(q)$ is continuous (for the topology induced by $I \times \mathbb{R}$ on $\mathcal{I}_{(m,n)}(\mathbb{R}) \times \mathbb{R}$).*
- **(iii)** *$\mathcal{I}_{(m,n)}(\mathbb{R})$ is a closed subset of I.*

Proof. **(i)** The existence of such γ_ε follows from the definition of $\mathcal{I}_{(m,n)}(\mathbb{R})$. The unicity comes from Proposition 13. Note that γ_ε is Lipschitz continuous with a Lipschitz constant depending on ε, but which can be chosen uniformly for any ε lying in a compact subinterval $I' \subset I$ by Corollary 16. This property will be useful for the rest of the proof.

(ii) Choose $\varepsilon \in \mathcal{I}_{(m,n)}(\mathbb{R})$ and a sequence $(\varepsilon_j)_j \subset \mathcal{I}_{(m,n)}(\mathbb{R})$ converging to ε. Let us how that $(\gamma_{\varepsilon_j})_j$ converges to γ_ε in the uniform topology. As noticed in item (i), each γ_{ε_j} as well as γ_ε is Lipschitz continuous, and they share the same Lipschitz constant (*ie* equi-Lipschitz).

We do the proof in two steps:

• **Step 1.** We prove that any subsequence of $(\gamma_{\varepsilon_j})_j$ converging in the space of 1-periodic continuous maps $\mathbb{R} \to \mathbb{R}$ for the uniform topology has γ_ε as a limit;

• **Step 2.** We show that $(\gamma_{\varepsilon_j})_j$ has at least one converging subsequence (applying Ascoli-Arzelà theorem).

These two steps imply the assertion of item (ii).

Proof of Step 1. Assume that there is a subsequence $(\gamma_{\varepsilon_{j_k}})_k$ converging to a 1-periodic map $\gamma_\infty : \mathbb{R} \to \mathbb{R}$ in the uniform topology. By Proposition 15, graph $(\gamma_\infty) \subset \mathbb{A}^{\varepsilon}_{I,p^\pm}$. The identity $F^n_{\varepsilon_{j_k}} = \mathrm{Id} + (m, 0)$ is satisfied on graph $\gamma_{\varepsilon_{j_k}}$, hence, by continuity, $F^n_\varepsilon = \mathrm{Id} + (m, 0)$ on graph (γ_∞). Let us mention that γ_∞ is Lipschitz continuous since all $\gamma_{\varepsilon_{j_k}}$ are Lipschitz continuous with the same Lipschitz constant. Hence graph γ_∞ is a (m, n)-periodic graph of F_ε, which is invariant by Proposition 13. Therefore $\gamma_\infty = \gamma_\varepsilon$, which follows from the unicity of (m, n)-periodic graphs, again by Proposition 13.

Proof of Step 2. Let us check that the assumptions of Ascoli-Arzelà theorem are satisfied by the sequence of maps $(\gamma_{\varepsilon_j})_j$. By Proposition 15, the maps γ_{ε_j} are contained in a compact subset K of $\mathbb{A}_{I,p^\pm}$ which implies that they are bounded by the same constant (*ie* equibounded). Moreover, we noticed already that they are Lipshitz continuous with a uniform Lipschitz constant. Hence Step 2 follows from Ascoli-Arzelà theorem, and therefore point (2) is proven.

(iii) Consider a sequence of $\varepsilon_j \in \mathcal{I}_{(m,n)}(\mathbb{R})$ converging to a $\varepsilon \in I$. As in item (ii), the family of maps γ_{ε_j} is equibounded and equi-Lipschitz. By extracting a subsequence as in Step 2 of item (ii), we can suppose that γ_{ε_j} converges to a 1-periodic Lipschitz continuous map $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$, such that $\{\varepsilon\} \times \mathrm{graph}\,(\gamma_\varepsilon) \subset \mathbb{A}_{I,p^\pm}$ (Proposition 15), and which is (m, n)-periodic by continuity of F as a map of (ε, q, p). By Proposition

13, this implies that $\varepsilon \in \mathcal{I}_{(m,n)}(\mathbb{R})$.

□

We can now prove the following stronger result on the topological structure of the set $\mathcal{I}_{(m,n)}(\mathbb{R})$.

Lemma 19. *Under the assumptions of Theorem 8, the set $\mathcal{I}_{(m,n)}(\mathbb{R})$ is either the whole I or it has empty interior.*

Proof. Assume that $\mathcal{I}_{(m,n)}(\mathbb{R})$ has non-empty interior, and let us show that $\mathcal{I}_{(m,n)}(\mathbb{R}) = I$. Let us define $a = \inf I$ and $b = \sup I$ so that $I \cap (a,b) = (a,b)$..

Consider a connected component $A \subset \mathcal{I}_{(m,n)}(\mathbb{R})$ which is not reduced to a point. Let $\beta = \sup A$, and suppose that $\beta < b$. We will raise a contradiction.

First note that necessarily $\beta \in \mathcal{I}_{(m,n)}(\mathbb{R})$ since $\beta \in I$ and the set $\mathcal{I}_{(m,n)}(\mathbb{R})$ is closed in I by Lemma 18; therefore, $\beta \in A$.

Applying again Lemma 18, we can then define a family of Lipschitz continuous maps $(\gamma_\varepsilon)_{\varepsilon\in A}$ such that for all $\varepsilon \in A$, $\mathrm{graph}(\gamma_\varepsilon)$ is an (m,n)-periodic graph invariant by F_ε. Moreover the map $\Gamma : (\varepsilon, q) \mapsto \gamma_\varepsilon(q)$ is continuous, again by Lemma 18. We will show that we can extend Γ to the open set $(\beta - r, \beta + r) \times \mathbb{R}$, with $r > 0$, thus leading to a contradiction, since this would imply that $(\beta - r, \beta + r) \subseteq A$, thus contradicting the maximality of A.

Let us apply the implicit function theorem to the map

$$\Delta_1(\varepsilon, q, p) := \pi_q \circ F_\varepsilon^n(q,p) - q - m.$$

Since F_β has no conjugate points on $\mathrm{graph}(\gamma_\beta)$ (see Proposition 13), $\partial_p \Delta_1$ do not vanish on the set $\{(\beta, q, \gamma_\beta(q)) \,|\, q \in \mathbb{R}\}$. Hence we can define an analytic map $(\varepsilon, q) \in (\beta - r, \beta + r) \times \mathbb{R} \mapsto \eta_\varepsilon(q)$, with $r > 0$, such that $\eta_\beta = \gamma_\beta$ and for any (ε, q, p) close to $(\beta, q, \eta_\beta(q))$ we have

$$\pi_q \circ F_\varepsilon(q,p) = q + m \qquad \Leftrightarrow \qquad p = \eta_\varepsilon(q).$$

The latter implies – together with the continuity of γ – that for $\varepsilon < \beta$ and sufficiently close to β, we have $\eta_\varepsilon = \gamma_\varepsilon$.

Now consider, the map $\Delta_2 : (\beta - r, \beta + r) \times \mathbb{R} \to \mathbb{R}$ defined by

$$(\varepsilon, q) \mapsto \pi_p \circ F_\varepsilon^n(q, \eta_\varepsilon(q)) - \eta_\varepsilon(q).$$

Since $\eta_\varepsilon = \gamma_\varepsilon$ for $\beta - r < \varepsilon \le \beta$ and graph γ_ε is (m,n)-periodic, Δ_2 vanishes on $(\beta - r, \beta] \times \mathbb{R}$. Since Δ_2 is analytic and $(\beta - r, \beta + r) \times \mathbb{R}$ is connected, Δ_2 vanishes on $(\beta - r, \beta + r) \times \mathbb{R}$.

Combining the two results on Δ_1 and Δ_2, we obtain that, for any $\varepsilon \in (\beta - r, \beta + r)$, graph$(\eta_\varepsilon)$ is an (m,n)-periodic graph of F_ε. This contradicts the fact that ε cannot be bigger than β. □

Proof of Theorem 8. Let us suppose that $\mathcal{I}_{(m,n)}(\mathbb{R})$ has an accumulation point $\beta \in I$ and show that in this case $\mathcal{I}_{(m,n)}(\mathbb{R}) = I$. The proof of Lemma 19 can be adapted to this context: there is a sequence $(\varepsilon_n)_n$ converging to β with $\varepsilon_n \neq \beta$ satisfying

$$\forall (n,q) \in \mathbb{N} \times \mathbb{R} \qquad\qquad \Delta_2(\varepsilon_n, q) = 0.$$

Hence Δ_2 is flat at any $(\beta, q) \in \{\beta\} \times \mathbb{R}$, meaning that its partial derivatives of any order in ε and q vanish. By analyticity of Δ_2 and 1-periodicity in q, it vanishes on an open set of the form $J \times \mathbb{R}$. Hence for ε sufficiently close to β, η_ε parametrizes an (m,n)-periodic F_ε-invariant graph. We conclude that $\mathcal{I}_{(m,n)}(\mathbb{R})$ has non-empty interior, and by Lemma 19 that $\mathcal{I}_{(m,n)}(\mathbb{R}) = I$. □

4. Proof of Theorem 11 using Main Theorem

Let $(\Omega_\varepsilon)_{\varepsilon \in I}$ be an analytic family of strictly convex domains with analytic boundary. By applying homotheties to the different Ω_ε, we can suppose that each Ω_ε has perimeter 1 (note that homotheties do not break the property of the billiard map in a domain of having an (m,n)-periodic invariant graph of a fixed rotation number).

For each $\varepsilon \in I$, one can consider a parametrization $s \in \mathbb{R} \mapsto \gamma_\varepsilon(s)$ of γ_ε by arc-length which by assumption one can assume analytic in (ε, s). The billiard map inside Ω_ε induces an exact symplectic twist map $F_\varepsilon : \mathbb{R} \times (-1,1) \to \mathbb{R} \times (-1,1)$ defined for all $(s,\sigma) \in \mathbb{R} \times (-1,1)$ by

$$F_\varepsilon(s,\sigma) = (s_1, \sigma_1)$$

where, if $\sigma = -\cos\varphi$ and $\varphi \in (0,\pi)$, then s_1 and σ_1 are defined by following requirements a) and b):

a) $\gamma_\varepsilon(s_1)$ is the second point of intersection with $\partial\Omega_\varepsilon$ of the line ℓ passing through $\gamma_\varepsilon(s)$ and making an angle φ with the tangent line of $\partial\Omega$ at $\gamma_\varepsilon(s)$;

b) the line ℓ makes an angle φ_1 with the tangent line of $\partial\Omega$ at $\gamma_\varepsilon(s_1)$ and this defines $\sigma_1 = -\cos\varphi_1$.

Let us check that the corresponding family of billiard maps $(F_\varepsilon)_\varepsilon$ satisfies the assumptions of Theorem 8.

It is well-known that the billiard map written in the coordinates (s, σ) is an exact symplectic twist map as defined in Definition 1.

Given $\varepsilon \in I$ and $(s, \varphi) \in \mathbb{R} \times (-1, 1)$, the corresponding line ℓ (as in a)) is transverse to the boundary of Ω_ε at $\gamma_\varepsilon(s_1)$. A simple computation shows that it allows to apply the implicit function theorem, namely there is a neighborhood U of $(\varepsilon, s, \varphi) \in I \times \mathbb{R} \times (-1, 1)$ and an analytic map $\varphi : U \to \mathbb{R}$ such that $s_1(\varepsilon', s', \sigma') = \varphi(\varepsilon', s', \sigma')$ for $(\varepsilon', s', \sigma') \in U$. The analytic regularity of σ_1 comes from the fact that it can be written as

$$\sigma_1 = \frac{\gamma_\varepsilon(s_1) - \gamma_\varepsilon(s)}{\|\gamma_\varepsilon(s_1) - \gamma_\varepsilon(s)\|} \cdot \gamma'_\varepsilon(s_1)$$

where $u \cot v$ denotes the scalar product of two vectors u and v. Hence Assumption (i) of Main Theorem is satisfied.

It is well-known that since each Ω_ε is strictly convex, the map F_ε extends to a continuous map $\mathbb{R} \times [-1, 1]$ satisfying for any $s \in \mathbb{R}$ the equalities $F_\varepsilon(s, 0) = (s, 0)$ and $F_\varepsilon(s, 1) = (s + 1, 1)$. This implies that $\mathrm{TI}(F_\varepsilon) = (0, 1)$.

Hence the assumptions of Main Theorem are satisfied and it implies the result.

5. Proof of Theorem 12 using Main Theorem

Let $(\Omega_\varepsilon)_{\varepsilon \in I}$ be an analytic family of strictly convex domains with analytic boundary. We introduce the so-called *enveloppe coordinates* on each $\partial\Omega_\varepsilon$, see [7]. They are defined as follows.

By applying translations to the domains, one can assume that there is a point O which remains inside all domains. Consider a fixed direction Ox. For each $\varepsilon \in I$ and any angle $\theta \in \mathbb{R}$, one can associate the oriented line $\mathcal{L}_\theta$ to $\partial\Omega$ which makes an angle $\theta + \pi/2$ with Ox and is tangent to it at a point $\alpha(\theta)$ where the orientations of $\partial\Omega$ and $\mathcal{L}_\theta$ are the same. Let $p_\varepsilon(\theta)$ be the distance from O to $\mathcal{L}_\theta$. Under the assumptions of Theorem 12, one can assume that p is analytic in (ε, θ).

For any point $p \in \mathbb{R}^2 \setminus \Omega$, one can find a unique θ such that the vector $\alpha(\theta) - z$ and $\mathcal{L}_\theta$ are colinear with the same orientation. The pair (θ, γ) where $\gamma = \|\alpha(\theta) - z\|^2/2$ is called the enveloppe coordinate of p and uniquely determines p.

The outer billiard map outside Ω_ε acts on the space of enveloppe coordinates $\mathbb{R}\times(0,+\infty)$ and induces an exact symplectic twist map $F_\varepsilon:\mathbb{R}\times(0,+\infty)\to\mathbb{R}\times(0,+\infty)$ defined for all $(\theta,\gamma)\in\mathbb{R}\times(0,+\infty)$ by

$$F_\varepsilon(\theta,\gamma)=(\theta_1,\gamma_1).$$

Let us check that the corresponding family of outer billiard maps $(F_\varepsilon)_\varepsilon$ satisfies the assumptions of Theorem 8.

Given $\varepsilon\in I$ and $p\in\mathbb{R}^2\setminus\overline{\Omega}_\varepsilon$, consider $G_\varepsilon(p)$ to be the image of p after one outer billiard reflection on Ω_ε. The map G is well-defined and analytic. It is then a consequence of the implicit function theorem that the enveloppe coordinates of a point q depend analytically on q. Hence Assumption (i) of Main Theorem is satisfied.

It is well-known that the twist interval of a dual billiard map is $\mathrm{TI}(F_\varepsilon)=(0,1/2)$. Hence the assumptions of Main Theorem are satisfied and it implies the result.

6. Appendix

Proposition 20. *Let $f:\mathbb{T}^1\times(0,1)\to\mathbb{T}^1\times(0,1)$ be the billiard map written in $(s,-\cos\varphi)$ coordinates and Γ be an (m,n)-periodic Lipschitz graph Γ of f. Then*
(i) *Γ invariant by f;*
(ii) *the projection of an orbit of Γ is a minimal configuration;*
(iii) *Γ is as smooth as f is.*

Proof. We show first (ii); then (i) and (iii) will follow from Proposition 13.

(ii) Minimality. First let us show that the projection $\underline{s}=(s_p)_{p\in\mathbb{Z}}$ of an orbit $(s_p,y_p)_p$ such that $(s_0,y_0)\in\Gamma$ is minimal (here y stands for $-\cos\varphi$).

Let us write $\Gamma=\{(s,\phi(s))\,|\,s\in L\mathbb{T}^1\}$ where $\phi:\mathbb{T}^1\to(0,1)$ is a Lipschitz continuous map and L is the perimeter of the billiard boundary. Denote by $h:D\to\mathbb{R}$ the generating function of the billiard where $D=\{(s,s')\,|\,s\le s'\le s+1\}$. For any $p\le q$, define

$$\mathcal{E}_{pq}(x_p,\dots,x_q)=\sum_{k=p}^{q-1}h(x_k,x_{k+1})-\int_{x_p}^{x_q}\phi(u)du.$$

In the case of the billiard map, $\mathcal{E}_{pq}$ is well-defined and continuous on the compact set $K=\{(x_p,\dots,x_q)\,|\,x_0\in[0,1],\ \forall k\ x_k\le x_{k+1}\le x_k+1\}$.

Hence it has a global minimal value which is reached at a certain point $\underline{x} = (x_p, \dots, x_q)$. Now by triangular inequality, we have the important fact that

$$\underline{x} \in \mathrm{int}(K).$$

This property is specific to the billiard map and due to the fact that $h(x,y) + h(y,z) < h(x,x) + h(x,z)$ for any $x < y < z$. Hence $\underline{x}$ is a critical point of $\mathcal{E}_{pq}$ and by a classical argument, $\underline{x}$ is a stationary configuration corresponding to an orbit $(x_j, y_j)_{p \le j \le q}$ of the billiard map such that $y_p = \phi(x_p)$ and $y_q = \phi(x_q)$.

It follows that our initial configuration $\underline{s}$ minimizes each $\mathcal{E}_{pn,qn}$ among all configurations with the same endpoints s_{pn} and s_{qn}, where $p \le q$. Hence s minimizes the action considered between the indices pn and qn, among all configurations with the same endpoints. By another classical argument, $\underline{s}$ is minimal. □

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