Intermittency and multifractal formalism

1 Gaussian description of the Kolmogorov (1941) theory

Let us consider the random field defined by

$$u_\eta^g(x) = \int_{\mathbb{R}} \varphi_L(x-y) \frac{dW(y)}{\sqrt{|x-y|^2+\eta^2}^{1/2-H}},$$

for a given exponent $0 < H < 1$, $\eta > 0$, W a Gaussian white measure, and φ_L a cutoff function at scale L (which we assume smooth and compactly supported). We shall admit the following rules:

$$\mathbb{E}\left[dW(y)\right] = 0 \text{ and } \mathbb{E}\left[dW(y)dW(y')\right] = \delta(y-y')dydy'.$$

1. For fixed $\eta > 0$, justify that u_η^g is well-defined and smooth. Draw the graph of a possible choice for φ_L. Give a physical interpretation of the parameter η.
2. Justify that the random field u_η^g is gaussian.
3. Show that it is homogeneous and has zero mean, i.e. $\mathbb{E}[u_\eta^g] = 0$.
4. Compute its variance and show that it admits a finite (non-zero) limit when $\eta \to 0$. We shall denote u^g the limit process.

Let us define the velocity increment at scale ℓ:

$$\delta_\ell u^g(x) = u^g(x+\ell) - u^g(x) = \int_{\mathbb{R}} \phi_\ell(x-y)dW(y),$$

with

$$\phi_\ell(x) = \varphi_L(x+\ell)\frac{1}{|x+\ell|^{1/2-H}} - \varphi_L(x)\frac{1}{|x|^{1/2-H}}.$$

4. Show that the random field $\delta_\ell u^g$ is gaussian.
5. Show that the variance $\mathbb{E}[(\delta_\ell u^g)^2]$ is finite, and determine its asymptotic value for $\ell \to 0$. Under which condition on H is it compatible with the Kolmogorov theory?
6. Show that the velocity increment can be written as

$$\delta_\ell u^g \underset{\text{law}}{=} \sigma \left(\frac{\ell}{L}\right)^H \times \omega,$$

   where ω is a Gaussian random variable with zero mean and unit variance, with $\sigma^2 = \langle(\delta_L u)^2\rangle$ the variance of the velocity increment at large scale. The above equality means that the two random variables have the same probability law.
7. Compute $\mathbb{E}[(\delta_\ell u^g)^3]$.
8. Show that $\mathbb{E}[(\delta_\ell u^g)^{2n}]$ is proportional to $\left(\mathbb{E}[(\delta_\ell u^g)^2]\right)^n$. Deduce the scaling exponents ζ_n for the structure functions, defined by $\mathbb{E}(\delta_\ell u^g)^{2n} \underset{\ell\to 0}{\sim} \ell^{\zeta_{2n}}$.

2 Intermittency and non-gaussianity

Let us now study the properties of velocity increments with a random variance at each scale. More precisely, let us assume that the velocity increment can be written as the product of two independent random variables:

$$\delta_\ell u \underset{\text{law}}{=} \beta_\ell \times \omega,$$

where ω is still a standard normally distributed random variable and β_ℓ is a positive random variable of the form

$$\beta_\ell = \sigma \left(\frac{\ell}{L}\right)^h,$$

where $\sigma^2 = \langle(\delta_L u)^2\rangle$ is the variance of velocity increments at large scale, and h is a random scaling exponent (called *Hölder exponent*), with density

$$\mathcal{P}_h^{(\ell)}(h) = \frac{1}{\mathcal{Z}(\ell)} \left(\frac{\ell}{L}\right)^{1-\mathcal{D}(h)},$$

and $\mathcal{D}(h)$ is a function, independent of the scale ℓ but which may depend on parameters, and $\mathcal{Z}(\ell)$ is a normalization constant.

1. Show that the Probability Distribution Function (PDF) of the velocity increment $\delta_\ell u$ can be written as

   $$\mathcal{P}_\ell(\delta u) = \int_{h_{\min}}^{h_{\max}} \frac{1}{\sigma}\left(\frac{\ell}{L}\right)^{-h} \mathcal{P}_\omega \left[\frac{\delta u}{\sigma}\left(\frac{\ell}{L}\right)^{-h}\right] \mathcal{P}_h^{(\ell)}(h) dh,$$

   where $\mathcal{P}_\omega(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the standard normal distribution.

   After recalling the shape of a *self-similar* PDF, suggest an interpretation of this model.

2. Compute the structure function of order n, $S_n(\ell) = \mathbb{E}|\delta_\ell u|^n$. We shall use that $\mathbb{E}|\omega|^n = \Gamma\left(\frac{n+1}{2}\right)/\sqrt{2^n \pi}$.

3. Show that in the small scale limit $\ell \to 0$, the structure functions follow a power-law scaling:

   $$S_n(\ell) \sim \left(\frac{\ell}{L}\right)^{\zeta_n}.$$

   Assuming $\min_h [1 - \mathcal{D}(h)] = 0$, show that

   $$\zeta_n = \min_h \left[nh + 1 - \mathcal{D}(h)\right].$$

4. Let us now consider the *log-normal* model corresponding to

   $$\mathcal{D}^{\mathrm{LN}}(h) = 1 - \frac{(h - c_1)^2}{2c_2}.$$

   The coefficient c_2 is called the *intermittency coefficient*.

   Why is this model called *log-normal*?

   Compute ζ_n^{LN}.

   Is this model compatible with the 2/3-law and the 4/5-law?

5. Finally we consider the *She-Lévêque* model:

   $$\mathcal{D}^{\mathrm{SL}}(h) = -1 + 3\left[\frac{1 + \ln(\ln(3/2))}{\ln(3/2)} - 1\right](h - 1/9) - \frac{3}{\ln(3/2)}(h - 1/9)\ln(h - 1/9).$$

   Compute ζ_n^{SL}.

   Is this model compatible with the 2/3-law and the 4/5-law?