Turbulent transport of a passive scalar

In this problem, we study the effect of turbulence on the transport of quantities like temperature or chemical species in a fluid. Such quantities can be represented in an abstract manner by a dimensionless scalar field, governed by an advection-diffusion equation:

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta, \tag{1}$$

where κ is the molecular diffusivity of the scalar, and $\mathbf{u}$ a given (deterministic or random) velocity field, assumed divergence-free. For definiteness, we assume that the domain is a 3D cube of side L with periodic boundary conditions. The scalar is said *passive* because it does not affect the properties of the velocity field (e.g. it does not appear in the Navier-Stokes equations, or the prescribed statistics of $\mathbf{u}$ do not depend on θ). Here, we shall simply assume that the statistics of the velocity field $\mathbf{u}$ are well described by Kolmogorov theory. We denote by ν the molecular viscosity of the fluid and let $\mathrm{Pr} = \frac{\nu}{\kappa}$.

We shall further assume that the statistics of all the fields are stationary, homogeneous and isotropic, and denote by $\langle \cdot \rangle$ the average value with respect to the invariant measure.

We define the mean energy dissipation rate $\varepsilon = \nu \langle \|\nabla \mathbf{u}\|^2 \rangle = \nu \langle \partial_i u_j \partial^i u^j \rangle$.

1 Phenomenology

1. We define the variance of the passive scalar $\Xi = \langle \theta^2 \rangle / 2$, and we introduce a source term Q in the right-hand side of (1). Show that

$$\frac{d\Xi}{dt} = \langle Q\theta \rangle - \varepsilon_\theta, \tag{2}$$

   with $\varepsilon_\theta = -\kappa \langle \theta \Delta \theta \rangle$ the mean scalar variance dissipation rate. What is the sign of ε_θ?

2. When $Q = \kappa = 0$, Ξ is a conserved quantity. Under the same conditions, are there other invariants?

3. In practice, the scalar may be forced for instance by imposing a mean scalar gradient Γ in an arbitrary direction (say the z direction); what is the form of the source term Q in that case?

   In the sequel, we shall assume that the passive scalar source acts at a spatial scale comparable to the scale ℓ_F of the mechanism generating the turbulent flow. That scale is assumed to be close to the size of the domain L. What is the corresponding condition on Γ?

4. Explain qualitatively how the term $\mathbf{u} \cdot \nabla \theta$ acts to transfer scalar variance across scales.

5. How many (independent) non-dimensional numbers characterize the statistical properties of solutions of (1)? By analogy with the first Kolmogorov assumption of universality for the velocity field, list the parameters upon which the statistics (two-point correlation functions) of the scalar field should depend.

6. We introduce the scalar variance spectrum $F_\theta(k)$ such that $\Xi = \int_0^{+\infty} F_\theta(k)dk$.

   Using dimensional analysis, show that if there exists a range of scales where dissipative effects can be discarded, the spectrum should take the form $F_\theta(k) = C_\theta \varepsilon_\theta^{-3\beta} \varepsilon^\beta k^{2\beta-1}$ in that range, with C_θ a non-dimensional constant. Why is there a free parameter β?

7. (a) By analogy with the Kolmogorov scale η, define a scale η_θ where inertial and diffusive effects equilibrate.
   (b) By considering separately velocity increments in the inertial and dissipative range, establish two scaling laws for η_θ/η with respect to the non-dimensional numbers.
   (c) By comparing η_θ and η, show that these two scaling laws correspond to the two regimes $\mathrm{Pr} < 1$ and $\mathrm{Pr} > 1$.

8. We assume that $\mathrm{Pr} < 1$ and we consider the inertial-convective range of scales $\ell_F \geq \ell \geq \eta_\theta$.
   (a) Estimate the typical time scale for velocity fluctuations (the eddy-turnover time) in that range of scales, as a function of the scale k and the energy spectrum $E(k)$.
   (b) Similarly, relate the scalar variance at scale k to the scalar variance spectrum $F_\theta(k)$, estimate the scalar dissipation rate ε_θ, and deduce the expression of $F_\theta(k)$ as a function of $\varepsilon_\theta, \varepsilon$ and k only. This is the *Kolmogorov-Obhukov-Corrsin* spectrum. Is it compatible with dimensional analysis?
   (c) Draw a schematic picture of the scalar variance spectrum.

9. We now consider the $\mathrm{Pr} > 1$ regime.
   (a) Is the *Kolmogorov-Obhukov-Corrsin* spectrum still relevant in this case?
   (b) We consider the viscous-convective range $\eta_\theta \leq \ell \leq \eta$. Using the typical timescale for viscous dissipation instead of the eddy-turnover time, derive the expression of the passive scalar spectrum $F_\theta(k)$. This is the *Batchelor* spectrum. Is it compatible with dimensional analysis? Why?
   (c) Draw a schematic picture of the scalar variance spectrum.

2 Karman-Howarth equation

Let us define the passive scalar covariance:

$$R(\mathbf{x}, \mathbf{x}+\mathbf{r}, t) = \langle \theta(\mathbf{x}, t)\theta(\mathbf{x}+\mathbf{r}, t)\rangle , \tag{3}$$

and the velocity and scalar increments $\delta\mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)$ and $\delta\theta = \theta(\mathbf{x}+\mathbf{r}, t) - \theta(\mathbf{x}, t)$.

1. Derive the evolution equation for R. Simplify it as much as possible by exploiting statistical homogeneity.

2. Still making use of the symmetries, show that, in the stationary state,

$$-\nabla_{\mathbf{r}} \cdot \left\langle \delta\theta^2 \delta\mathbf{u} \right\rangle = 2\left\langle Q(\mathbf{x}, t)[\theta(\mathbf{x}+\mathbf{r}, t) + \theta(\mathbf{x}-\mathbf{r}, t)]\right\rangle - 2\kappa\Delta_{\mathbf{r}} \left\langle \delta\theta^2 \right\rangle . \tag{4}$$

   Give a physical interpretation of this relation.

   Justify that for $r \ll \ell_F$, the first term in the right-hand side can be replaced by $4\varepsilon_\theta$.

3. By integrating (4) on the ball with center $\mathbf{x}$ and radius r, show that

$$\left\langle \delta\theta^2 \delta u_\parallel \right\rangle = -\frac{4}{3}\varepsilon_\theta r + 2\kappa \frac{\mathrm{d}\left\langle \delta\theta^2\right\rangle}{\mathrm{d}r}. \tag{5}$$

   What can you deduce about the direction of the cascade of passive scalar variance?